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Coefficient Regions for Univalent Trinomials, II

Obszar zmienności współczynników trójmianów jednolistnych II

Область изменения коэффициентов однолистных триполиноиов

In connection with his work on the Picard Theorem, Landau ([7], [8]) proved that every trinomial

$$(1) \qquad 1 + z + a_n z^n, \qquad n \geqslant 2,$$

has at least one zero in the circle $|z| \leqslant 2$. Using a simple rule due to Bohl [1], Herglotz [6] and Biernacki [2] showed (also see [5, p. 53]) that the trinomial

$$(2) \qquad 1 + z^{n_1} + a_{n_2} z^{n_2}, \qquad 1 \leqslant n_1 < n_2$$

has at least one zero in

$$|z| \leqslant \begin{cases} \left(\dfrac{n_2}{n_2 - n_1}\right)^{1/n_1} & \text{if } n_2 \text{ is an integral multiple of } n_1 \\ 1 & \text{if } n_2 \text{ is not an integral multiple of } n_1. \end{cases}$$

It is easily seen that the result of Herglotz and Biernacki is equivalent to the following

THEOREM A. *If*

$$1 + a_{n_1} z^{n_1} + a_{n_2} z^{n_2}, \qquad 1 \le n_1 < n_2 \tag{3}$$

does not vanish in $|z| < 1$, *then*

$$|a_{n_1}| \le \begin{cases} \dfrac{n_2}{n_2 - n_1} & \text{if } n_2 \text{ is an integral multiple of } n_1 \\ 1 & \text{if } n_2 \text{ is not an integral multiple of } n_1. \end{cases} \tag{4}$$

The examples

$$p(z) = 1 - \frac{k}{k-1} z^{n_1} + \frac{1}{k-1} z^{kn_1} = (1 - z^{n_1})\left(1 - \frac{1}{k-1} \sum_{j=1}^{k-1} z^{jn_1}\right)$$

and

$$q(z) = 1 + (1 - \varepsilon) z^{n_1} + \frac{\varepsilon}{2} z^{n_2}, \qquad \varepsilon > 0$$

show that (4) is best possible. However, we can claim more precisely (see [10]) that if G denotes the region determined by the curve

$$\varphi \longrightarrow e^{-in_1\varphi} + a_{n_2} e^{i(n_2 - n_1)\varphi}, \qquad 0 \le \varphi \le 2\pi$$

and containing the origin, then (3) is $\ne 0$ in $|z| < 1$ if and only if $-a_{n_1} \in \overline{G}$. This observation was used to deal with a related and in fact more difficult problem of Cowling and Royster [4], namely the determination of the precise region of variability of (a_2, a_k) for the univalent trinomial $z + a_2 z^2 + a_k z^k$ where $k \ge 3$. In fact, we considered arbitrary trinomials $z + a_p z^p + a_q z^q$ where $p < q$. Denoting the region determined by the curve

$$(5) \quad w(\varphi) = e^{-i(p-1)\varphi} + t\,\frac{\sin q\theta}{\sin\theta}\,e^{i(q-p)\varphi}, \quad 0 \le \varphi \le 2\pi, \quad 0 \le t \le \frac{1}{q}$$

and containing the origin by $G_\theta = G_\theta(p,q,t)$ where $G_0(p,q,\frac{1}{q})$ stands for the interval $[-2,2]$ if $q = 2p - 1$, and for $\{0\}$ otherwise, we proved [10]:

THEOREM B. *The trinomial*

$$f_t(z) = z - a_p z^p + tz^q, \qquad (p < q, \quad 0 < t \le \frac{1}{q})$$

is univalent in $|z| < 1$ *if and only if*

$$(6) \qquad a_p \in \bigcap_{0 \le \theta \le \frac{\pi}{2}} \frac{\sin\theta}{\sin p\theta}\,\overline{G}_\theta$$

where for $\theta = \frac{\pi}{p}, \; 2\frac{\pi}{p}, \ldots, [\frac{p}{2}]\frac{\pi}{p}$, $\frac{\sin\theta}{\sin p\theta}\,\overline{G}_\theta = \mathbb{C}$.

Besides, we carried out a closer study of trinomials of the forms

(i) $z - a_2 z^2 + tz^4$

(ii) $z - a_3 z^3 + tz^4$

(iii) $z - a_2 z^2 + tz^5$

(iv) $z - a_4 z^4 + tz^5$

which along with the previously known result ([11], [9]) about polynomials of the form $z + a_p z^p + a_{2p-1} z^{2p-1}$, gave us a reasonably good understanding of the coefficient region for univalent trinomials of degree ≤ 5.

Here we carry our investigation further and prove the following results.

THEOREM 1. <u>Let</u> G_θ <u>be as defined above. If</u> $2p-1>q>p$, <u>then the trinomial</u>

$$f_t(z) = z - a_p z^p + tz^q, \qquad (0<t\leq\frac{1}{q})$$

<u>is univalent in</u> $|z|<1$ <u>if and only if</u>

$$a_p \in \frac{1}{p}\overline{G_0}.$$

THEOREM 2. <u>Again let</u> G_θ <u>be as defined above. If</u> $q>2p-1$, <u>then provided</u> $q-1$ <u>is not an integral multiple of</u> $p-1$, <u>the trinomial</u>

$$f_t(z) = z - a_p z^p + tz^q, \qquad (0<t\leq\frac{1}{q})$$

<u>is univalent in</u> $|z|<1$ <u>if and only if</u>

$$a_p \in \frac{1}{p}\overline{G_0}.$$

The conclusion of Theorems 1 and 2 does not hold in general if $q-1$ is a multiple of $p-1$. However, it is known ([3], [4], [10]) that according as q is equal to 3, 4 or 5 the trinomial

$$f_t(z) = z - a_2 z^2 + tz^q, \qquad (t>0)$$

is univalent in $|z|<1$ if and only if

$$a_2 \in \frac{1}{2}\overline{G_0} = \frac{1}{2}\overline{G_0(2,q,t)}$$

provided t does not exceed 1/5, 1/16 or 1/35 respectively. Here we prove

THEOREM 3. *The trinomial*

$$f_t(z) = z - a_2z^2 + tz^q, \qquad (q \geq 3)$$

is univalent in $|z|<1$ *if and only if*

$$a_2 \in \frac{1}{2}\,\overline{G_0(2,q,t)}$$

provided $0<t \leq \dfrac{3}{q(q^2-4)}$.

Since $\frac{1}{p}\,\overline{G_0(p,q,\frac{1}{q})} = \{0\}$ if $q \neq 2p-1$, it is an immediate consequence of Theorem B that

$$f_{1/q}(z) = z - a_pz^p + \frac{1}{q}\,z^q, \qquad (q \neq 2p-1)$$

is univalent in $|z|<1$ if and only if $f'_{1/q}(z)$ does not vanish there. This proves Theorems 1 and 2 in the case $t = 1/q$ and so hereafter we will restrict ourselves to values of $t \in (0, \frac{1}{q})$.

We need various auxiliary results which we collect as lemmas.

LEMMA 1. *If* $\ell - 1$ *and* $m-1$ *are relatively prime, then the set of points*

$$(7) \qquad \exp\left(-i\,\frac{2\mu(\ell-1)\pi}{m-1}\right), \qquad \mu = 0,1,2,\ldots$$

is identical with the set

$$(8) \qquad \exp\left(-i\,\frac{2\mu\pi}{m-1}\right), \qquad \mu = 0,1,2,\ldots,m-2\ .$$

P r o o f. First, let us observe that for $\mu = 0,1,2,\ldots,m-2$ the points $\exp\left(-i\,\frac{2\mu(\ell-1)\pi}{m-1}\right)$ are all distinct. In fact

$$\exp\left(-i\,\frac{2\mu(\ell-1)\pi}{m-1}\right) = \exp\left(-i\,\frac{2\nu(\ell-1)\pi}{m-1}\right)$$

for some μ, ν such that $0 \leq \mu < \nu \leq m-2$ if and only if

(9) $$\exp\left(\frac{\ell - 1}{m - 1}(\nu - \mu)2\pi i\right) = 1$$

Since, by hypothesis, $\ell - 1$ and $m - 1$ have no common factors and $\nu - \mu \leq m - 2$ it is easily seen that $\frac{\ell - 1}{m - 1}(\nu - \mu)$ cannot be an integer and so (9) cannot hold.

On the other hand, the numbers (7) are of the form

$$\{\exp(-i(\ell - 1)2\mu\pi)\}^{1/(m-1)}, \qquad \mu = 0,1,2,\dots ,$$

i.e. they are amongst the $(m - 1)$-st roots of unity. In other words, the set of numbers (7) is a subset of the set (8).

The above two considerations show that the sets (7) and (8) are identical.

LEMMA 2. *Let* $\frac{p-1}{q-1} = \frac{\ell - 1}{m-1}$, *where* $\ell - 1$ *and* $m - 1$ *are relatively prime. Then there exists a positive integer* n *such that*

$$\exp\left(-i\frac{p-1}{q-1}2n\pi\right) = \exp\left(i\frac{2\pi}{m-1}\right).$$

Proof. According to Lemma 1 there exists a positive integer n such that

$$\exp\left(-i\frac{2(m-2)\pi}{m-1}\right) = \exp\left(-i\frac{2n(\ell - 1)\pi}{m-1}\right).$$

Hence

$$\exp\left(i\frac{2\pi}{m-1}\right) = \exp\left(-i\frac{2(m-2)\pi}{m-1}\right) = \exp\left(-i\frac{2n(\ell - 1)\pi}{m-1}\right) =$$
$$= \exp\left(-i\frac{p-1}{q-1}2n\pi\right).$$

The region G_θ is determined by a curve of the form

(10) $$w(\varphi) = w(b,\varphi) = e^{-i(p-1)\varphi} + be^{i(q-p)\varphi}, \qquad 0 \leq \varphi \leq 2\pi$$

where $-b_0 \leq b < 1$ with $0 < b_0 < 1$. In [10] we noted some important properties of the curve Γ_b defined by (10). For example, a point w lies on Γ_b if and only if its conjugate does. This in conjunction with the fact that $0 \in G_\theta$ implies:

LEMMA 3. _The region_ G_θ _is symmetrical about the real axis._

Here we prove

LEMMA 4. _If_ $\frac{p-1}{q-1} = \frac{\ell-1}{m-1}$ _where_ $\ell - 1$ _and_ $m-1$ _are relatively prime then the curve_ Γ_b _and hence the region_ G_θ _is symmetrical about the line_

$$\operatorname{Im}\{we^{-i\pi/(m-1)}\} = 0 .$$

P r o o f. Let n be as in Lemma 2. If we define $w(\varphi)$ outside the interval $[0,2\pi]$ by periodicity, then

$$w\left(\frac{2n\pi}{q-1} - \varphi\right) = \exp\left\{-i(q-1)\left(\frac{2n\pi}{q-1} - \varphi\right)\right\} +$$

$$+ b\exp\left\{i(q-p)\left(\frac{2n\pi}{q-1} - \varphi\right)\right\} =$$

$$= e^{2\pi i/(m-1)}e^{i(p-1)\varphi} + be^{2n\pi i}e^{2\pi i/(m-1)}e^{-i(q-p)\varphi} =$$

$$= e^{2\pi i/(m-1)}\left\{e^{i(p-1)\varphi} + be^{-i(q-p)\varphi}\right\} = e^{2\pi i/(m-1)}\overline{w(\varphi)} .$$

This means that a point w lies on Γ_b if and only if $e^{2\pi i/(m-1)}\overline{w(\varphi)}$ does. Hence we have the desired result.

We are now ready to prove

LEMMA 5. _Let_ $\frac{p-1}{q-1} = \frac{\ell-1}{m-1}$, _where_ $\ell - 1$ _and_ $m-1$ _are relatively prime. Then_ $G_\theta(p,q,t)$ _is symmetrical about the lines_

(11) $$\operatorname{Im}\left\{w \exp\left(-i\frac{k\pi}{m-1}\right)\right\} = 0, \qquad k = 0,1,2,\dots,2m-3.$$

P r o o f. From the definition of $w(\varphi)$ it is readily seen that

$$w\left(\varphi + \frac{2\pi}{q-1}\right) \equiv w(\varphi)\exp\left(-i\frac{2(p-1)\pi}{q-1}\right).$$

Hence a point w lies on Γ_b if and only if the points

$$w\exp\left(-i\frac{2\mu(\ell-1)\pi}{m-1}\right), \qquad \mu = 0,1,2,\dots$$

do. But according to Lemma 1 this set of points is identical with the set

$$w\exp\left(-i\frac{2\mu\pi}{m-1}\right), \qquad \mu = 0,1,2,\dots,m-2 .$$

The desired result is now a simple consequence of Lemmas 3 and 4.

The next four lemmas give some useful information about the curve Γ_b and the region G_θ.

LEMMA 6. _Let_

$$g(z) = z^{-(p-1)} + bz^{q-p}, \qquad (q > p > 1)$$

where $-1 < b < 1$. _If_ $2p - 1 > q$ _then the vector_ $g(e^{i\varphi})$ _turns monotonically in the clockwise direction as_ φ _increases from_ 0 _to_ 2π.

P r o o f. It is enough to show that

(12) $$\operatorname{Re}\{zg'(z)/g(z)\} < 0 \qquad \text{for } |z| = 1.$$

Writing $z = e^{i\varphi}$ we see that (12) holds if and only if

$$L(b,\varphi) := b^2(q-p) - b(2p-1-q)\cos\{(q-1)\varphi\} - (p-1) < 0$$

for $0 \leq \varphi \leq 2\pi$.

But clearly

$$L(b,\varphi) \leq b^2(q-p) + |b|(2p-1-q) - (p-1),$$

and so for $-1 < b < 1$

$$L(b,\varphi) < (q-p) + (2p-1-q) - (p-1) = 0$$

LEMMA 7. _Under the conditions of Lemma 6 the tangent to the curve_

$$w(\varphi) = g(e^{i\varphi}), \qquad 0 \leq \varphi \leq 2\pi$$

turns monotonically in the clockwise direction as φ _increases from_ 0 _to_ 2π.

P r o o f. It is clearly enough to verify that

$$\mathrm{Re}\{1 + zg''(z)/g'(z)\} < 0 \qquad \text{for } |z| = 1, \tag{13}$$

or equivalently

$$b^2(q-p)^3 + b(q-p)(p-1)(2p-1-q)\cos\{(q-1)\varphi\} - (p-1)^3 < 0 \qquad \text{for } 0 \leq \varphi \leq 2\pi. \tag{14}$$

But the expression on the left hand side of (14) cannot exceed

$$(q-p)^3 + (q-p)(p-1)(2p-1-q) - (p-1)^3$$

which is negative since it can be written in the form

$$-(2p-1-q)\{(q-p)^2 + (p-1)^2\}.$$

LEMMA 8. *Let*

$$g(z) = z^{-(p-1)} + bz^{q-p}, \qquad (q > p > 1).$$

If $2p - 1 < q$ *then for* $-(p-1)/(q-p) \leq b \leq (p-1)/(q-p)$ *the vector* $g(e^{i\varphi})$ *turns monotonically in the clockwise direction as* φ *increases from* 0 *to* 2π.

P r o o f. We observe that if $-(p-1)/(q-p) < b < (p-1)/(q-p)$ then (12) holds, or equivalently

$$L(b, \varphi) := b^2(q-p) + b(q-2p+1)\cos\{(q-1)\varphi\} -$$

$$-(p-1) < 0 \qquad \text{for } 0 \leq \varphi \leq 2\pi.$$

In fact

$$L(b, \varphi) \leq b^2(q-p) + |b|(q-2p+1) - (p-1) =$$

$$= \{(q-p)|b| - (p-1)\}(|b| + 1) < 0$$

$$\text{if } -(p-1)/(q-p) < b < (p-1)/(q-p).$$

If $b = \pm(p-1)/(q-p)$ then $L(b, \varphi) < 0$ except at the points where $\cos\{(q-1)\varphi\} = \frac{b}{|b|}$. At such points $L(b, \varphi) = 0$. Hence the lemma holds.

LEMMA 9. *Let*

$$g(z) = z^{-(p-1)} + bz^{q-p}, \qquad (q > p > 1, \quad -1 < b < 1).$$

If $2p - 1 < q$ *then for* $|b| \geq (p-1)/(q-p)$ *the tangent to the curve*

$$w(\varphi) = g(e^{i\varphi}), \qquad 0 \leq \varphi \leq 2\pi$$

turns monotonically in the counter-clockwise direction as φ increases from 0 to 2π.

P r o o f. We observe that if $|b| > (p-1)/(q-p)$ then

$$\text{(13')} \qquad \mathrm{Re}\{1 + zg''(z)/g'(z)\} > 0 \quad \text{for} \quad |z| = 1,$$

or equivalently

$$\mathcal{L}(b,\varphi) := \bar{b}^2(q-p)^3 -$$
$$- b(q-p)(p-1)(q-2p+1)\cos\{(q-1)\varphi\} -$$
$$- (p-1)^3 > 0 \qquad \text{for} \quad 0 \le \varphi \le 2\pi .$$

In fact

$$\mathcal{L}(b,\varphi) \ge b^2(q-p)^3 - |b|(q-p)(p-1)(q-2p+1) -$$
$$- (p-1)^3 = \{|b|(q-p)^2 + (p-1)^2\}\{|b|(q-p) -$$
$$- (p-1)\} > 0 \qquad \text{if} \quad |b| > (p-1)/(q-p).$$

If $b = \pm (p-1)/(q-p)$ then $\mathcal{L}(b,\varphi) > 0$ except at the points where $\cos\{(q-1)\varphi\} = \frac{b}{|b|}$. At such points $\mathcal{L}(b,\varphi) = 0$. Hence Lemma 9 holds.

We will also need

LEMMA 10. Let $\frac{p-1}{q-1} = \frac{\ell-1}{m-1}$ where $\ell - 1$ and $m - 1$ are relatively prime. Further, let $\frac{p-1}{\ell-1} = \frac{q-1}{m-1} = s$, and for $k = 0,1,2,\dots,m-2$

$$\text{(15)} \qquad \psi_k = \begin{cases} -\dfrac{\ell-1}{m-1}(2k+1)\pi & \text{if} \quad t\,\dfrac{\sin q\theta}{\sin\theta} > 0 \\[2ex] -\dfrac{\ell-1}{m-1}\,2k\pi & \text{if} \quad t\,\dfrac{\sin q\theta}{\sin\theta} < 0 \end{cases}$$

Then the part of the boundary of G_{Θ} contained in the sector $|\arg w - \psi_k| \leq \frac{\pi}{m-1}$ is the image of some subinterval $I_{\theta,k} := [\alpha_{\theta,k}, \beta_{\theta,k}]$ by the mapping (10) with $b = t\,\frac{\sin q\theta}{\sin\theta}$.

P r o o f. Since $w(\varphi + \frac{2\pi}{s}) \equiv w(\varphi)$ for all real φ,

$$w(\varphi) = e^{-i(p-1)\varphi} + be^{i(q-p)\varphi}, \quad 0 \leq \varphi \leq 2\pi/s$$

is a closed curve γ_b whose trace is the same as that of the curve Γ_b.

Now let $b > 0$. Note that the minimum distance between the origin and a point on the boundary of G_{Θ} is 1-b and the points of the boundary for which this distance is attained are precisely the points

(16) $$(1-b)e^{i\psi_k}, \qquad k = 0,1,2,\ldots,m-2 .$$

In the same way as for Lemma 1 it can be shown that this set of points is identical with the set

$$(1-b)\exp(-i\,\frac{2\mu\pi}{m-1}), \qquad \mu = 0,1,2,\ldots,m-2$$

or the set

$$(1-b)\exp(-i\,\frac{(2\mu+1)\pi}{m-1}), \qquad \mu = 0,1,2,\ldots,m-2$$

according as $\ell - 1$ is even or odd.

The region G_{Θ} being symmetrical about the lines

$$\operatorname{Im}\{w\exp(-i\,\frac{k\pi}{m-1})\} = 0, \qquad \mu = 0,1,2,\ldots,2m-3$$

the part $\gamma_{b,k}$ of its boundary lying in the sector $|\arg w - \psi_k| \leq \frac{\pi}{m-1}$ is either the image of an interval $I_{\theta,k} \subset [0, 2\pi/s]$ by $w(\varphi)$ or else it contains at least two points w^*, $\overline{w^*}e^{2i\psi_k}$ not lying on the rays $\arg w = \psi_k \pm \frac{\pi}{m-1}$ where the curve γ_b cuts itself. Clearly then, the curve γ_b cuts itself also in the points $\left\{w^*\exp(i\,\frac{2\mu\pi}{m-1})\right\}_{\mu=1}^{m-2}$ and $\left\{\overline{w^*}e^{2i\psi_k}\exp(i\,\frac{2\mu\pi}{m-1})\right\}_{\mu=1}^{m-2}$. Thus, there are at least $4(m-1)$ values of φ in $[0, 2\pi/s]$ such that $|w(\varphi)| = |w^*|$. However, this is impossible. In fact, the curve γ_b is the union of $m-1$ congruent arcs C_k described by the moving point $w(\varphi)$ as φ increases from $\frac{k}{m-1}\frac{2\pi}{s}$ to $\frac{k+1}{m-1}\frac{2\pi}{s}$, $k = 0,1,2,\dots,m-2$. On each of these arcs $|w(\varphi)|$ decreases from $1+b$ to $1-b$ and then increases to $1+b$. Hence $|w(\varphi)|$ cannot assume any value more than twice in the interval $\left[\frac{k}{m-1}\frac{2\pi}{s}, \frac{k+1}{m-1}\frac{2\pi}{s}\right]$ and can assume any given value at most $2(m-1)$ times in $[0, 2\pi/s]$.

The argument is similar in the case $b<0$.

In addition we will need the following lemma which is proved in [10].

LEMMA 11. <u>Let</u> $F(z,x)$ <u>be a complex valued function of</u> z <u>(complex) and</u> x <u>(real) having the following properties:</u>

(i) <u>there exists an absolute constant</u> $\alpha > 0$ <u>such that for each</u> x <u>belonging to the interval</u> $I := \{x : a < x \leq b\}$, $F(z,x)$ <u>is analytic in the annulus</u> $A_\alpha := \{z : 1-\alpha < |z| < 1+\alpha\}$ <u>and is univalent on the arc</u>

$$\gamma_x := \{z = e^{i\varphi} : \varphi_1(x) \leq \varphi \leq \varphi_2(x)\},$$

<u>where</u> $\varphi_1(x)$, $\varphi_2(x)$ <u>are continuous functions of</u> x <u>satisfying</u> $0 < \varphi_2(x) - \varphi_1(x) < 2\pi$,

(ii) <u>for each</u> z_0 <u>lying on</u> γ_{x_0} <u>where</u> x_0 <u>is an arbitrary point of</u> I <u>there exists a left-hand neighbourhood</u>

$$N(x_0; \delta(z_0)) := \{x : x_0 - \delta(z_0) < x \leq x_0\}$$

of x_0 in which $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x^2}$, $\frac{\partial^2 F}{\partial x \partial z}$ exist and are bounded,

(iii) there exists an absolute constant M such that for all $x \in I$ and $z \in \overline{A}_{\alpha/2}$,

$$|F(z,x)| < M.$$

For each $x \in I$, let C_x be the arc

$$w = F(e^{i\varphi}, x), \qquad \varphi_1(x) \leq \varphi \leq \varphi_2(x).$$

Now, if

$$\mathrm{Re}\left[\frac{\partial}{\partial x} F(z,x) / \left\{z \frac{\partial}{\partial z} F(z,x)\right\}\right] > 0 \tag{17}$$

for all $x \in I$, $z \in \gamma_x$, then the arcs C_{x_1}, C_{x_2} where $x_1 \in I$, $x_2 \in I$ do not intersect each other if $|x_1 - x_2|$ is sufficiently small. In particular, if the arcs C_x, except for their end points, remain confined to the interior of a fixed angle $\alpha_1 < \psi < \alpha_2$ of opening $< 2\pi$ whereas, each arc has its initial point on $\psi = \alpha_2$ and its terminal point on $\psi = \alpha_1$, then the sectorial region bounded by C_x and the two rays $\psi = \alpha_1$, α_2 shrinks as x increases.

P r o o f of Theorem 1. First of all we wish to prove that

$$\bigcap_{0 \leq \theta < \pi/q} \overline{G_\theta} = \overline{G_0}.$$

It is clearly enough to show that the part of G_θ lying in the sector $|\arg w - \psi_0| \leq \frac{\pi}{m-1}$, where ψ_0 is defined in (15), shrinks monotonically as θ decreases from π/q to 0.

For this we apply Lemma 11 to the function

$$F(z,x) = z^{-(p-1)} + t\,\frac{\sin q\theta}{\sin\theta}\,z^{q-p}, \qquad x = \cos\theta$$

where for γ_x we take $\{z = e^{i\varphi}\ :\ \varphi \in [\bar{\alpha}_{\theta,0}, \bar{\beta}_{\theta,0}]\}$. The numbers $\bar{\alpha}_{\theta,0}$, $\bar{\beta}_{\theta,0}$ are the same as in the statement of Lemma 10. The part of the boundary of G_θ lying in the sector $|\arg w - \psi_0| \le \frac{\pi}{m-1}$ is then the arc C_x of Lemma 11. A simple calculation shows that condition (17) is equivalent to

$$(18)\qquad (q\cos q\theta\sin\theta - \cos\theta\sin q\theta)\Big\{-(p-1)\cos(q-1)\varphi + t(q-p)\frac{\sin q\theta}{\sin\theta}\Big\} < 0 .$$

The quantity within the first pair of brackets is negative for $\theta \in (0, \pi/q)$ whereas the quantity within the second pair of brackets is positive for $\varphi \in \left(\frac{\pi}{2(q-1)}, \frac{3\pi}{2(q-1)}\right)$ and $\theta \in (0, \pi/q)$.

Now let us show that

$$(19)\qquad (\bar{\alpha}_{\theta,0}, \bar{\beta}_{\theta,0}) \subset \left(\frac{\pi}{2(q-1)}, \frac{3\pi}{2(q-1)}\right).$$

If we denote by $\operatorname{Arg} w$, the value of the argument lying in $[-2\pi, 0)$, then

$$\operatorname{Arg} w(\bar{\alpha}_{\theta,0}) = -\frac{p-1}{q-1}\pi + \frac{\pi}{m-1},$$

$$\operatorname{Arg} w(\bar{\beta}_{\theta,0}) = -\frac{p-1}{q-1}\pi - \frac{\pi}{m-1},$$

$$\operatorname{Arg} w\left(\frac{\pi}{2(q-1)}\right) = -\frac{p-1}{2(q-1)}\pi + \psi^*,$$

$$\operatorname{Arg} w\left(\frac{3\pi}{2(q-1)}\right) = -\frac{3(p-1)}{2(q-1)}\pi - \psi^*$$

where ψ^* is the unique root of the equation $\tan\psi = t\,\frac{\sin q\theta}{\sin\theta}$ in $(0, \pi/4]$.

In order to prove (19) it is enough, in view of Lemma 6, to verify that

(20) $$\operatorname{Arg} w(\bar{\alpha}_{\theta,0}) < \operatorname{Arg} w\left(\frac{\pi}{2(q-1)}\right),$$

(21) $$\operatorname{Arg} w\left(\frac{3\pi}{2(q-1)}\right) < \operatorname{Arg} w(\bar{\beta}_{\theta,0}).$$

It is easily seen that inequalities (20), (21) hold if and only if

(22) $$\frac{\pi}{m-1} < \frac{1}{2}\,\frac{\ell-1}{m-1}\,\pi + \psi^*$$

The hypothesis $2p-1>q$ which is equivalent to $\frac{\ell-1}{m-1} = \frac{p-1}{q-1} > \frac{1}{2}$ implies that $\ell - 1 \geqslant 2$. Hence (22) does hold and in turn so do (20), (21).

Thus (18) certainly holds for $\varphi \in (\bar{\alpha}_{\theta,0}, \bar{\beta}_{\theta,0})$, i.e. the curves C_x do not intersect each other as x varies from $\cos(\pi/q)$ to 1. Indeed we have shown that the region G_θ shrinks monotonically as θ decreases from π/q to 0.

Since $\frac{1}{p} \leqslant \frac{\sin\theta}{\sin p\theta}$ for $\theta \in [0, \pi/q)$ and $\overline{G_0} \subset \overline{G_\theta}$ for all θ in this range it follows that $\frac{1}{p}\overline{G_0}$ is a fortiori contained in $\frac{\sin\theta}{\sin p\theta}\overline{G_\theta}$, i.e.

$$\bigcap_{0 \leqslant \theta \leqslant \pi/q} \frac{\sin\theta}{\sin p\theta}\,\overline{G_\theta} = \frac{1}{p}\,\overline{G_0}.$$

The theorem will be completely proved if we show that $\frac{1}{p}\overline{G_0} \subset \frac{\sin\theta}{\sin p\theta}\overline{G_\theta}$ for all $\theta \in \left[\frac{\pi}{q}, \frac{\pi}{2}\right]$. We shall in fact show that

(23) $$\frac{1}{p}\max_{w\in\overline{G_0}} |w| \leqslant \frac{\sin\theta}{|\sin p\theta|}\min_{w\in\partial G} |w| \quad \text{for } \theta \in \left[\frac{\pi}{q}, \frac{\pi}{2}\right],$$

and thereby complete the proof of the theorem.

There are $m - 1$ points on ∂G_0 where $\max_{w \in \bar{G}_0} |w|$ is attained. If w_0 is such a point, then $\arg w_0 \pm \frac{\pi}{m-1}$ are two of the directions in which $\min_{w \in \partial G_0} |w| = 1 - tq$ is attained. Lemmas 6, 7 imply that the region G_0 is convex, from which it readily follows that

$$\max_{w \in \bar{G}_0} |w| \leq (1 - tq) \sec \frac{\pi}{m - 1}. \tag{24}$$

Since $\min_{w \in \partial G} |w| = 1 - t\left|\frac{\sin q\theta}{\sin \theta}\right|$ inequality (23) will proved if we show that

$$\frac{1}{p}(1 - tq)\sec \frac{\pi}{m - 1} \leq \frac{\sin \theta}{|\sin p\theta|}\left(1 - t\frac{|\sin q\theta|}{\sin \theta}\right)$$

$$\text{for } \theta \in \left[\frac{\pi}{q}, \frac{\pi}{2}\right].$$

We shall indeed prove that for $\theta \in \left[\frac{\pi}{q}, \frac{\pi}{2}\right]$ the stronger inequality

$$\frac{|\sin p\theta|}{\sin \theta} < p \cos \frac{\pi}{m - 1} \tag{25}$$

holds.

First let $\pi/q \leq \theta < \pi/p$. Then, in view of the hypothesis $2p - 1 > q$ we have $\frac{\pi}{2} + \frac{\pi}{2q} < p\theta \leq \pi$ and so

$$0 \leq \sin p\theta < \cos \frac{\pi}{2q}, \qquad \sin \theta \geq \sin \frac{\pi}{q}.$$

Consequently $\frac{\sin p\theta}{\sin \theta} < 1/(2 \sin \frac{\pi}{2q})$ and for (25) to be true for $\pi/q \leq \theta \leq \pi/p$ it is enough that the inequality

$$2p \sin \frac{\pi}{2q} \cos \frac{\pi}{m - 1} \geq 1 \tag{26}$$

hold for values of p, q and m under consideration. Now if $m - 1 \geq 4$ then also $q - 1 \geq 4$ and the hypothesis $2p - 1 > q$

implies that $p \geq 3$. Hence, the left-hand side of (26) is at least equal to $\sqrt{2}\, p \sin \frac{\pi}{4p}$. Now using the fact that $\frac{1}{x} \sin(\frac{\pi}{4} x)$ is a decreasing function of x in $(0,2)$ we obtain

$$\sqrt{2}\, p \sin \frac{\pi}{4p} \geq 3\sqrt{2} \sin \frac{\pi}{12} > 1$$

In the case $m - 1 = 3$ we write $p = 1 + s(\ell - 1)$ and $q = 1 + s(m - 1)$ where of course $\ell - 1 = 2$ and s is a positive integer. The left-hand side of (26) becomes $(1 + 2s)\sin \frac{\pi}{2(1 + 3s)}$ which is larger than $(1 + 2s)\sin\frac{\pi}{3(1+2s)}$. Again using the fact that $\frac{1}{x} \sin(\frac{\pi}{3} x)$ is a decreasing function of x in $(0,\frac{3}{2})$ we conclude that

$$(1 + 2s) \sin \frac{\pi}{3(1 + 2s)} \geq 3 \sin \frac{\pi}{9} > 1.$$

With this the proof of (25) for $\theta \in [\pi/q, \pi/p]$ is complete.

If $\pi/p \leq \theta \leq \pi/2$ then $\sin\theta \geq \sin \frac{\pi}{p}$ and so (25) will be proved if we show that

$$p \sin \frac{\pi}{p} \cos \frac{\pi}{m - 1} \geq 1 . \tag{27}$$

The hypothesis $2p - 1 > q$ implies that $m - 1$ is necessarily ≥ 3 and so is p. Hence the left-hand side of (27) is at least equal to $\frac{3\sqrt{3}}{4}$ and is therefore greater than 1. Here again we have used the fact that $\frac{1}{x} \sin(\pi x)$ is a decreasing function of x in $(0,1/2)$.

The following result which is quite surprising is a simple consequence of Theorem 1.

COROLLARY 1. *If* $2p - 1 > q$, *then the trinomial*

$$z + a_p z^p + a_q z^q$$

is univalent in $|z|<1$ if and only if its derivative does not vanish there.

REMARK. From (24) it readily follows that if the trinomial

$$1 + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} \qquad (n_1 < n_2 < 2n_1)$$

does not vanish in $|z|<1$ and $\frac{n_1}{n_2} = \frac{\vartheta_1}{\vartheta_2}$ where ϑ_1, ϑ_2 are relatively prime, then

$$|a_{n_1}| \le (1 - |a_{n_2}|) \sec \frac{\pi}{\vartheta_2} . \tag{28}$$

We can, in fact, prove the following result which is to be compared with Theorem A.

THEOREM A′. *If*

$$1 + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} \qquad (n_1 < n_2 < 2n_1)$$

does not vanish in $|z|<1$ and $\frac{n_1}{n_2} = \frac{\vartheta_1}{\vartheta_2}$ where ϑ_1, ϑ_2 are relatively prime, then

$$|a_{n_1}| \le \begin{cases} \min\left\{(1 - |a_{n_2}|)\sec \frac{\pi}{\vartheta_2} ,\ 1 - |a_{n_2}| + |a_{n_2}|^2 \right\} & \text{if } \vartheta_1 \ge 3 \\ 1 - |a_{n_2}|^2 & \text{if } \vartheta_1 = 2 . \end{cases} \tag{29}$$

P r o o f. In view of (28) and Corollary 1 it is enough to prove that if

$$z + a_p z^p + t z^q \qquad (p < q < 2p - 1, \quad 0 < t \le \tfrac{1}{q})$$

is univalent in $|z|<1$ and $\frac{p-1}{q-1} = \frac{\ell - 1}{m - 1}$ where $\ell - 1$ and $m - 1$ are relatively prime, then

$$(30)\qquad p|a_p| \leqslant \begin{cases} 1 - tq + t^2q^2 & \text{if } \ell - 1 \geqslant 3 \\ 1 - t^2q^2 & \text{if } \ell - 1 = 2 . \end{cases}$$

There are $m - 1$ points on the boundary of G_0 whose absolute value is equal to $\max_{w \in \overline{G}_0} |w|$. There is one whose Argument is equal to $-\frac{\ell - 1}{m - 1}\pi + \frac{\pi}{m - 1}$. Call it w_0. The point w_0 lies on the portion of Γ_{tq} described by the moving point

$$w(\varphi) = e^{-i(p-1)\varphi} + tqe^{i(q-p)\varphi}$$

as φ increases from 0 to $\frac{\pi}{q-1}$. Since $|w(\varphi)|$ decreases monotonically from $1 + tq$ to $1 - tq$ as φ increases from 0 to $\frac{\pi}{q-1}$ there is a unique value of φ, say φ_0, in $(0, \frac{\pi}{q-1})$ such that $w(\varphi_0) = w_0$, and the points lying on the portion γ of Γ_{tq} which is the image of $[0, \varphi_0]$ must be of modulus $\geqslant \max_{w \in \overline{G}_0} |w|$. Now we wish to show that

$$(31)\qquad w\left(\frac{2}{3}\frac{\pi}{q-1}\right) \in \gamma$$

which would imply that

$$(32)\qquad \max_{w \in \overline{G}_0} |w| \leqslant \left|w\left(\frac{2}{3}\frac{\pi}{q-1}\right)\right| .$$

Since $\operatorname{Arg} w(\varphi)$ decreases from 0 to $-\frac{\ell - 1}{m - 1}\pi + \frac{\pi}{m - 1}$ as φ increases from 0 to φ_0 it is enough to show that

$$(33)\qquad \operatorname{Arg} w\left(\frac{2}{3}\frac{\pi}{q-1}\right) > \operatorname{Arg} w_0 .$$

If α_0 is the unique root of the equation

$$\tan \alpha = \frac{(\sqrt{3}/2)tq}{1 - (1/2)tq}$$

in $(0,\frac{\pi}{3}]$ then

$$\operatorname{Arg} w\left(\frac{2}{3}\frac{\pi}{q-1}\right) = -\frac{2}{3}\frac{\ell-1}{m-1}\pi + \alpha_0$$

and (33) is equivalent to

$$\frac{1}{3}\frac{\ell-1}{m-1}\pi + \alpha_0 > 0$$

which is certainly true for $\ell \geq 4$. The case $\ell - 1 \geq 3$ of inequality (30) is now an immediate consequence of (32) since

$$\left|w\left(\frac{2}{3}\frac{\pi}{q-1}\right)\right| = 1 - tq + t^2q^2$$

If $\ell - 1 = 2$, then $m - 1$ is necessarily equal to 3 and in that case it follows from our study of the coefficient region of univalent trinomials of the form $z - a_3z^3 + tz^4$, $0 < t \leq \frac{1}{4}$ that (see [10, Corollary 2])

$$p|a_p| \leq \max_{w \in \overline{G}_0} |w| \leq 1 - t^2q^2$$

which completes the proof of (30) and in turn that of Theorem A'.

P r o o f of Theorem 2. First we observe that

$$\bigcap_{0 \leq \theta < \pi/q} \frac{\sin\theta}{\sin p\theta}\,\overline{G}_\theta = \frac{1}{p}\,\overline{G}_0 .$$

The reasoning used in the first part of the proof of Theorem 1 to prove this fact in the case $2p - 1 > q$ remains valid. Indeed, the condition $2p - 1 > q$ was used only to conclude that $\ell - 1 \geq 2$ but that is true here as well since, by hypothesis, $q - 1$ is not a multiple of $p - 1$.

What we need to show now is that

(34) $$\frac{1}{p}\overline{G_0} \subseteq \frac{\sin\theta}{\sin p\theta}\overline{G_\theta} \qquad \text{for all } \theta\in\left[\frac{\pi}{q},\frac{\pi}{2}\right].$$

This would follow if we could show that

(35) $$\max_{w\in\overline{G_0}}|w| \le \min_{w\in\partial G_\theta}|w| \qquad \text{for all } \theta\in\left[\frac{\pi}{q},\frac{\pi}{2}\right].$$

Since we do not know the precise value of $\max_{w\in\overline{G_0}}|w|$ we look for a good enough upper estimate. For this let w_0 be the point of ∂G_0 such that $\max_{w\in\overline{G_0}}|w| = |w_0|$, and $\operatorname{Arg} w_0 = -\frac{\ell-1}{m-1}\pi + \frac{\pi}{m-1}$. Denote by γ_{tq} the portion of the curve Γ_{tq} described by

$$w(\varphi) = e^{-i(p-1)\varphi} + tqe^{i(q-p)\varphi}$$

as φ increases from 0 to $\frac{\pi}{q-1}$. Thus the initial and terminal points of γ_{tq} are $1 + tq$ and $(1-tq)\exp(-i\frac{\ell-1}{m-1}\pi)$ respectively. As φ increases from 0 to $\frac{\pi}{q-1}$, $|w(\varphi)|$ decreases monotonically from $1 + tq$ to $1 - tq$ and according to Lemma 8 the vector $w(\varphi)$ turns monotonically in the clockwise direction provided $tq \le \frac{\ell-1}{m-\ell}$. From the expression for $w(\varphi)$ and Lemma 9 it follows that if $tq > \frac{\ell-1}{m-\ell}$ then $\operatorname{Im}\{w(\varphi)\}$ first increases and then decreases monotonically as φ increases from 0 to $\frac{\pi}{q-1}$. Now set $\varphi_\lambda = \lambda\frac{\pi}{q-1}$ where $0<\lambda<1$. If $\arg^* w$ denotes the value of the argument lying in $[-\frac{3\pi}{2}, \frac{\pi}{2})$ then in view of the above mentioned properties of γ_{tq} we may take $|w(\varphi_\lambda)|$ as an upper estimate for $|w_0|$ provided

(36) $$\arg^* w(\varphi_\lambda) \ge \arg^* w_0 = -\frac{\ell-1}{m-1}\pi + \frac{\pi}{m-1}.$$

Inequality (36) holds if and only if

(37) $$\alpha^* + \{(\ell - 1)(1 - \lambda) - 1\}\frac{\pi}{m - 1} \geqslant 1 .$$

where α^* is the unique root of the equation

(38) $$\tan \alpha = \frac{tq \sin(\lambda\pi)}{1 + tq \cos(\lambda\pi)}$$

in the interval $[0, \frac{\pi}{2}]$.

Now let us set $\lambda = 1 - \frac{\varepsilon}{\ell - 1}$ $(0 < \varepsilon \leqslant 1)$. Then (37) takes the form

(39) $$\alpha^* \geqslant \frac{\pi}{m - 1}(1 - \varepsilon) .$$

Using (38) we see that (39) is true if

(40) $$t \geqslant \frac{1}{q} \frac{\tan(\frac{\pi}{m - 1}(1 - \varepsilon))}{\sin(\frac{\pi}{\ell - 1}\varepsilon) + \cos(\frac{\pi}{\ell - 1}\varepsilon) \tan(\frac{\pi}{m - 1}(1 - \varepsilon))}$$

Thus we may use the estimate

(41) $$\max_{w \in G_0} |w|^2 \leqslant |w(\varphi_\lambda)| = 1 + t^2q^2 - 2tq \cos(\frac{\pi}{\ell - 1}\varepsilon)$$

provided (40) holds. In particular,

$$\max_{w \in G_0} |w|^2 \leqslant 1 + t^2q^2 - 2tq \cos \frac{\pi}{\ell - 1} \quad \text{for all} \quad t \in [0, \tfrac{1}{q}]$$

Besides,

$$\min_{w \in \partial G_\theta} |w| = 1 - t\frac{|\sin q\theta|}{\sin\theta} \geqslant 1 - t/(\sin\tfrac{\pi}{q}) \quad \text{for } \theta \in [\tfrac{\pi}{q}, \tfrac{\pi}{2}]$$

Hence inequality (35) will be proved for all $t \in [0, \frac{1}{q}]$ if it turns out that

(42) $$1 + t^2q^2 - 2tq \cos \frac{\pi}{\ell - 1} \leqslant \{1 - t/(\sin \tfrac{\pi}{q})\}^2 .$$

After simplification inequality (42) takes the form

$$(43)\qquad t\{q^2 - 1/(\sin\tfrac{\pi}{q})^2\} + 2/(\sin\tfrac{\pi}{q}) \leq 2q\cos\frac{\pi}{\ell - 1}\,.$$

Using the estimate $\frac{1}{1-x} \leq 1 + \frac{1}{1-a}x$ which is valid for $0 \leq x \leq a < 1$ we obtain

$$(44)\qquad 1/(\sin\tfrac{\pi}{q}) < \tfrac{q}{\pi}(1 + 1.048\,\frac{\pi^2}{6q^2}) \qquad \text{for all } q \geq 6\,.$$

Hence (43) would hold for $q \geq 6$ if the inequality

$$(45)\qquad tq(1 - \frac{1}{\pi^2}) + \frac{2}{\pi} + 1.048\,\frac{\pi}{q^2} \leq 2\cos\frac{\pi}{\ell - 1}$$

were true. Inequality (45) turns out to be true if $\ell - 1 \geq 5$ since in that case $q \geq 12$. Thus (34) holds if $\ell - 1 \geq 5$.

Now let $\ell - 1 = 4$. Then clearly $q \geq 10$ and it is a matter of simple verification that (45) (and so (34)) holds for $tq < 0.75$. In order to deal with the case $0.75 < tq \leq 1$ we take $\varepsilon = \frac{2}{3}$ in (41) and obtain the estimate

$$(46)\qquad \max_{w \in \overline{G}_0} |w|^2 \leq 1 + t^2q^2 - \sqrt{3}\,tq$$

valid for $1 \geq tq \geq \frac{2\tan(\pi/27)}{1 + \sqrt{3}\tan(\pi/27)}$ and so certainly for $1 \geq tq > 0.75$. Thus (35) would hold if

$$(47)\qquad 1 + t^2q^2 - \sqrt{3}\,tq \leq \{1 - t/(\sin\tfrac{\pi}{q})\}^2$$

were true for $1 \geq tq > 0.75$ and $q \geq 10$. That it is indeed the case can be easily checked using the estimate (44). Hence (34) holds also if $\ell - 1 = 4$.

If $\ell - 1 = 3$ then $q \geq 8$ and (45) holds for $tq \leq 0.36$ though not for all $tq \leq 1$. Setting $\varepsilon = \frac{1}{2}$ in (41) we see that in the case $1 \geq tq > 0.36$ we can use the estimate (46) for $\max_{w \in \overline{G}_0} |w|^2$. Hence (35) would hold if (47) were true for $1 \geq tq > 0.36$ and $q \geq 8$. It does indeed turn out to be the case and so (35) and in turn (34) holds for $\ell - 1 = 3$ as well.

The case $\ell - 1 = 2$ cannot be handled in quite the same way. We will, in fact, need a couple of additional lemmas.

LEMMA 12. The function $\frac{\sin p\theta}{\sin\theta}$ decreases from p to 0 as θ increases from 0 to π/p.

Since $\cos t$ is a decreasing function of t in $(0,\pi)$ the conclusion follows immediately from the fact that

$$\frac{\sin p\theta}{\sin\theta} = \begin{cases} 1 + 2\cos 2\theta + 2\cos 4\theta + \ldots + 2\cos(p-1)\theta & \text{if } p \text{ is odd} \\ 2\cos\theta + 2\cos 3\theta + \ldots + 2\cos(p-1)\theta & \text{if } p \text{ is even.} \end{cases}$$

LEMMA 13. If $\ell - 1$ $(= 2)$, $m - 1$ are relatively prime, then a point w lies on the curve

$$\Gamma_b : w_1(\varphi) = e^{-2si\varphi} + be^{i(m-3)s\varphi}, \quad 0 \le \varphi \le 2\pi$$

if and only if it lies on the curve

$$\Gamma_{-b} : w_2(\varphi) = e^{-2si\varphi} - be^{i(m-3)s\varphi}, \quad 0 \le \varphi \le 2\pi .$$

P r o o f. Since 2, $m - 1$ do not have common divisors, $m - 1$ and so $m - 3$ must be odd. Hence

$$w_1(\varphi + \frac{\pi}{s}) = \exp\{-2si(\varphi + \frac{\pi}{s})\} + b\exp\{i(m-3)s(\varphi + \frac{\pi}{s})\} =$$
$$= e^{-2si\varphi} + be^{i(m-3)s\varphi}\, e^{i(m-3)\pi} =$$
$$= e^{-2si\varphi} - be^{i(m-3)s\varphi} = w_2(\varphi) .$$

The case $\ell - 1 = 2$ of Theorem 2. We already know that

(48) $$\frac{1}{p}\,\overline{G_0} \subseteq \frac{\sin\theta}{\sin p\theta}\,\overline{G_\theta} \qquad \text{for} \quad \theta \in (0, \frac{\pi}{q}],$$

where we may refer to Theorem A for the case $\theta = \frac{\pi}{q}$. Next we wish to prove that

(49) $$\frac{1}{p}\,\overline{G_0} \subsetneq \frac{\sin\theta}{\sin p\theta}\,\overline{G_\theta} \qquad \text{for} \quad \theta \in (\tfrac{\pi}{q}, \tfrac{\pi}{p}].$$

Let us recall that G_θ is the region containing the origin and determined by the curve Γ_b where $b := t\,\frac{\sin q\theta}{\sin\theta}$. As θ increases from 0 to π/q, b decreases monotonically (and continuously) from tq to 0. Hence if we take a θ arbitrary in $(\frac{\pi}{q}, \frac{\pi}{p}]$, then in view of Lemma 13 there exists a $\theta^* \in (0, \frac{\pi}{q}]$ such that $G_\theta = G_{\theta^*}$. Thus (49) is equivalent to

(50) $$\frac{1}{p}\,\overline{G_0} \subsetneq \frac{\sin\theta}{\sin p\theta}\,\overline{G_{\theta^*}}\,.$$

But by (48) we have

$$\frac{1}{p}\,\overline{G_0} \subsetneq \frac{\sin\theta^*}{\sin p\theta^*}\,\overline{G_{\theta^*}}$$

which implies (50) since the regions G_θ are starlike and

$$\frac{\sin\theta^*}{\sin p\theta^*} \le \frac{\sin\theta}{\sin p\theta}$$

by Lemma 12.

Finally, we shall prove that

(51) $$\frac{1}{p}\,\overline{G_0} \subsetneq \frac{\sin\theta}{\sin p\theta}\,\overline{G_\theta} \qquad \text{for} \quad \theta \in (\tfrac{\pi}{p}, \tfrac{\pi}{2}].$$

For this it is enough to verify the inequality

(52) $$\frac{1}{p}\,(1 + tq) \le \frac{\sin\theta}{|\sin p\theta|}\,\Bigl(1 - t\,\frac{|\sin q\theta|}{\sin\theta}\Bigr)\,.$$

But (52) would certainly hold if

(53) $$1 + tp + tq \leq p \sin \frac{\pi}{p}$$

were true. As it is easily checked, (53) is indeed true for $p \geq 5$ and therefore so does (52). That (52) holds also in the only remaining case $p = 3$ is seen by noting that

$$\frac{\sin \theta}{|\sin 3\theta|} = \frac{1}{4 \sin^2 \theta - 3} \geq 1,$$

$$\frac{|\sin q\theta|}{\sin \theta} \leq \frac{1}{\sin \theta} \leq \frac{2}{\sqrt{3}}$$

and $t \leq \frac{1}{q} \leq \frac{1}{6}$.

As an immediate consequence of Theorem 2, we have

COROLLARY 2. If $q > 2p - 1$, then provided $q - 1$ is not an integral multiple of $p - 1$, the trinomial

$$z + a_p z^p + a_q z^q$$

is univalent in $|z| < 1$ if and only if its derivative does not vanish there.

P r o o f of Theorem 3. Since the result is already known to be true for $q = 3$, 4 and 5 we shall assume $q \geq 6$.

It is easily checked that

$$w(\varphi) = e^{-i\varphi} + t \frac{\sin q\theta}{\sin\theta} e^{i(q-2)\varphi}, \qquad 0 \leq \varphi \leq 2\pi$$

defines a Jordan curve for $0 < t \leq \frac{1}{q(q-2)}$. According to Lemma 8 it is also starlike. We wish to show that as θ decreases from π/q to 0 the region $\frac{1}{2 \cos\theta} G_\theta$ shrinks monotonically to the region $\frac{1}{2} G_0$. In view of Lemma 5 it is enough to show that the subregion

$$\Delta_\theta := \left\{ w : -\frac{2}{q-1} < \operatorname{Arg} w < 0 \right\} \cap \frac{1}{2\cos\theta} G_\theta$$

shrinks monotonically as θ decreases from π/q to 0. For this we apply Lemma 11 to the function

$$F(z,x) = F(z, \cos\theta) := \frac{(\sin\theta)z^{-1} + t(\sin q\theta)z^{q-2}}{\sin 2\theta}$$

and take for γ_x the arc $z = e^{i\varphi}$, $0 \le \varphi \le \frac{2\pi}{q-1}$. Computing $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial z}$ we see that if

$$A = \sin 2\theta \cos\theta - 2\sin\theta\cos 2\theta ,$$

$$B = 2\sin q\theta \cos 2\theta - q\sin 2\theta \cos q\theta ,$$

then (17) is equivalent to

$$(54) \qquad -A - Bt^2(q-2)\frac{\sin q\theta}{\sin\theta} + \left\{ B + A(q-2)\frac{\sin q\theta}{\sin\theta} \right\} t\cos(q-1)\varphi < 0$$

$$\text{for } 0 \le \varphi \le \frac{2\pi}{q-1}$$

It is easily checked that both A and B are positive for $0 < \theta \le \pi/q$. So (54) will certainly hold if

$$-A - Bt^2(q-2)\frac{\sin c\theta}{\sin\theta} + \left\{ B + A(q-2)\frac{\sin q\theta}{\sin\theta} \right\} t < 0,$$

i.e.

$$(A - Bt)\left\{ -1 + t(q-2)\frac{\sin q\theta}{\sin\theta} \right\} < 0 .$$

Since $0 < t \le \frac{3}{q(q^2-4)}$, the second factor is negative and so it is sufficient to show that $A - Bt$ is positive, i.e.

$$(55)\quad \sin 2\theta\cos\theta - 2\sin\theta\cos 2\theta - \\ - \frac{3}{q(q^2-4)}(2\sin q\theta\cos 2\theta - q\sin 2\theta\cos q\theta) > 0$$

The expression on the left-hand side of (55) vanishes for $\theta = 0$ and its derivative which is equal to $\frac{3}{q}(\sin 2\theta)\cdot \cdot(q\sin\theta - \sin q\theta)$ is positive for $0 < \theta \leq \pi/q$. Hence (55) holds for $\theta \in (0, \pi/q]$ and in turn so does (54). Thus we have proved that

$$\bigcap_{0\leq\theta\leq\pi/q} \frac{1}{2\cos\theta}\overline{G_\theta} = \frac{1}{2}\overline{G_0}$$

Now we shall show that if $0 < t \leq \frac{3}{q(q^2-4)}$, then for $\frac{\pi}{q} \leq \theta \leq \frac{\pi}{2}$,

$$\frac{1}{2}\overline{G_0} \subseteq \frac{1}{2\cos\theta}\overline{G_\theta},$$

so that for such values of t

$$\bigcap_{0\leq\theta\leq\pi/2} \frac{1}{2\cos\theta}\overline{G_\theta} = \frac{1}{2}\overline{G_0}.$$

Since

$$\frac{1}{2}\overline{G_0} \subseteq \left\{w : |w| \leq \frac{1}{2}\left(1 + \frac{3}{q^2-4}\right)\right\}$$

and

$$\left\{w : |w| \leq \frac{1}{2\cos\theta}\left(1 - \frac{3}{q(q^2-4)}\frac{|\sin q\theta|}{\sin\theta}\right)\right\} \subseteq \frac{1}{2\cos\theta}\overline{G_\theta}$$

we will simply check that

$$1 + \frac{3}{q^2-4} \leq \frac{1}{\cos\theta}\left(1 - \frac{3}{q(q^2-4)}\frac{|\sin q\theta|}{\sin\theta}\right) \quad \text{for } \frac{\pi}{q} \leq \theta < \frac{\pi}{2}$$

For values of θ under consideration

$$\frac{1}{\cos\theta} \geqslant \frac{1}{\cos\frac{\pi}{q}}, \qquad \frac{|\sin q\theta|}{\sin\theta} \leq \frac{1}{\sin\frac{\pi}{q}} .$$

Hence it is enough to verify that

$$(56) \qquad 1 + \frac{3}{q^2 - 4} \leqslant \frac{1}{\cos\frac{\pi}{q}} \left(1 - \frac{3}{q^2 - 4}\, \frac{1}{q \sin\frac{\pi}{q}}\right) .$$

Since $q \sin\frac{\pi}{q} \geqslant 3$ for $q \geqslant 6$ the expression on the right-hand side of (56) is $\geqslant \frac{1}{\cos\frac{\pi}{q}} \frac{q^2 - 5}{q^2 - 4}$, and so (56) would certainly hold if

$$\cos\frac{\pi}{q} \leqslant \frac{q^2 - 5}{q^2 - 4}$$

were true. Since this latter inequality is indeed true Theorem 3 is completely proved.

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STRESZCZENIE

W niniejszej pracy zajmujemy się określeniem warunków koniecznych i dostatecznych na to by wielomian $f_t(z) = z - a_p z^p + tz^q$ był jednolistny w kole $|z|<1$. Podajemy też warunki na to by wielomian $f_t(z)$ lokalnie jednolistny był również globalnie jednolistny w kole $|z|<1$.

Резюме

В данной работе определены необходимые и достаточные условия для того, чтобы полином $f_t(z)=z-a_p z^p+tz^q$ был однолистный в круге $|z|<1$. Они дают также условия к тому, чтобы локально однолистный полином $f_t(z)$ являлся также глобально однолистным в круге $|z|<1$.