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# Some Integral Inequalities for Entire Functions of Exponential Type

Pewne nierówności całkowe dla funkcji całkowitych typu wykładniczego

Некоторые интегральные неравенства для целых функций экспоненциального типа

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $f(z)$ be an entire function of exponential type $\tau$. The following integral inequalities (for references, see [1, pp. 211, 98]) are well known.

THEOREM A. If $f(z)$ is an entire function of exponential type $\tau$ belonging to $L^p$ $(1 \le p < \infty)$ on the real axis, then

$$(1.1) \qquad \int_{-\infty}^{\infty} |f'(x)|^p \, dx \le \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx$$

and

$$(1.2) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^p dx \le e^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p dx, \quad -\infty < y < \infty .$$

If $h_f(\pi/2) = 0$ ($h_f(\theta) = \limsup\limits_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}$ is the indicator function of $f(z)$) and $f(z) \neq 0$ for $\operatorname{Im} z > 0$, then the inequality analogous to (1.1) has been obtained by Rahman [5]. No inequality analogous to (1.2) is known, but if $p = 2$, it has been proved by Rahman [6] that for $y<0$,

$$(1.3) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq \frac{e^{2\tau|y|}+1}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

An inequality analogous to (1.1) for functions of exponential type not vanishing in $\operatorname{Im} z > k$ $(k \leq 0)$ has been obtained by Govil and Rahman [2]. In this paper we consider the class of entire functions of exponential type $\tau$ satisfying $f(z) \equiv \omega(z)$, where $\omega(z) = e^{i\tau z}\overline{\{f(\bar{z})\}}$ and prove the following

THEOREM 1. *Let* $f(z)$ *be an entire function of exponential type* $\tau$ *belonging to* $L^p$ $(1 \leq p < \infty)$ *on the real axis. If* $f(z) \equiv \omega(z)$, *then we have*

$$(1.4) \qquad \int_{-\infty}^{\infty} |f'(x)|^p dx \leq C_p \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx$$

*and*

$$(1.5) \qquad \int_{-\infty}^{\infty} |f'(x)|^p dx \geq (1 - C_p^{\frac{1}{p}})^p \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx ,$$

*where*

$$C_p = \frac{2\pi}{\int_0^{2\pi} |1 + e^{i\alpha}|^p \, d\alpha}$$

THEOREM 2. If $f(z)$ is an entire function of exponential type $\tau$ belonging to $L^2$ on the real axis and satisfying $f(z) \equiv \omega(z)$, then

$$(1.6) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leqslant \frac{e^{-2\tau y}+1}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx, \quad -\infty<y<\infty$$

THEOREM 3. Let $f(z)$ be an entire function of exponential type $\tau$ $(\geqslant 1)$ and periodic on the real axis with period $2\pi$. If $f(z) = \omega(z)$, then

$$(1.7) \qquad \int_{-\pi}^{\pi} |f'(x)|^2 dx \leqslant \frac{\tau^2}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

and

$$(1.8) \qquad \int_{-\pi}^{\pi} |f(x+iy)|^2 dx \leqslant \frac{e^{-2\tau y}+1}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx, \quad -\infty<y<\infty$$

We also prove

THEOREM 4. Let $f(z)$ be an entire function of exponential type $\tau$ $(\geqslant 1)$ and periodic on the real axis with period $2\pi$. If $f(z) = e^{i\tau z} f(-z)$, then

$$(1.9) \qquad \int_{-\pi}^{\pi} |f'(x)|^2 dx \leqslant \frac{\tau^2}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx .$$

and

$$(1.10) \qquad \int_{-\pi}^{\pi} |f(x+iy)|^2 dx \leqslant \frac{e^{-2\tau y}+1}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx, \quad -\infty<y<\infty$$

## 2. LEMMAS

LEMMA 1. *If* $f(z)$ *is regular and of exponential type in the upper half plane,* $h_f(\pi/2) \leqslant c$ *and* $|f(x)| \leqslant M$, $-\infty < x < \infty$ *then*

(2.1) $$|f(x+iy)| \leqslant Me^{cy}, \quad -\infty < x < \infty, \quad 0 \leqslant y < \infty$$

This Lemma is due to Pólya and Szegö [4, p. 36], Boas [1, p. 82].

LEMMA 2. *If* $f(z)$ *is an entire function of exponential type* $\tau$ *belonging to* $L^p$ $(1 \leqslant p < \infty)$ *on the real axis, then*

(2.2) $$\int_{-\infty}^{\infty} \left| i\tau f(x) + f'(x) + e^{i\alpha}\{-i\tau f(x) + f'(x)\} \right|^p dx \leqslant (2\tau)^p \int_{-\infty}^{\infty} |f(x)|^p dx, \qquad \alpha \in [0, 2\pi).$$

This Lemma is due to Rahman [5, inequality (3.18), p. 300]

## 3. PROOFS OF THEOREMS

P r o o f of Theorem 1. Since $f(z)$ is an entire function of exponential type $\tau$, belonging to $L^p$ $(1 \leqslant p < \infty)$ on real axis, there exists a constant $M$ {Boas [1, Th. 6.7.1]} such that $|f(x)| \leqslant M$, $-\infty < x < \infty$. Further since $f(z) = \omega(z)$ we have $h_f(\pi/2) \leqslant 0$. Now if $f_1(z)$ denotes the function $e^{-i\tau z/2} f(z)$, then $f_1(z)$ is of exponential type $\tau/2$ and belongs to $L^p (1 \leqslant p < \infty)$. Hence applying Lemma 2 to $f_1(z)$, we get

$$\int_{-\infty}^{\infty}\Big|i\,\frac{\tau}{2}\,f_1(x)+f_1'(x)+e^{i\alpha}\Big\{-i\,\frac{\tau}{2}\,f_1(x)+f_1'(x)\Big\}\Big|^p dx \leq \tau^p\int_{-\infty}^{\infty}|f_1(x)|^p dx, \quad (p\geq 1)$$

which gives

$$\int_{-\infty}^{\infty}\Big|f'(x)e^{-i\tau x/2}+e^{i\alpha}\Big\{-i\tau e^{-i\tau x/2}f(x)+ e^{-i\tau x/2}f'(x)\Big\}\Big|^p dx \leq \tau^p\int_{-\infty}^{\infty}|f(x)|^p dx \qquad (p\geq 1).$$

Consequently

$$\int_{-\infty}^{\infty}\Big|f'(x)+e^{i\alpha}\{-i\tau f(x)+f'(x)\}\Big|^p dx \leq \tau^p\int_{-\infty}^{\infty}|f(x)|^p dx, \quad (p\geq 1).$$

Integrating both sides with respect to $\alpha$ from $0$ to $2\pi$, we get

$$\int_0^{2\pi}d\alpha\int_{-\infty}^{\infty}\Big|f'(x)+e^{i\alpha}\{-i\tau f(x)+f'(x)\}\Big|^p dx \leq 2\pi\,\tau^p\int_{-\infty}^{\infty}|f(x)|^p dx \qquad (p\geq 1). \tag{3.1}$$

Note that $f'(x)$ can be zero only at a countable number of points. Besides, we can clearly invert the order of integration on the left side of (3.1). Therefore

$$\int_0^{2\pi}d\alpha\int_{-\infty}^{\infty}\Big|f'(x)+e^{i\alpha}\{-i\tau f(x)+f'(x)\}\Big|^p dx =$$

$$= \int_0^{2\pi}d\alpha\int_{-\infty}^{\infty}|f'(x)|^p\Big|1+e^{i\alpha}\,\frac{-i\tau f(x)+f'(x)}{f'(x)}\Big|^p dx = \tag{3.2}$$

$$= \int_{-\infty}^{\infty} |f'(x)|^p dx \int_0^{2\pi} \left|1 + e^{i\alpha} \frac{-i\tau f(x) + f'(x)}{f'(x)}\right|^p d\alpha =$$

$$= \int_{-\infty}^{\infty} |f'(x)|^p dx \int_0^{2\pi} \left|1 + e^{i\alpha} \frac{B(x)}{A(x)}\right|^p d\alpha,$$

where $B(x) = -i\tau f(x) + f'(x)$ and $A(x) = f'(x)$.

Further since $f(z) \equiv \omega(z)$, we have for real $x$

$$\begin{aligned} |A(x)| &= |f'(x)| \\ &= |\omega'(x)| \\ &= |-i\tau f(x) + f'(x)| \\ &= |B(x)|, \end{aligned}$$

i.e. $\left|\frac{B(x)}{A(x)}\right| = 1$. Thus for a fixed real $x$ and every $p > 0$

$$(3.3) \qquad \int_0^{2\pi} \left|1 + e^{i\alpha} \frac{B(x)}{A(x)}\right|^p d\alpha = \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^p d\alpha, \quad (p>0)$$

Combining inequality (3.1) and equalities (3.2), (3.3), we get

$$\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^p d\alpha \int_{-\infty}^{\infty} |f'(x)|^p dx \leq 2\pi\, \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx, \quad (p \geq 1)$$

which is (1.4).

To prove (1.5), note that $\omega(z)$ is an entire function of exponential type $\tau$ satisfying $\omega(z) = e^{i\tau z}\{\overline{\omega(\bar{z})}\}$. Hence using (1.4), we get

$$\int_{-\infty}^{\infty} |\omega'(x)|^p dx \leq C_p \tau^p \int_{-\infty}^{\infty} |\omega(x)|^p dx,$$

which is equivalent to

$$(3.4) \qquad \left(\int_{-\infty}^{\infty}|-i\tau f(x)+f'(x)|^p dx\right)^{\frac{1}{p}} \leqslant \tau C_p^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}}, \quad (p\geqslant 1).$$

Therefore by Minkowski's inequality

$$\left(\int_{-\infty}^{\infty}|\{-i\tau f(x)+f'(x)\}+\{-f'(x)\}|^p dx\right)^{\frac{1}{p}}$$
$$\leqslant \left(\int_{-\infty}^{\infty}|\{-i\tau f(x)+f'(x)\}|^p dx\right)^{\frac{1}{p}} + \left(\int_{-\infty}^{\infty}|f'(x)|^p dx\right)^{\frac{1}{p}},$$

which gives

$$(3.5) \qquad \tau\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}} \leqslant \left(\int_{-\infty}^{\infty}|-i\tau f(x)+f'(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{-\infty}^{\infty}|f'(x)|^p dx\right)^{\frac{1}{p}}.$$

On combining (3.4) and (3.5), we get

$$\tau(1-C_p^{\frac{1}{p}})\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}} \leqslant \left(\int_{-\infty}^{\infty}|f'(x)|^p dx\right)^{\frac{1}{p}}.$$

from which (1.5) follows.

P r o o f of Theorem 2. Since $f(z)\in L^2$ on the real axis, we have by Paley-Wiener Theorem [3, pp. 499-501]

$$(3.6) \qquad f(z) = \int_0^{\tau} e^{izt}\varphi(t)dt, \qquad \varphi\in L^2(0,\tau).$$

Now

$$(3.7) \qquad \omega(z) = e^{i\tau z}\int_0^{\tau} e^{-izt}\,\overline{\varphi(t)}\,dt$$

$$= \int_0^{\tau} e^{iz(\tau - t)} \overline{\varphi(t)}\, dt$$

Since $f(z) \equiv \omega(z)$, hence

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx = \frac{1}{2} \int_{-\infty}^{\infty} |\omega(x+iy)|^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} |f(x+iy)|^2 dx =$$

$$= \pi \int_0^{\tau} e^{-2y(\tau - t)} |\varphi(t)|^2 dt + \pi \int_0^{\tau} e^{-2yt} |\varphi(t)|^2 dt$$

$$\leq \pi (e^{-2\tau y} + 1) \int_0^{\tau} |\varphi(t)|^2 dt$$

$$= \frac{(e^{-2\tau y} + 1)}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx,$$

which is (1.6).

P r o o f of Theorem 3. Since $f(z)$ is an entire function of exponential type $\tau$ and is periodic on the real axis with period $2\pi$, we have (see Boas [1, p. 109])

(3.8) $$f(z) = \sum_{k=-n}^{n} a_k e^{ikz}, \qquad n \leq \tau$$

and since $f(z) \equiv \omega(z)$, we have $h_f(\pi/2) \leq 0$. Hence we get from (3.8)

(3.9) $$f(z) = \sum_{k=0}^{n} a_k e^{ikz}, \qquad n \leq \tau.$$

Further

(3.10) $$\omega(z) = e^{i\tau z} \sum_{k=0}^{n} \overline{a}_k e^{-ikz} = \sum_{k=0}^{n} \overline{a}_k e^{i(\tau - k)z}$$

Therefore

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = \frac{1}{2} \int_{-\pi}^{\pi} |f'(x)|^2 dx + \frac{1}{2} \int_{-\pi}^{\pi} |\omega'(x)|^2 dx =$$

$$= \pi \sum_{k=0}^{n} k^2 |a_k|^2 + \pi \sum_{k=0}^{n} (\tau - k)^2 |a_k|^2,$$

(by (3.9) and (3.10))

$$\leqslant \pi \tau^2 \sum_{k=0}^{n} |a_k|^2 = \frac{\tau^2}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx,$$

which is (1.7).

To prove (1.8), we have

$$\int_{-\pi}^{\pi} |f(x+iy)|^2 dx = \frac{1}{2} \int_{-\pi}^{\pi} |f(x+iy)|^2 dx + \frac{1}{2} \int_{-\pi}^{\pi} |\omega(x+iy)|^2 dx =$$

$$= \pi \sum_{k=0}^{n} e^{-2ky} |a_k|^2 + \pi \sum_{k=0}^{n} e^{-2(\tau-k)y} |a_k|^2$$

$$\leqslant \pi (1 + e^{-2\tau y}) \sum_{k=0}^{n} |a_k|^2 =$$

$$= \frac{1 + e^{-2\tau y}}{2} \int_{-\pi}^{\pi} |f(x)|^2 dx,$$

which is (1.8).

P r o o f of Theorem 4. Here $f(z) = e^{i\tau z} f(-z)$. Hence we get $h_f(\pi/2) \leqslant 0$. And so here also, the representation (3.8) of $f(z)$ will reduce to the representation (3.9). Then the proof follows on the lines similar to that of Theorem 3.

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## STRESZCZENIE


W pracy udowodniono nierówność (1)

$$\int_{-\infty}^{+\infty} |f'(x)|^p dx \leq c_p \tau^p \int_{-\infty}^{+\infty} |f(x)|^p dx \tag{1}$$

dla funkcji całkowitej $f \in L^p$, $p \geq 1$ typu wykładniczego $\tau$ oraz przy warunku, że $f(z) = e^{i\tau z}\overline{f(\bar{z})}$, nierówność (2)

$$\int_{-\infty}^{+\infty} |f'(x)|^2 dx \leq \frac{e^{-2\tau y} + 1}{2} \int_{-\infty}^{+\infty} |f(x)|^2 dx, \quad y < 0. \tag{2}$$

Ponadto otrzymano nierówność przeciwną do (1) z zamianą $c_p$ na $(1 - c_p^{\frac{1}{p}})^p$ oraz kilka innych analogicznych nierówności do (1) i (2).

## Резюме

В работе доказано неравенство (1) $\int_{-\infty}^{+\infty}|f'(x)|^p dx \leqslant c_p \tau^p \int_{-\infty}^{+\infty}|f(x)|^p dx$ для целой функции $f \in L^p$, $p \geqslant 1$ экспоненциального типа $\tau$, а также, при условии $f(z)=e^{i\tau z}\overline{f(\bar z)}$, неравенство (2)

(2) $$\int_{-\infty}^{+\infty}|f'(x)|^2 dx \leqslant \frac{e^{-2\tau y}+1}{2}\int_{-\infty}^{+\infty}|f(x)|^2 dx, \quad y<0.$$

Кроме того получено неравенство противоположное к (1) с заменой $c_p$ на $(1-c_p^{\frac{1}{p}}\,p)$, а также несколько других аналогических неравенств к (1) и (2).