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An Internal Geometric Characterization of Strongly Starlike Functions

Wewnętrzna charakteryzacja geometryczna funkcji mocno gwiaździstych

Abstract. The authors prove that a univalent function f is strongly starlike of order α if and only if for every $w \in f(\mathbf{D})$ a certain lens-shaped region with vertices 0 and w is contained in $f(\mathbf{D})$. They also obtain sharp estimates for both the coefficient functional $|a_3 - \mu a_2^2|$ and the quantity $|\operatorname{Arg}\{f(z)/z\}|$ for the family of strongly starlike functions of order α.

1. Introduction. A holomorphic and univalent function f defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if f is normalized by $f(0) = f'(0) - 1 = 0$ and satisfies

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi\alpha}{2} \quad (z \in \mathbf{D}).$$

We denote the set of all such functions by $S^*(\alpha)$. This class was introduced by Brannan and Kirwan [2], and independently by Stankiewicz ([8], [9], [10]), who called functions in the class α-angularly starlike. Brannan, Clunie and Kirwan [1] gave sharp upper bounds on the second and third coefficients of functions in this class. In general, sharp coefficient bounds for this class remain open. In [2] Brannan and Kirwan obtained a geometric condition, which they called δ-visibility, which is sufficient for $f \in S^*(\alpha)$. Precisely, they proved that if f is holomorphic and univalent in $\mathbf{D}$, normalized by $f(0)=f'(0) - 1 = 0$, and for every r, $0 < r < 1$, and every point $\omega \in f(\{z : |z| = r\})$, the set $\Delta(\omega, \delta(r)) \subseteq f(|z| \leq r)$, then $f \in S^*(\alpha)$. Here $\Delta(\omega, \delta(r))$ is the closed convex hull of the union of the circle $\{w : |w| = \delta(r)\}$ and the two line segments from ω which are tangent to this circle and $\delta(r) = \cos(\pi\alpha/2)\max\{|f(z)| : |z| = r\}$. Stankiewicz [9] presented an external geometric characterization of strongly starlike functions; it says that a normalized holomorphic and univalent function f belongs to the class $S^*(\alpha)$ if and only if every point $w \in \mathbf{C} \setminus f(\mathbf{D})$ is the vertex of an angular sector with opening of measure

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$(1-\alpha)\pi/2$ which is contained in $\mathbf{C}\setminus f(\mathbf{D})$ and bisected by the radius vector through w. He obtained sharp growth and distortion theorems for the family $S^*(\alpha)$.

$S^* = S^*(1)$ is clearly the well-known class of normalized starlike functions. There is a simple, internal geometric characterization of starlike functions. A starlike function f is defined to be a univalent function such that $f(\mathbf{D})$ is starlike with respect to the origin; this means that the line segment $[0,w]$ between 0 and w is contained in $f(\mathbf{D})$ for every $w \in f(\mathbf{D})$. Note that every strongly starlike function of order α also has this simple geometric property, but this condition is not sufficient for a function to be strongly starlike of order α when $\alpha < 1$. The main purpose of this note is to give a simple, internal geometric characterization for strongly starlike functions of order α that is a natural refinement of the ordinary notion of starlikeness. Roughly speaking, we prove that a univalent function f is strongly starlike of order α if and only if for every $w \in f(\mathbf{D})$ a certain lens-shaped region with vertices 0 and w is contained in $f(\mathbf{D})$. We also obtain sharp estimates for both the coefficient functional $|a_3 - \mu a_2^2|$ and the quantity $|\operatorname{Arg}\{f(z)/z\}|$ for the family $S^*(\alpha)$.

2. The classes of k-starlike regions and functions. In order to introduce the concept of k-starlikeness, we define certain standard lens-shaped regions. For $0 < k \le 2$, we denote by $E_k = E_k[0,1]$ the intersection of the two closed disks of radii $1/k$ both of which have 0 and 1 on their boundaries. When $k = 0$, we define $E_0 = E_0[0,1]$ to be simply $[0,1]$, the straight line segment between 0 and 1. This type of lens-shaped region plays an important role in the study of euclidean k-convex functions ([6], [5]). Also, for each complex number w, we set $wE_k = \{w\eta : \eta \in E_k\}$. A geometric property of the lens-shaped regions E_k will play an important role in our later work. Note that as the point z traverses the upper half σ of the boundary of E_k in a counterclockwise direction from 1 to 0, the angle between the vector iz (which is a normal vector to the radial vector from 0 to z) and the tangent vector to σ at z increases strictly. The maximum value $\pi/2$ occurs at the origin, while the minimum value $\arccos(k/2)$ is achieved at 1.

For $k \in [0,2]$ a region Ω in the complex plane that contains the origin is called k-starlike (with respect to the origin) if for every $w \in \Omega$, $wE_k \subseteq \Omega$. Thus, a 0-starlike region is just an ordinary starlike region while for $k > 0$, a k-starlike region contains the lens-shaped region wE_k joining 0 and w rather than just the line segment $[0,w]$ for every $w \in \Omega$. Observe that if $0 \le k' \le k \le 2$ and Ω is k-starlike, then Ω is also k'-starlike since $E_k' \subseteq E_k$. If Ω is k-starlike for some $k \in [0,2]$, set

$$k(\Omega) = \sup\{k \in [0,2] : \Omega \text{ is } k\text{-starlike }\}.$$

Note that any disk centered at the origin is 2-starlike. A conformal mapping f of the unit disk $\mathbf{D}$ onto a region Ω with $f(0) = 0$ is called k-starlike if Ω is k-starlike. In this case we set $k(f) = k(\Omega)$. The hereditary property of starlike functions extends to k-starlike functions.

Theorem 1. *Suppose $f : \mathbf{D} \to \Omega$, $f(0) = 0$, is a k-starlike conformal mapping. Then for every $r \in (0,1)$, the region $f(\{z : |z| < r\})$ is k-starlike.*

Proof. Fix $r \in (0,1)$ and set $\Omega_r = f(\{z : |z| < r\})$. For each $w \in \Omega_r$ we want to show that $wE_k \subseteq \Omega_r$. Note that for each $\eta \in E_k$ and all $w \in \Omega$ we have $\eta w \in \Omega$.

Therefore, $h(z) = f^{-1}(\eta f(z))$ is holomorphic in $\mathbf{D}$ with $h(0) = 0$ and $h(\mathbf{D}) \subseteq \mathbf{D}$. Schwarz' Lemma implies that h maps $\{z : |z| < r\}$ into itself. Hence, $\eta\Omega_r \subseteq \Omega_r$. Since $\eta \in E_k$ is arbitrary, we conclude that $E_k\Omega_r \subseteq \Omega_r$. In particular, $wE_k \subseteq \Omega_r$, so Ω_r is k-starlike.

3. An internal geometric characterization. To state our main result, we introduce some notation and terminology. First, for a function f starlike with respect to $f(0) = 0$, not necessarily normalized by $f'(0) = 1$, we define

$$\alpha(f) = \sup\left\{\frac{2}{\pi}\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| : z \in \mathbf{D}\right\}.$$

Starlike functions are characterized by $\operatorname{Re}\{zf'(z)/f(z)\} > 0$, $z \in \mathbf{D}$, so it makes sense to use the principal branch of the logarithm. It is obvious that $0 \leq \alpha(f) \leq 1$, and $\alpha(f) = 0$ if and only if $f(z) = cz$, where c is a non-zero constant.

Now we are ready to state our main result.

Theorem 2. *Suppose f is starlike with respect to $f(0) = 0$ in $\mathbf{D}$. Then $\alpha(f) = \alpha$ if and only if $k(f) = 2\cos(\pi\alpha/2)$. Consequently, $S^*(\alpha) = \{f : f$ is holomorphic and univalent in $\mathbf{D}$ with normalization $f(0) = f'(0) - 1 = 0$ and for every $w \in f(\mathbf{D})$, $wE_k \subseteq f(\mathbf{D})\}$, where $k = 2\cos(\pi\alpha/2)$.*

Proof. First, we shall show that

$$\alpha(f) \leq \frac{2}{\pi}\arccos\left(\frac{k(f)}{2}\right).$$

Without loss of generality we may assume that $k(f) > 0$. Actually, it suffices to show

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \arccos\left(\frac{k}{2}\right) \qquad (z \in \mathbf{D})$$

for any value $k > 0$ such that $\Omega = f(\mathbf{D})$ is k-starlike. This inequality trivially holds at the origin. Fix $a \in \mathbf{D} \setminus \{0\}$ and set $r = |a|$. Theorem 1 implies that $f(\{z : |z| < r\})$ is k-starlike. A simple limit argument then shows that $f(a)E_k \subseteq f(\{z : |z| \leq r\})$. This implies that the tangent line to the starlike curve $\gamma = f(|z| = r)$ at $f(a)$ cannot intersect the interior of $f(a)E_k$. Therefore, the angle φ between the tangent $iaf'(a)$ to the curve γ at $f(a)$ (when γ traversed in a counterclockwise direction) and the straight line $\Gamma : w = (1 + it)f(a)$, $t \in \mathbf{R}$ (which is normal to the radial path from the origin to $f(a)$), is less than or equal to $\arccos(k/2)$. It is not difficult to show that

$$\varphi = \left|\operatorname{Arg}\left\{\frac{af'(a)}{f(a)}\right\}\right|.$$

Therefore, we have established the desired inequality.

Next, we established the reverse inequality

$$\alpha(f) \geq \frac{2}{\pi}\arccos\left(\frac{k(f)}{2}\right),$$

or equivalently,

$$k(f) \geq 2\cos(\frac{\pi\alpha(f)}{2}).$$

In fact, it is enough to show that f is k-starlike for $k = 2\cos(\pi\alpha(f)/2)$; we fix this value of k. Set $\alpha(f) = \alpha$. If $\alpha = 0$, then $f(z) = cz$ and $k(f) = 2$. So we only need to consider the possibility that $0 < \alpha \leq 1$. In this case $zf'(z)/f(z)$ is nonconstant and we have

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi\alpha}{2} \quad (z \in \mathbf{D}).$$

From Theorem 1 it follows that f is a k-starlike if and only if for every r, $0 < r < 1$, and every $w \in f(\{z : |z| \leq r\})$, that $wE_k \subseteq f(\{z : |z| \leq r\})$, where $k = 2\cos(\pi\alpha/2)$. If the function f were not k-starlike, then it would follow that there exists $r \in (0,1)$ and a point $w \in f(\{z : |z| \leq r\})$ such that wE_k is not contained in $f(\{z : |z| \leq r\})$. However, since f is starlike, the curve $\gamma = f(|z| = r)$ is strictly starlike, so there exists $j > 0$ such that wE_j is contained in $f(\{z : |z| \leq r\})$. Let

$$J = \sup\{j : wE_j \text{ is contained in } f(\{z : |z| \leq r\})\} .$$

Then $J < k$ and the boundary of wE_J is tangent to the curve γ at some point which is an interior point $\omega = f(\zeta)$, $|\zeta| = r$, of one of the circular arcs bounding wE_J. As we noted at the start of Section 2, the angle between the normal to $[0, \omega]$ and the tangent to the boundary of wE_J at ω is at least $\arccos(J/2)$. This implies that

$$\left|\operatorname{Arg}\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\}\right| \geq \arccos\frac{J}{2} > \arccos\frac{k}{2} = \frac{\pi\alpha}{2}.$$

This is a contradiction, so f is in fact k-starlike. The proof is now complete.

Note that our result implies the sufficient condition of Brannan and Kirwan [2] because $f(z)E_k \subseteq \Delta(f(z), |f(z)|\cos(\pi\alpha/2)) \subseteq \Delta(f(z), \delta(r))$ follows from $k = 2\cos(\pi\alpha/2)$.

4. Sharp bounds on $|a_3 - \mu a_2^2|$ and $|\operatorname{Arg}\{f(z)/z\}|$. Define the function k_α on the unit disk by $k_\alpha(0) = k'_\alpha(0) - 1 = 0$ and

$$\frac{zk'_\alpha(z)}{k_\alpha(z)} = \left(\frac{1+z}{1-z}\right)^\alpha .$$

Then it is clear that $k_\alpha \in S^*(\alpha)$ and

$$k_\alpha(z) = z\exp\left\{\int_0^z \frac{1}{t}\left[\left(\frac{1+t}{1-t}\right)^\alpha - 1\right] dt\right\}.$$

For many extremal problems for the class $S^*(\alpha)$ this function plays the role of Koebe function (see [2], [9], [10]). Moreover, we define $g_\lambda(z)$ and $h_\lambda(z)$, $0 \leq \lambda \leq 1$, by $g_\lambda(0) = h_\lambda(0) = g'_\lambda(0) - 1 = h'_\lambda(0) - 1 = 0$ and

$$\frac{zg'_\lambda(z)}{g_\lambda(z)} = \left(\lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^\alpha,$$

$$\frac{zh'_\lambda(z)}{h_\lambda(z)} = \left(\lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^{-\alpha},$$

respectively. Then it is obvious that both g_λ and h_λ, $0 \le \lambda \le 1$, belong to $S^*(\alpha)$.

Theorem 3. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S^*(\alpha)$. Then we have the sharp bounds*

$$|a_3 - \mu a_2^2| \le \begin{cases} \alpha^2(3-4\mu), & -\infty < \mu \le (3-1/\alpha)/4; \\ \alpha, & (3-1/\alpha)/4 \le \mu \le (3+1/\alpha)/4; \\ \alpha^2(4\mu-3), & (3+1/\alpha)/4 \le \mu < \infty. \end{cases}$$

For $-\infty < \mu < (3-1/\alpha)/4$ and $(3+1/\alpha)/4 < \mu < \infty$, equality holds if and only if f is a rotation of k_α. If $(3-1/\alpha)/4 < \mu < (3+1/\alpha)/4$, then equality holds if and only if f is a rotation of $g_{1/2}$. For $\mu = (3-1/\alpha)/4$, equality holds if and only if f is a rotation of g_λ, $0 \le \lambda \le 1$. Finally, equality holds if and only if f is a rotation of h_λ, $0 \le \lambda \le 1$, when $\mu = (3+1/\alpha)/4$.

In the case $\alpha = 1$, the bounds above were given by Keogh and Merkes [4].

Proof. First, we note that to prove the desired inequalities it is enough to show either $\mathrm{Re}\{a_3 - \mu a_2^2\}$ or $\mathrm{Re}\{\mu a_2^2 - a_3\}$ has the given upper bounds. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S^*(\alpha)$, then there exists a holomorphic function p with $p(0) = 1$ and positive real part in $\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = p(z)^\alpha .$$

Assume $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. We express $\mathrm{Re}\{a_3 - \mu a_2^2\}$ in terms of the coefficients of p. It is easy to verify that

$$a_2 = \alpha b_1$$

and

$$a_3 = \frac{1}{2}\alpha b_2 + \frac{1}{4}\alpha(3\alpha - 1)b_1^2$$

so that

$$a_3 - \mu a_2^2 = \frac{\alpha}{2}\Big(b_2 + \frac{1}{2}(3\alpha - 4\mu\alpha - 1)b_1^2\Big) .$$

By the Herglotz representation formula for holomorphic functions on the unit disk with positive real part [7, p.40], there exists a probability measure $\nu(t)$ on $[0, 2\pi]$ such that

$$p(z) = \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z}\, d\nu(t) .$$

Thus,

$$b_n = 2\int_0^{2\pi} e^{-int} d\nu(t) \qquad (n = 1, 2, \ldots)$$

and

$$\mathrm{Re}\{a_3 - \mu a_2^2\} =$$

$$\alpha\Big\{\int_0^{2\pi} \cos(2t)d\nu(t) + (3\alpha - 4\mu\alpha - 1)\Big[\Big(\int_0^{2\pi}\cos(t)d\nu(t)\Big)^2 - \Big(\int_0^{2\pi}\sin(t)d\nu(t)\Big)^2\Big]\Big\}.$$

Now, we consider various cases according to the value of μ. If $\mu \le (3-1/\alpha)/4$, then $3\alpha - 4\mu\alpha - 1 \ge 0$ and

$$\operatorname{Re}\{a_3 - \mu a_2^2\} \le \alpha\Big\{\int_0^{2\pi} \cos(2t)d\nu(t) + (3\alpha - 4\mu\alpha - 1)\Big(\int_0^{2\pi} \cos(t)d\nu(t)\Big)^2\Big\}$$

$$\le \alpha\Big\{\int_0^{2\pi} \cos(2t)d\nu(t) + (3\alpha - 4\mu\alpha - 1)\int_0^{2\pi} \cos^2(t)d\nu(t)\Big\} \le \alpha^2(3-4\mu) .$$

The second case is $\mu \ge (3-1/\alpha)/4$, or equivalently, $4\mu\alpha + 1 - 3\alpha \ge 0$. Then

$$\operatorname{Re}\{\mu a_2^2 - a_3\} \le \alpha\Big\{-\int_0^{2\pi} \cos(2t)d\nu(t) + (4\mu\alpha + 1 - 3\alpha)\Big(\int_0^{2\pi} \cos(t)d\nu(t)\Big)^2\Big\}$$

$$\le \alpha\Big\{-\int_0^{2\pi} \cos(2t)d\nu(t) + (4\mu\alpha + 1 - 3\alpha)\int_0^{2\pi} \cos^2(t)d\nu(t)\Big\}$$

$$\le \alpha\int_0^{2\pi}\left[1 + (4\mu\alpha - 1 - 3\alpha)\cos^2(t)\right]d\nu(t) ,$$

which is less than or equal to α if $\mu \le (3+1/\alpha)/4$, and less than or equal to $\alpha^2(4\mu-3)$ if $\mu \ge (3+1/\alpha)/4$.

Now we determine all possible extremal functions. It is elementary to check that equality holds for those functions as stated in the theorem. Note that if equality holds in one of the inequalities for some f, then there is a rotation of f that is extremal for the functional $\operatorname{Re}\{a_3 - \mu a_2^2\}$ or $\operatorname{Re}\{\mu a_2^2 - a_3\}$.

Suppose $-\infty < \mu < (3-1/\alpha)/4$. If $\operatorname{Re}\{a_3 - \mu a_2^2\} = \alpha^2(3-4\mu)$, then we have

$$\int_0^{2\pi} \cos(2t)d\nu(t) = \Big(\int_0^{2\pi} \cos(t)d\nu(t)\Big)^2 = 1 ,$$

which implies that

$$\int_0^{2\pi} \cos^2(t)d\nu(t) = 1 .$$

Thus $\nu = \lambda\nu_0 + (1-\lambda)\nu_\pi$, where $0 \le \lambda \le 1$ and ν_0 and ν_π are point masses at 0 and π, respectively. Therefore,

$$1 = \Big(\int_0^{2\pi} \cos(t)d\nu(t)\Big)^2 = (2\lambda - 1)^2 .$$

This implies that λ must be equal to 1 or 0, that is, either $\nu = \nu_0$ or $\nu = \nu_\pi$. Equivalently, $f(z) = k_\alpha(z)$ or $-k_\alpha(-z)$.

In a similar fashion, we can show the desired result when μ satisfies the inequalities $(3+1/\alpha)/4 < \mu < \infty$.

Now assume $(3-1/\alpha)/4 < \mu < (3+1/\alpha)/4$. If $\operatorname{Re}\{\mu a_2^2 - a_3\} = \alpha$, then

$$\int_0^{2\pi} \cos^2(t)d\nu(t) = 0 ,$$

which yields $\nu = \lambda\nu_{\pi/2} + (1-\lambda)\nu_{3\pi/2}$, $0 \le \lambda \le 1$. Also,

$$0 = \int_0^{2\pi} \sin(t)d\nu(t) = \lambda - (1-\lambda)$$

gives that $\lambda = 1/2$, or equivalently, $f(z) = -ig_{1/2}(iz)$.

If $\mu = (3-1/\alpha)/4$ and $\mathrm{Re}\{a_3 - \mu a_2^2\} = \alpha$, then

$$1 = \int_0^{2\pi} \cos(2t)d\nu(t) = -1 + 2\int_0^{2\pi} \cos^2(t)d\nu(t)$$

implies that $\nu = \lambda\nu_0 + (1-\lambda)\nu_\pi$, $0 \le \lambda \le 1$. This yields $f(z) = g_\lambda(z)$.

If $\mu = (3+1/\alpha)/4$ and $\mathrm{Re}\{\mu a_2^2 - a_3\} = \alpha$, then $\alpha = \frac{\alpha}{2}\mathrm{Re}\{b_1^2 - b_2\}$. Define $q(z)$ by $q(z) = 1/p(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then q also has positive real part in $\mathbf{D}$ and $c_2 = b_1^2 - b_2$. We know that (for example, see [7, p.41]) Re $c_2 = 2$ if and only if

$$q(z) = \lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}, \quad 0 \le \lambda \le 1 .$$

Consequently, $f(z) = h_\lambda(z)$. This completes the proof of Theorem 3.

Next, we give the sharp upper bound on $|\mathrm{Arg}\{f(z)/z\}|$ for the family $S^*(\alpha)$.

Theorem 4. *Let $f \in S^*(\alpha)$ and $|z| = r < 1$. Then*

$$|\mathrm{Arg}\{f(z)/z\}| \le \max_{|z|=r} \mathrm{Arg}\{k_\alpha(z)/z\} .$$

Equality holds for some $|z| = r$, $0 < r < 1$, if and only if f is equal to k_α or one of its rotations.

Proof. If $f \in S^*(\alpha)$, then there exists a holomorphic function $p(z)$ with $p(0) = 1$ and positive real part in $\mathbf{D}$ such that

$$\frac{zf'(z)}{f(z)} = p(z)^\alpha .$$

This implies that

$$f(z) = z\exp\left\{\int_0^z \frac{1}{t}\left[p(t)^\alpha - 1\right] dt\right\}$$

and

$$\mathrm{Arg}\{f(z)/z\} = \mathrm{Im}\left\{\int_0^z \frac{1}{t}\left[p(t)^\alpha - 1\right] dt\right\}.$$

Since $p(z)^\alpha - 1$ is subordinate to the convex univalent function $\left(\frac{1+z}{1-z}\right)^\alpha - 1$, we have [3] (see also [7, p.50]) that the function

$$\int_0^z \frac{1}{t}\left[p(t)^\alpha - 1\right] dt$$

is subordinate to

$$\int_0^z \frac{1}{t}\left[\left(\frac{1+t}{1-t}\right)^\alpha - 1\right] dt \equiv G(z) .$$

Because $zG'(z)$ is convex, we see that $G(z)$ is convex; also, the power series for G has real coefficients. Thus, by using the subordination principle, we get

$$|\mathrm{Arg}\{f(z)/z\}| \leq \left|\mathrm{Im}\left\{ \int_0^z \frac{1}{t}\left[p(t)^\alpha - 1\right] dt\right\}\right|$$
$$\leq \max_{|z|=r} \mathrm{Im}\, G(z) = \max_{|z|=r} \mathrm{Arg}\{k_\alpha(z)/z\} .$$

When we used the subordination principle, the inequality becomes an equality only if $p(z)$ is equal to $\frac{1+e^{i\theta}z}{1-e^{i\theta}z}$ for some $\theta \in \mathbf{R}$, so equality can hold in our theorem only for some rotation of $k_\alpha(z)$. On the other hand, it is clear that the inequality becomes an equality for some $|z| = r$ if $f(z)$ is $k_\alpha(z)$ or one of its rotations. This completes our proof.

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STRESZCZENIE

Autorzy wykazują, że funkcja jednolistna f jest mocno gwiaździsta rzędu α wtedy i tylko wtedy, gdy dla każdego punktu $w \in f(\mathbf{D})$ pewien obszar o kształcie soczewki z wierzchołkami $0, w$ zawiera się w $f(\mathbf{D})$. Otrzymali oni również dokładne oszacowanie dla współczyników funkcjonału $|a_3 - \mu a_2^2|$ oraz dla wartości $|\mathrm{Arg}\{f(z)/z\}|$ w rodzinie funkcji mocno wypukłych rzędu α.

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