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On Lipschitz Projection – a Geometrical Approach

O rzutowaniu lipszycowskim – podejście geometryczne

О липшицевой проекции – геометрический подход

Let X be a Banach space over reals. Let Y be a subspace of X. We say that M is a Lipschitz projection on Y if $MX = Y$, $Mx = x$ for $x \in Y$ and

$$\| M(x_1) - M(x_2) \| \leq L \| x_1 - x_2 \| \tag{1}$$

The infimum of constants satisfying (1) is called the norm.

Lindenstrauss [1] proved that if X is a reflexive space and if there is a Lipschitz projection of norm L then there is also a linear projection of norm L. In particular it holds for all finite dimensional spaces. However, even in finite dimensional spaces the proof of Lindenstrauss is going via construction of a projection of an infinite dimensional space onto finite dimensional. In this note I shall prove the Lindenstrauss theorem for the case when X is three-dimensional space and Y is two-dimensional and for the norm equal one by a geometrical method. The proof is based on the following lemmas:

Lemma 1. *Let P_0, P_1 be convex closed figures in R^2. We assume moreover that P_0 is centrally symmetric and that* $(0, 0)$ *is its center of symmetry. Then there are two possibilities either there is a translation a_0 such that*

$$a_0 + P_1 \subset P_0 \tag{2}$$

or there is a translation a_1 such that

$$(0, 0) \in \operatorname{conv} [(a_1 + P_1) \setminus P_0] \tag{3}$$

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Proof. Suppose that (2) does not hold. Let a_i^0 be chosen in such a way that $f(a_i^0) = \min f(a)$, where

$$f(a) = \sup_{x \in a+P_1} \inf_{y \in P_0} \| x - y \|$$

be minimal. The existence of such a number follows from compactness arguments. Suppose that (3) does not hold. Hence, there is a real line L such that conv $[(a_i^0 + P_1) \setminus P_0]$ is on a one side of L and a projection P_L of direction a_L of norm one on L. This shifting P_1 into direction of projection we obtain that

$$f(a_i^0 - ta_L) < f(a_i^0) \tag{4}$$

for sufficiently small t and it leads to a contradiction with the choice of a_i^0.

Lemma 2. *Let X be a three-dimensional Banach space. Let H be a two-dimensional subspace in X. Let K_1 denote the unit ball in X and let K^0 be a half-ball obtained by the section of K_1 by H. If there is not a projection of norm one on H, then there is a projection P of K_1 on H such that*

$$0 \in \mathrm{conv}\,(PK_1 \setminus K \cap H).$$

Proof. Let H_ϵ be a plane parallel to H being in the distance ϵ from H and on this same side of H as K^0. Let K_2 denote $K_\epsilon = K \cap H_\epsilon$. Suppose that for each ϵ there is a linear projection P_ϵ such that $P_\epsilon K_\epsilon \subset K \cap H$. Since K_1 is convex it implies that for $\eta > \epsilon$ $P_\epsilon K_\eta \subset \subset P_\epsilon K_\epsilon$. Thus we obtain that $\| P_\epsilon \|$ tends to one for a tending to 0. By compactness argument we obtain in this way that there is a linear projection of norm one.

Suppose now that there is no linear projection of norm one on H. Then there is K_ϵ such that for any projection P, PK_ϵ is not contained in $K \cap H$. Observe that by changing of projection we are making translations of PK_ϵ. Therefore, by Lemma 1 we can find a projection P_0 such that $(0, 0, \varepsilon) \in \mathrm{conv}\,(P_0 K_\epsilon \setminus K \cap H)$.

Theorem ([1]). *Let X be a three-dimensional space. Let Y be a two-dimensional subspace. If there is a Lipschitz projection of norm one of X onto Y, then there is a linear projection of norm one.*

Proof. Suppose that there is no projection of norm one on Y. Using Lemma 2 we can find a linear projection P and points p_1, p_2, p_3 such that

$$p_i \in PK^0,\ i = 1, 2, 3$$

$$p_i \notin K,\ i = 1, 2, 3$$

there are $\alpha_1, \alpha_2, \alpha_3 \geqslant 0$ such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$$

Take a direction p_i, for every point $a \in P^{-1}(0)$ and lying on this same side of H as K^0 we can find positive numbers t_a^i, r_a^i such that a belongs to the boundary of the ball $K(-t_a^i p_i, r_a^i)$ of radius r_a^i and the center at $-t_a^i p_i$ and $P^{-1}(0)$ is tangential to this ball. It obviously implies that

$$P(a) = 0 \in \partial P(K(-t_a^i p_i, r_a^i))$$

but

$$P(a) \notin K(-t_a^i p_i, r_a^i)$$

It implies that

$$H \cap \bigcap_{i=1}^{3} K(-t_a^i p_i, r_a^i) = \emptyset . \tag{5}$$

Suppose that there is a Lipschitz projection M of norm one. Since $\| a + t_a^i p_i \| = r_a^i$ and M is Lipschitz projection of norm one

$$M(a) \in H \cap K(-t_a^i p_i, r_a^i) \; i = 1, 2, 3$$

but it is impossible by (5).

REFERENCES

[1] Lindenstrauss, *On non-linear projection in Banach spaces*, Michigan Math. Journal, 11 (1964), 263–287.

STRESZCZENIE

Mówimy, że M jest lipszycowskim rzutowaniem o normie L przestrzeni rzeczywistej Banacha X na jej podprzestrzeń, jeśli $MX = Y$, $Mx = x$ dla $x \in X$, $\| M(x_1) - M(x_2) \| \leq L \| x_1 - x_2 \|$, przy czym stałej L nie da się zastąpić przez liczbę mniejszą. Lindenstrauss wykazał, że dla refleksywnych przestrzeni Banacha rzutowanie M można zastąpić rzutowaniem liniowym o normie L. W pracy podano geometryczny dowód twierdzenia Lindenstraussa w przypadku $L = 1$, $X = R^3$, $Y = R^2$.

РЕЗЮМЕ

Скажем, что M – липшицевая проекция с нормой L, действительного банахого пространства X на ее подпространство Y, если $MX = Y$, $Mx = x$ для $x \in Y$, $\| M(x_1) - M(x_2) \| \leq L \| x_1 - x_2 \|$, и при этом, константу L нельзя заменить меньшим числом. Линденштраусс установил, что в случае рефлексивных банаховых пространств, проекцию M можно заменить линейной проекцией с нормой L. В работе приведено геометрическое доказательство теоремы Линденштраусса для случая $L = 1$, $X = R^3$, $Y = R^2$.