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On Differential Inclusions with an Advanced Argument

Abstract. In the paper the following initial problem $x'(t) \in F(t, x(t), x(\nu(t)))$, $t \geq 0$, $x(0) = x_0$, where $\nu(t) \geq t$, is considered. We proved that under suitable boundedness conditions on multifunction F and real function ν the problem has at least one solution.

1. Introduction. Recently there have been published many papers devoted to the existence of solutions of multivalued differential inclusions. Most of them are concerned with the existence of solutions on a finite interval but when we examine differential inclusions with an advanced argument it is generally necessary to consider infinite intervals or to add some additional conditions on the deviation function.

In this paper we study differential inclusions of the form

$$x'(t) \in F(t, x(t), x(\nu(t))) , \quad t \geq 0 \tag{1}$$

with the initial condition

$$x(0) = x_0 \tag{2}$$

in a separable and reflexive Banach space X. Here $t \in I = \langle 0, \infty)$ and $t \to \nu(t)$ denotes a continuous function such that $\nu(t) \geq t$ for $t \in I$ and F is a multifunction from $I \times X \times X$ into the set of all nonempty compact and convex subsets of X.

We start with some lemmas and then under a boundedness conditions on F we prove that the problem (1)–(2) has at least one solution. Moreover we get an evaluation of the growth of this solution.

It is worth to notice that our existence result generalizes the one of Bielecki [2] which was proved for $X = \mathbf{R}$ and a differential equation with deviated argument.

2. Preliminaries. Let us start with our notations, definitions and some of the basic results which will be nedded in the subsequent section.

Let Y, Z be arbitrary nonempty sets. A set valued function $F : Y \to 2^Z$ is called *multifunction* with a domain Y and a range contained in 2^Z, where 2^Z is the family of all nonempty subsets of Z. If Σ is a σ-field of subsets Y and Z is a topological space, then *a multifunction* $F : Y \to 2^Z$ *is said to be* Σ*-measurable* whenever $F^-(B) = \{y \in Y : F(y) \cap B \neq \emptyset\} \in \Sigma$ for each closed subset B of Z. Similarly we will say that *a function* $f : Y \to Z$ *is* Σ*-measurable* if $f^{-1}(B) = \{y \in Y : f(y) \in B\}$ is measurable for every closed subset B of Z. In case of separable metric spaces Z this notion of measurability of f is equivalent to the strong measurability of f (see Lemma 2.5 of [4]). If both Y and Z are topological spaces then *a multifunction* $F : Y \to 2^Z$ is called *upper semi-continuous (=usc)* whenever $F^-(B)$ is closed in Y for every closed subset B of Z.

For compact valued multifunctions F we have the following equivalent condition of upper semicontinuity:

Lemma 1. [5, Proposition 4.1, p.48]. *Suppose that Y, Z are metric spaces and a multifunction $F : Y \to 2^Z$ has compact values i.e. for each $y \in Y$ the set $F(y)$ is a compact subset of Z. Then F is usc if and only if for every sequence $\{y_n\}_{n\in\mathbf{N}}$ of points belonging to Y convergent to y_0 and for every sequence $\{z_n\}_{n\in\mathbf{N}}$ such that $z_n \in F(y_n)$ there is a subsequence $\{z_{n_k}\}_{k\in\mathbf{N}}$ convergent to z_0 such that $z_0 \in F(y_0)$.*

In the proof of our main theorem we will also apply the following fixed point theorem due to Sek Wui Seah [8]:

Lemma 2. *Let C be a nonempty closed bounded and convex subset of a locally convex topological linear space Y. If $T : C \to 2^C$ having nonempty convex and closed values is usc and $\overline{T(C)}$ is compact then there exists $y \in C$ such that $y \in T(y)$.*

Let us observe that this lemma is a consequence of the well-known result of Fan [3].

Now let us denote by X a real infinite dimensional separable and reflexive Banach space with the norm $\|\cdot\|$ and the zero element θ. The open ball centered at $x_0 \in X$ and of radius r will be denoted by $K(x_0, r)$ and the closed one by $\overline{K}(x_0, r)$. For A being arbitrary subset of X the symbols $\overline{A}$, $K(A, r)$ will respectively stand for the closure of A and the ball centered at A and of radius r i.e.

$$K(A,r) = \bigcup_{a\in A} K(a,r)\ .$$

The norm of a bounded nonempty set $A \subset X$ is the number

$$\|A\| = \sup\{\|a\| : a \in A\}\ .$$

The usual algebraical operations on sets are defined as follows

$$A + B = \{a + b \in X : a \in A\ ,\ b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}\ ,\quad \lambda \in \mathbf{R}\ .$$

We will simply write $a + B$ instead of $\{a\} + B$. Let us observe here that if A is a convex set and such that $\theta \in A$ then

$$0 \le \lambda_1 < \lambda_2 \quad \text{implies} \quad \lambda_1 A \subset \lambda_2 A\ .$$

The convex hull and the closed convex hull of A will be denoted by conv A, $\overline{\text{conv}}\ A$ respectively.

Further, denote by

$cf\ X-$ the family of all nonempty closed and convex subsets of X ,

$ccf\ X-$ the family of all nonempty compact and convex subsets of X .

Now, we can define an integral for the multifunction $G : \langle a,b\rangle \to ccf\ X$ by the following formula

$$\int_a^b G(t)\,dt = \left\{ \int_a^b \lambda(t)\,dt : \lambda \text{ is a measurable selection of } G(\cdot) \right\} ,$$

where $\int_a^b \lambda(t)\,dt$ is the usual Bochner integral. This multivalued integral is called Aumann's integral.

We will say that the multifunction G is integrably bounded if there is Lebesgue integrable function g such that

$$\|G(t)\| \le g(t) \quad \text{for almost every } t \in \langle a,b\rangle \ .$$

Using the Rädstrom embedding theorem (see [6], Theorem 17.2.1 p. 189), we see that $\int_a^b G(t)\,dt$ is a nonempty convex and compact set in X.

3. The main result. Let us begin with the following

Lemma 3. *Let X be a real infinite dimensional Banach space and let $\{\psi_n\}_{n\in\mathbf{N}}$ be a sequence of absolutely continuous functions $\psi_n : \langle 0,\infty) \to X$ which are differentiable almost everywhere on $\langle 0,\infty)$. If*

(a) $\psi_n(t) \to \psi(t)$ as $n \to \infty$ for all $t \in \langle 0,\infty)$ where $\psi : \langle 0,\infty) \to X$;

(b) $\|\psi_n'(t)\| \le g(t)$ almost everywhere on $\langle 0,\infty)$ for all $n \in \mathbf{N}$, where $g : \langle 0,\infty) \to \langle 0,\infty)$ is locally Lebesgue integrable on $\langle 0,\infty)$;

(c) $\{\psi_n'(t) : n \ge 1\}$ is relatively compact in X

then ψ is absolutely continuous on $\langle 0,\infty)$, almost everywhere differentiable on $\langle 0,\infty)$ and

$$\psi'(t) \in \bigcap_{n=1}^{\infty} \overline{\mathrm{conv}} \bigcup_{k=n}^{\infty} \{\psi_k'(t)\}$$

for almost all $t \in \langle 0,\infty)$.

Proof. In particular, conditions (a), (b), (c) are satisfied on the intervals $\langle 0,i\rangle$ for each $i \in \mathbf{N}$. Hence by Theorem 1.3 [9, p.16] ψ is absolutely continuous on $\langle 0,i\rangle$ and almost everywhere differentiable on $\langle 0,i\rangle$ and

$$\psi'(t) \in \bigcap_{n=1}^{\infty} \overline{\mathrm{conv}} \bigcup_{k=n}^{\infty} \{\psi_k'(t)\} \quad \text{for almost all } t \in \langle 0,i\rangle \ .$$

It follows that ψ is absolutely continuous on $\langle 0,\infty)$.

Now let

$$D_i = \{t \in \langle 0, i) : \psi'(t) \notin \bigcap_{n=1}^{\infty} \overline{\text{conv}} \bigcup_{k=n}^{\infty} \{\psi_k'(t)\}$$

and

$$D = \{t \in \langle 0, \infty) : \psi'(t) \notin \bigcap_{n=1}^{\infty} \overline{\text{conv}} \bigcup_{k=n}^{\infty} \{\psi_k'(t)\} .$$

Then D_i is of measure zero and $D = \bigcup_{k=n}^{\infty} D_i$ being the union of denumerable collection of such sets is of measure zero too. Therefore

$$\psi'(t) \in \bigcap_{n=1}^{\infty} \overline{\text{conv}} \bigcup_{k=n}^{\infty} \{\psi_k'(t)\} \quad \text{for almost all } t \in \langle 0,\infty) .$$

This completes the proof.

Remark. Contrary to the case of numerical functions the X-valued absolutely continuous function need not necessary be almost everywhere differentiable.

However, if the space X is reflexive we have

Lemma 4. [1, Theorem 3.4, p.53]. *Let X be a reflexive Banach space. Then every X-valued absolutely continuous on $\langle a,b\rangle$ function ψ is almost everywhere differentiable on $\langle a,b\rangle$ and can be represented as*

$$\psi(t) = \psi(a) + \int_a^t \psi'(s)\,ds\,, \quad t \in \langle a,b\rangle$$

where ψ' is a strong derivative.

Now let us state the following

Lemma 5. *Let $\mathcal{L}$ be the σ-field of all Lebesgue measurable subsets of $I = \langle 0,\infty)$, $\mathcal{B}(X\times X)$ the σ-field of all Borel subsets of $X\times X$ and $\mathcal{L}\otimes\mathcal{B}(X\times X)$ the product σ-field of $\mathcal{L}$ and $\mathcal{B}(X\times X)$ i.e. the smallest σ-field of subsets of $I\times X\times X$ containing all sets $A\times B$ where $A\in\mathcal{L}$ and $B\in\mathcal{B}(X\times X)$. If $F: I\times X\times X \to ccf\ X$ is $\mathcal{L}\otimes\mathcal{B}(X\times X)$-measurable multifunction such that for every $t\in I$ $F(t,\cdot,\cdot)$ is usc and if $\varphi: I\to X$, $\nu: I\to I$ are continuous functions then the multifunction $G: I\to ccf\ X$ defined as follows*

$$G(t) = F(t,\varphi(t),\varphi(\nu(t)))\,, \quad t\in I$$

is $\mathcal{L}$-measurable.

Proof. Put $Y = X\times X$ and denote by $\widehat{F}$ a multifunction from $I\times Y$ into $ccf\ X$ such that $\widehat{F}(t,u) = F(t,x,y)$ for $u=(x,y)$. Then $\widehat{F}$ is upper semi-Carathéodory multifunction i.e. $\widehat{F}(\cdot, u)$ is $\mathcal{L}$-measurable and $\widehat{F}(t, \cdot)$ is usc. Moreover $\widehat{F}$ is $\mathcal{L}\otimes\mathcal{B}(Y)$-measurable.

Now let us define a function $h : I \to Y$ by the formula

$$h(t) = (\varphi(t), \varphi(\nu(t))) , \quad t \in I .$$

Obviously h is $\mathcal{L}$-measurable. Invoking Theorem 1 of [10] we get that the multifunction $G(\cdot) = \widehat{F}(\cdot, h(\cdot)) = F(\cdot, \varphi(\cdot), \varphi(\nu(\cdot)))$ is $\mathcal{L}$-measurable and thus we are done.

So, we can now begin to formulate our main result.

Let $\nu : I \to I$ be a continuous function such that $\nu(t) \geq t$ for $t \in I$.Let $k, m, n : I \to I$ be locally Lebesgue integrable functions. Set

$$l(t) = k(t) + m(t) + n(t) \quad \text{for } t \in I$$

(3) and

$$\widetilde{l}(t) = \int_0^t l(s)\,ds \quad \text{for } t \in I .$$

Suppose that the following inequality is satisfied

$$\widetilde{l}(\nu(t)) \leq \alpha^{-1}(\widetilde{l}(t) + b^{-1}\ln b) \tag{4}$$

where $0 < \alpha \leq 1$, $b > 1$ are fixed.

Now let $F : I \times X \times X \to ccf\ X$ be $\mathcal{L}\otimes\mathcal{B}(X \times X)$-measurable multifunction such that for every $t \in I$ $F(t, \cdot, \cdot)$ is usc on $X \times X$. Suppose also that

$$F(t, x, y) \subset \widetilde{k}(t) + \|x\| M(t) + \|y\|^{\alpha} N(t) \tag{5}$$

where $\widetilde{k} : I \to X$; $M, N : I \to ccf\ X$ are locally integrable function and multifunctions bounded by k and m, n respectively i.e.

$$\begin{aligned} \|\widetilde{k}(t)\| &\leq k(t) , \\ \|M(t)\| &\leq m(t) , \\ \|N(t)\| &\leq n(t) \quad \text{for } t \in I \end{aligned} \tag{6}$$

and such that $\theta \in M(t)$ and $\theta \in N(t)$, $t \in I$.

Theorem . *Under the above assumptions the problem (1)–(2) has at least one solution i.e. there is an absolutely continuous function φ such that $\varphi(0) = x_0$ and $\varphi'(t) \in F(t, \varphi(t), \varphi(\nu(t)))$ for almost all $t \in I$. Moreover this solution φ satisfies the inequality $\|\varphi(t)\| \leq a \exp(b\widetilde{l}(t))$ for $t \in I$.*

Proof. Denote by $C(I, X)$ the set of all X-valued functions which are continuous on I. Fix $\gamma > 0$ and denote by E a family of all functions $\varphi \in C(I, X)$ such that

$$|||\varphi||| = \sup\{\|\varphi(t)\| \exp(-b\widetilde{l}(t) - \gamma t) : t \in I\} < +\infty .$$

It is easy to verify that E with the norm $|||\cdot|||$ is a Banach space.

Next let

$$\Phi = \{\varphi \in C(I,X) : \|\varphi(t)\| \le a\exp(b\widetilde{l}(t)) : t \in I\} \tag{7}$$

where a is a fixed real number greater than $\max\{1, \|x_0\|\}$. Clearly Φ is nonempty bounded convex and closed subset of E.

Further, let us simply write $\varphi^*(t)$ instead of $\varphi(\nu(t))$.

Now we can define a multifunction $T : \Phi \to 2^\Phi$ in the following way:

$$(T\varphi)(t) = \{\psi \in C(I,X) : \psi(t) = x_0 + \int_0^t \lambda(s)\,ds \text{ where } \lambda : I \to X \text{ is a measurable selection of } F(\cdot,\varphi(\cdot),\varphi^*(\cdot))\} . \tag{8}$$

We claim that the multifunction T fulfils all the assumptions of Lemma 2. Really, by Lemma 5 and the well-known Kuratowski and Ryll–Nardzewski theorem [7] there is a measurable selection λ of $F(\cdot,\varphi(\cdot),\varphi^*(\cdot))$ and thus $T(\varphi) \ne \emptyset$.

Note that

$$\|\varphi^*(t)\| = \|\varphi(\nu(t))\| \le a\exp(b\widetilde{l}(\nu(t))) .$$

Hence by (4) we have

$$\|\varphi^*(t)\| \le a\exp(b\alpha^{-1}(\widetilde{l}(t) + b^{-1}\ln b)) = ab^{\alpha^{-1}}\exp(b\alpha^{-1}\widetilde{l}(t))$$

and finally

$$\|\varphi^*(t)\|^\alpha \le a^\alpha b\exp(b\widetilde{l}(t)) . \tag{9}$$

Also let

$$\psi(t) = x_0 + \int_0^t \lambda(s)\,ds \tag{10}$$

where

$$\begin{aligned}\lambda(s) \in F(s,\varphi(s),\varphi^*(s)) &\subset \widetilde{k}(s) + \|\varphi(s)\|M(s) + \|\varphi^*(s)\|^\alpha N(s) \subset \\ &\subset K(\theta, k(s) + \|\varphi(s)\|m(s) + \|\varphi^*(s)\|^\alpha n(s))\end{aligned}$$

because of (5) and (6). Thus by (7), (8) and (9)

$$\begin{aligned}\|\lambda(s)\| &\le k(s) + \|\varphi(s)\|m(s) + \|\varphi^*(s)\|^\alpha n(s) \le \\ &\le k(s) + a\exp(b\widetilde{l}(s))m(s) + a^\alpha b\exp(b\widetilde{l}(s))n(s) \le \\ &\le ab\exp(b\widetilde{l}(s)) \cdot l(s) = a(\exp(b\widetilde{l}(s)))' .\end{aligned}$$

Then making use of (10) we have

$$\begin{aligned}\|\psi(t)\| \le \|x_0\| + \int_0^t \|\lambda(s)\|\,ds &\le \|x_0\| + a(\exp(b\widetilde{l}(t)) - 1) \le \\ &\le a\exp(b\widetilde{l}(t))\end{aligned}$$

and thus we conclude that $\psi \in \Phi$ and therefore $T(\varphi) \subset \Phi$. As Φ is bounded subset of E so $T(\varphi)$ is bounded too. Clearly $T(\varphi)$ is convex. To show that $T(\varphi)$ is closed take $\psi_n \in T(\varphi)$, $\psi \in E$ such that $|||\psi_n - \psi||| \to 0$ as $n \to \infty$. So by the definition (8) there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of measurable selections of $F(\cdot, \varphi(\cdot), \varphi^*(\cdot))$ such that

$$\psi_n(t) = x_0 + \int_0^t \lambda_n(s)\, ds \quad \text{for } t \in I$$

Hence it follows, as in the above part of the proof, that

$$\|\psi_n(t)\| \le a \exp(b\widetilde{l}(t)) \quad \text{for } t \in I\,.$$

Of course ψ_n, $n \in \mathbb{N}$ are absolutely continuous functions. By virtue of Lemma 4 we have

$$\psi_n(t) = \psi_n(0) + \int_0^t \psi_n'(s)\, ds$$

and by the properties of Bochner integral we get

$$\psi_n'(t) \in F(t, \varphi(t), \varphi^*(t)) \quad \text{almost everywhere on } I\,.$$

Thus the assumption (a) and (b) of Lemma 3 are fulfilled. Observe that the compactness of the set $F(t, \varphi(t), \varphi^*(t))$ implies that the set $\{\psi_n'(t) : n \ge 1\}$ is relatively compact.

Hence in view of Lemma 3 ψ is absolutely continuous and almost everywhere on I differentiable. What is more, by convexity of $F(t, \varphi(t), \varphi^*(t))$, we get

$$\psi'(t) \in F(t, \varphi(t), \varphi^*(t)) \quad \text{for almost all } t \in I\,.$$

Now, invoking Lemma 4 once more, we have

$$\psi(t) = x_0 + \int_0^t \lambda(s)\, ds\,, \quad \text{where } \lambda(s) = \psi'(s) \text{ almost everywhere on } I\,.$$

Therefore $\psi \in T(\varphi)$ and $T(\varphi)$ is closed.

Now we will show that $\overline{T(\Phi)}$ is compact. To this aim take $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\{\psi_n\}_{n\in\mathbb{N}} \subset T(\Phi)$. Then there is a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ such that $\varphi_n \in \Phi$ and $\psi_n \in T(\varphi_n)$ and therefore by (8)

$$\psi_n(t) = x_0 + \int_0^t \lambda_n(s)\, ds \tag{11}$$

where λ_n is a measurable selection of $F(\cdot, \varphi_n(\cdot), \varphi_n^*(\cdot))$. We claim that $\{\psi_n : n \ge 1\}$ is relatively compact in $C(I, X)$ with respect to the almost uniform convergence topology. So, in order to prove it, take $t_1, t_2 \in I$, $t_2 > t_1$. Then, by similar estimations, we obtain

$$\|\psi_n(t_2) - \psi_n(t_1)\| \le \int_{t_1}^{t_2} a(\exp(b\widetilde{l}(s)))'\, ds = a[\exp(b\widetilde{l}(t_2)) - \exp(b\widetilde{l}(t_1))]$$

and therefore ψ_n are equicontinuous. In view of (5) and (11)

$$\psi_n(t) \in x_0 + \int_0^t (\widetilde{k}(s) + \|\varphi_n(s)\| M(s) + \|\varphi_n^*(s)\|^\alpha N(s))\, ds$$

but by (7) and (9)

$$\widetilde{k}(s) + \|\varphi_n(s)\| M(s) + \|\varphi_n^*(s)\|^\alpha N(s) \subset$$
$$\widetilde{k}(s) + a\exp(b\widetilde{l}(s))M(s) + a^\alpha b \exp(b\widetilde{l}(s))N(s) = G(s)$$

and therefore

$$\psi_n(t) \in x_0 + \int_0^t G(s)\, ds\,, \quad n \geq 1$$

where $G(s)$ is nonempty compact and convex. Hence, because of compactness of the integral $\int_0^t G(s)\,ds$, $x_0 + \int_0^t G(s)\,ds$ is compact too. Thus $\{\psi_n(t)\,,\ n \geq 1\}$ is relatively compact and by the well-known compactness criterion we conclude that $\{\psi_n : n \geq 1\}$ is relatively compact with respect to the almost uniform convergence topology in $C(I,X)$. Therefore there is a subsequence $\{\psi_{n_k}\}_{k\in\mathbb{N}}$ such that $\psi_{n_k} \to \psi$ as $k \to \infty$ in this topology. We can assume without loss of generality that $\psi_n \to \psi$. But our aim is to show that

$$|||\psi_n - \psi||| \to 0 \quad \text{as } n \to \infty\,.$$

To this goal fix $\varepsilon > 0$, put $\theta(\varepsilon) = \gamma^{-1}(\ln 4a - \ln\varepsilon)$ and observe that

$$\|\psi_i(t) - \psi_j(t)\| \leq \int_0^t \|\lambda_i(s) - \lambda_j(s)\|\, ds \leq$$
$$\leq \int_0^t (\|\lambda_i(s)\| + \|\lambda_j(s)\|)\, ds$$

and, as before, that

$$\|\psi_i(t) - \psi_j(t)\| \leq 2a\exp(b\widetilde{l}(t))\,.$$

Thus

$$\|\psi_i(t) - \psi_j(t)\| \leq a\exp(b\widetilde{l}(t) + \gamma t)\cdot\frac{\varepsilon}{2}\ \exp(-\gamma t)\frac{4}{\varepsilon} \leq$$
$$\leq \exp(b\widetilde{l}(t) + \gamma t)\cdot\frac{\varepsilon}{2}$$

for $a\frac{4}{\varepsilon}\exp(-\gamma t) \leq 1$ what means for

$$\gamma t \geq \ln 4a - \ln\varepsilon\,.$$

Summing up the above we get

$$\|\psi_i(t) - \psi_j(t)\| \leq \exp(b\widetilde{l}(t) + \gamma t)\cdot\frac{\varepsilon}{2} \quad \text{for } t \geq \theta(\varepsilon)$$

and therefore

$$\|\psi_i(t) - \psi_j(t)\| \exp(-b\widetilde{l}(t) - \gamma t) \le \frac{\varepsilon}{2} \quad \text{for } t \ge \theta(\varepsilon) \,. \tag{12}$$

But for $t \in \langle 0, \theta(\varepsilon)\rangle$

$$\|\psi_i(t) - \psi(t)\| < \frac{\varepsilon}{2} \quad \text{for } i \text{ sufficiently large} \tag{13}$$

(ψ_i converges uniformly to ψ on $\langle 0, \theta(\varepsilon)\rangle$).

Hence by (12) and (13)

$$|||\psi_i - \psi||| \le \sup\{\|\psi_i(t) - \psi(t)\| \exp(-b\widetilde{l}(t) - \gamma t) : t \in \langle 0, \theta(\varepsilon)\rangle\} +$$
$$+ \sup\{\|\psi_i(t) - \psi(t)\| \exp(-b\widetilde{l}(t) - \gamma t) : t \ge \theta(\varepsilon)\} \le \varepsilon$$

and thus $\psi_n \to \psi$ in the topology of E. There is nothing for us to do but to prove that T is usc. From the above considerations it follows that $T(\varphi)$ is compact for every $\varphi \in \Phi$. Therefore we can apply Lemma 1 to prove upper semiconitinuity of $T : \Phi \to ccf\,\Phi$. So, let $|||\varphi_n - \varphi||| \to 0$ as $n \to \infty$ and let $\psi_n \in T\varphi_n$. As $\psi_n \in \overline{T(\Phi)}$ so there is a subsequence $\{\psi_{n_k}\}_{k \in N}$ and $\psi \in \Phi$ such that $|||\psi_{n_k} - \psi||| \to 0$ as $k \to \infty$. We will show that $\psi \in T(\varphi)$. To this aim take $\varepsilon > 0$. Then by Lemma 4 and upper semicontinuity of $F(t, \cdot, \cdot)$ we have

$$\psi'_{n_k}(t) \in F(t, \varphi_{n_k}(t), \varphi^*_{n_k}(t)) \subset K(F(t, \varphi(t), \varphi^*(t)), \varepsilon)$$

for sufficiently large k. It implies that $\{\psi'_{n_k}(t)\}$ is relatively compact and after applying Lemma 3 we get

$$\psi'(t) \in \overline{K(F(t, \varphi(t), \varphi^*(t)), \varepsilon)}$$

and finally

$$\psi'(t) \in F(t, \varphi(t), \varphi^*(t))$$

($\varepsilon > 0$ was arbitrarily chosen), what means that

$$\psi \in T(\varphi) \,.$$

Hence, invoking Lemma 1, T is usc and thus by Lemma 2 there is $\varphi \in \Phi$ such that $\varphi \in T(\varphi)$, what finishes the proof.

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