ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXIV, 2 SECTIO A 1980

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A Variational Method for Grunsky Functions

Metoda wariacyjna dla funkcji Grunsky'ego

Вариационная формула для функций Грунского

1. Introductory remarks. Let $\widetilde{G}$ denote the class of functions of the form $f(z) = \sum_{n \geq 1} a_n z^n$ analytic in the unit disk $D = \{z : |z| < 1\}$ and satisfying the condition

$$f(z_1)\overline{f(z_2)} \neq -1 \qquad (1.1)$$

for z_1, z_2 in D.

This class of functions, analogous to the well-known class of Bieberbach-Eilenberg, was first considered by H. Grunsky [3] but the credit is often given to T. S. Shah [6].

It is well-known that any function of $\widetilde{G}$ is subordinate to a univalent function of the same class. Hence, in many extremal problems it is sufficient to consider such problems within the subclass G which consistsof all univalent Grunsky's functions.

In what follows we will be concerned with functions of the class G.

Recently J. A. Hummel and M. M. Schiffer [4] developed a variational technique for the class of Bieberbach-Eilenberg and solved some extremal problems within that class. Our aim here is to establish variational formulas for the class G and to give some applications. Our technique is slightly different from that of J. Hummel and M. Schiffer.

2. Variational formulas within G. Let E be a simply connected region in the complex plane. We say that E has the Grunsky's property (or is a Grunsky's region) if the following is true

$$w \in E \Rightarrow -\overline{w}^{-1} \notin E.$$

We begin with the following

Lemma. *Suppose that* (i) *E is a Grunsky's domain that contains the origin,* (ii) *Q is a domain which is symmetric w.r.t. the mapping* $w \to -\overline{w}^{-1}$, $0, \infty \notin \overline{Q}$ *and* $\partial E \subset Q$. *Let* ϕ *be an analytic function in Q subject to the condition*

$$\phi(w) = -\overline{\phi(-\overline{w}^{-1})} \tag{2. 1}$$

for w in Q. Then for any sufficiently small $|\epsilon|$ *the function*

$$w^*(w) = w \exp(\epsilon\,\phi(w)) \tag{2. 2}$$

is analytic and univalent in E and it maps the boundary of E onto the boundary of Grunsky's region.

Proof. The proof of our lemma is similar to the proof of a lemma of Hummel and Schiffer but we want to give it here in order to make the article selfsufficient.

Define the function

$$\psi(w, u) = \begin{cases} \dfrac{\phi(w) - \phi(u)}{w - u}, & w \neq u \\ \phi'(w) & \text{otherwise} \end{cases}$$

$\psi(w, u)$ is analytic and bounded on the compact set $\overline{Q} \times \overline{Q}$.

Suppose that there exist two points w_1, w_2 in Q such that $w_1 \neq w_2$ and $w^*(w_1) = w^*(w_2)$. Then, in view of (2. 2) one gets

$$w_1 - w_2 = w_1\,[\,1 - \exp\epsilon\,(w_1 - w_2)\,\psi\,(w_1, w_2)]\,.$$

Making use of the inequality

$$|\,1 - e^s\,| \leqslant |\,s\,|\,e^{|\,s\,|}$$

we obtain

$$|\,w_1 - w_2\,| \leqslant |\,\epsilon\,|\,(w_1 - w_2)\,w_1\,|\,|\,\psi(w_1, w_2)\,|\exp|\,\epsilon(w_1 - w_2)\,\psi\,(w_1, w_2)\,|.$$

This inequality can not hold for sufficiently small $|\,\epsilon\,|$. It proves the univalence of $w^*(w)$.

The function (2. 2) being univalent in Q, maps ∂E onto a boundary of a simply connected domain, say E^*. We want to show that if $E^0 = \{w : -\overline{w}^{-1} \in E^*\}$, then $E^* \cap E^0 = \emptyset$. To this end we assume that there exist points $w_1, w_2 \in E \cap Q$ such that

$$w^*(w_1)\,\overline{w^*(w_2)} = -1\,.$$

Then in view of (2. 2) we have

$$w_1 + \overline{w}_2^{-1} = w_1 [1 - \exp(\overline{\epsilon}\psi(w_1, w_2^{-1})(w_1 + w_2^{-1}))]$$

which is impossible for sufficiently small $|\epsilon|$. The lemma has been proved.

Corollary. *If $w_0 \in E$ and $0 < \epsilon$ is small enough, then the function $w^*(w)$ of the form*

$$w^*(w) = w + \epsilon e^{i\alpha} \frac{w}{w - w_0} + \epsilon e^{-i\alpha} \frac{w^2}{1 + \overline{w}_0 w} + o(\epsilon)$$

is analytic and univalent in certain neighbourhood of ∂E.

Proof. It is sufficient to notice that

$$\phi(w) = e^{i\alpha} \frac{1}{w - w_0} + e^{-i\alpha} \frac{w}{1 + \overline{w}_0 w} \tag{2.3}$$

satisfies (2.1).

We shall need the following result of G. M. Golusin [1] and G. G. Shlionsky [7].

Theorem. *Let $f(z)$, $f(0) = 0$, be a function regular and univalent in D and let $F(z, \epsilon)$ be a function analytic and univalent in the annulus $A = \{z : r \leq |z| < 1\}$ for all ϵ, $0 < \epsilon < \epsilon_0$, besides $F(z, \epsilon)$ suppose to be analytic for $z \in A$ and all fixed ϵ, $|\epsilon| < \epsilon_0$ and to have the form*

$$F(z, \epsilon) = f(z) + \epsilon g(z) + o(\epsilon).$$

Let D_ϵ^* be a simply connected domain which arises by adjoing to domain $F(A, \epsilon)$ the interior of the map of $|z| = r$ under $F(z, \epsilon)$. For all $\epsilon > 0$ small enough D_ϵ^* contains the origin and the function $f^*(z)$, $f^*(0) = 0$ mapping D onto the domain D_ϵ^* is of the form

$$f^*(z) = f(z) + \epsilon g(z) - \epsilon z f'(z) [S(z) + c + \overline{S(\overline{z}^{-1})} + \overline{c}] + o(\epsilon)$$

where c is an arbitrary constant, $S(z)$ denotes the sum of terms with negative powers of z in the Laurent's development of $g(z)/zf'(z)$ in the annulus $r < |z| < 1$.

We are now ready to prove.

Theorem 1. *Suppose $f \in G$, α is an arbitrary real number, z_0 is a fixed point of D. Then for any sufficiently small $\epsilon > 0$ there exists a function f^* of the form*

$$\begin{aligned} f^*(z) = f(z) &+ \epsilon e^{i\alpha} \frac{f(z)}{f(z) - f(z_0)} + e^{-i\alpha}\epsilon \frac{f^2(z)}{1 + \overline{f(z_0)} f(z)} - \\ &- \frac{1}{2}\epsilon e^{i\alpha} z f'(z) \frac{f(z_0)}{(z_0 f'(z_0))^2} \frac{z + z_0}{z - z_0} - \\ &- \frac{1}{2}\epsilon e^{-i\alpha} z f(z) \frac{f(z_0)}{(z_0 f'(z_0))^2} \frac{\overline{z}_0 z + 1}{\overline{z}_0 z - 1} + O(\epsilon^2) \end{aligned} \tag{2.4}$$

which belongs to the class G.

Proof. We put

$$F(z,\epsilon) = f(z)[1 + \epsilon\phi(f(z))]$$

where ϕ is given by (2. 3).

In view of the Lemma it is easy to see that $F(z, \epsilon)$ fulfils conditions of the Golusin-Shlionsky theorem. Simple computations yield

$$S(z) = e^{i\alpha}\frac{f(z_0)}{(z_0 f'(z_0))^2}\,\frac{z_0}{z - z_0}$$

and (2. 4) follows with

$$c = \frac{e^{i\alpha}}{2}\,\frac{f(z_0)}{(z_0 f'(z_0))^2}.$$

Theorem 2. *Let $f \in G$ and suppose that $w_0, -\overline{w}_0^{-1}$ do not belong to the set $f(D)$. Then for any sufficently small $\epsilon > 0$, for any real α there exist functions of the form*

$$f^*(z) = f(z) + \epsilon e^{i\alpha}\frac{f(z)}{f(z) - w_0} + \epsilon e^{-i\alpha}\frac{f^2(z)}{1 + \overline{w}_0 f(z)} + o(\epsilon) \qquad (2.\ 5)$$

which belong to G.

If z_0 is an arbitrary fixed point such that $|z_0| = 1$ and $0 < t < t_0$, $t_0 > 0$, then there exist functions of the form

$$f^0(z) = f(z) - tzf'(z)\frac{z_0 + z}{z_0 - z} + o(t) \qquad (2.\ 6)$$

which belong to G.

Proof. The proof of (2. 5) is similar to that of the formula (2. 4). One has only to notice that in this case $g(z) = f(z)\,\phi(z)$, $\phi(z)$ being given by (2. 3), is a regular function in A.

To prove (2. 6) we observe, that if $\omega(z, t)$ is a univalent function subject to Schwarz lemma conditions, then

$$g(z, t) = f(\omega(z, t)) \in G.$$

If $k(z) \equiv z(1 + \overline{z}_0 z)^{-2}$, *then* $\omega(z,t) = k^{-1}((1 - t)k(z))$, $0 < t < 1$, maps D in one-to-one manner onto D cut along some segment terminating at z_0.

The formula (2. 6) now follows by straightforward computations.

3. Applications. The class G is not compact, but if we adjoin function $f(z) \equiv 0$ then the new class, which is again denoted by G, is compact.

We want to give some applications of the formulas that we have just obtained. We proceed to solutions some extremal problems within the class G.

Theorem 3. *Let z be a fixed point of D and let f run over the whole class G. Then the disk*

$$|w| \leqslant |z|(1 - |z|^2)^{-1/2}$$

is the set of all possible values taken on at z by f, $f \in G$.

Proof. Let us consider the following extremal problem. For a fixed z, $z \in D$, $z \neq 0$ find $\sup\limits_{g \in G} |g(z)| = b(z)$. Since G is a compact family, there exists a function $f \in G$, $f \not\equiv 0$, for which $|f(z)| = b(z)$. Let us call such a function extremal. Without any loss of generality one may assume

$$z = r > 0, \quad f(r) = b(z) > 0.$$

1. Suppose that the points w_0, $-\overline{w}_0^{-1}$ do not belong to $\overline{f(D)}$, f being an extremal function, and construct a function $f^{\blacktriangle}$ according to (2. 5). We have

$$|f^{\blacktriangle}(r)|^2 = |f(r)|^2 + 2\mathrm{Re}\left\{\epsilon e^{i\alpha}\left[\frac{f(r)}{f(r) - w_0} + \frac{f^2(r)}{1 + w_0 f(r)}\right] f(r)\right\} + o(\epsilon)$$

and the choice of f yields

$$\frac{1}{b(r) - w_0} + \frac{b(r)}{1 + w_0 b(r)} \equiv 0$$

which leads to a contradiction. It follows, that if neither w_0 nor $-\overline{w}_0^{-1}$ belong to $f(D)$ then at least one of those points lies on $\partial f(D)$.

2. Applying the variational formula (2, 4), by a reasoning similar to that above we arrive at the following condition ($f(z_0) \equiv f(z)$)

$$\frac{b(r)}{b(r) - f(z)} - \frac{1}{2} rf'(r) \frac{f(z)}{(zf'(z))^2} \frac{z + r}{r - z} + \frac{b^2(r)}{1 + b(r) f(z)} =$$

$$= \frac{1}{2} rf'(r) \frac{f(z)}{(zf'(z))^2} \frac{zr + 1}{zr - 1}.$$

Putting $b(r) = b$, $f(z) = w$, $d = rf'(r)(1 - r^2)$ we may bring it to the form

$$\frac{b + b^3}{w(b - w)(1 + bw)} (dw)^2 = -\frac{d}{z(z - r)(1 - rz)} (dz)^2 \tag{2. 7}$$

By making use of (2.6) we can easily convince ourselves that $d > 0$ and that the r.h.s. is non-negative on $|z| = 1$.

Equation (2.7) is valid for $|z| < 1$. But it is well-known ([2] 36–44) that it holds on $|z| = 1$ except possibly for a finite number of points. The quadratic differential

$$Q(w)\,dw^2 \equiv \frac{b + b^3}{w(w-b)(1 + bw)}\,dw^2$$

has four simple poles only. There are exactly two critical trajectories that terminate at ∞ and b or 0 and $-b^{-1}$, respectively. The other trajectories are closed Jordan curves that are symmetric w.r.t. the real axis and separate 0 and $-b^{-1}$ from b and ∞. Two different trajectories do not intersect each other. This is the case of a ring-domain [5, Th. 3. 5]. Since f is bounded and the curve $f(|z| = 1)$ is a trajectory of $Q(w)\,dw^2$ we conclude that is necessary one of the closed Jordan curves described above.

From the previous considerations it follows that this trajectory passes through the points $\pm i$. One can easily check that the circumference $|w - b| = (1 + b^2)^{1/2}$ is a trajectory of $Q(w)\,dw^2$ and it passes through the points $\pm i$. Hence, we conclude that the extremal function maps D onto the disk $|w - b| < (1 + b^2)^{1/2}$ and it is necessary of the form $f(z) = \alpha z(1 + \beta z)^{-1}$. Some easy computations show that

$$\alpha = \sqrt{1 - r^2}, \quad \beta = -r \quad \text{and} \quad f(r) = r(1 - r^2)^{-1/2}$$

This proves the theorem.

This result has been earlier obtained by J. Jenkins [5] by means of the extremal metric technique.

3. Variability region of (a_1, a_2) within G. Denote by V_2 the variability region of (a_1, a_2) where a_1, a_2 are initial coefficients of $f, f \in G$.

Let $a_k = x_k + iv_k$, $k = 1, 2$, and let $F = F(a_1, a_2)$ be a real-valued function defined on an open set Q containing V_2. We assume that F has continuous partial derivatives on Q and, moreover,

$$|\operatorname{grad} F| > 0 \text{ in } Q.$$

Under those assumptions the function $F_{|V_2}$ attains its maximal value on the set ∂V_2.

Let $f(z) = a_1 z + a_2 z^2 + \ldots \in G$ be a function for which F attains its maximum. Since for any real α, β the function $e^{i\alpha} f(ze^{i\beta})$ is in G, we may assume a_1, a_2 to be real. Suppose that $f^*(z) = a_1^* z + a_2^* z^2 + \ldots$ is given by (2. 4) and $F^* = F(a_1^*, a_2^*)$, then

$$\Delta F = F - F^* = 2\operatorname{Re}\left\{F_1\,\Delta a_1 + F_2\,\Delta a_2\right\} + o(\epsilon) \leqslant 0$$

which is equivalent to

$$0 \geqslant \Delta F = 2\mathrm{Re}\epsilon e^{i\alpha}\left\{-\frac{a_1}{f(z_0)}F_1 + \frac{1}{2}\frac{f(z_0)}{(z_0 f'(z_0))^2}(a_1F_1 + \overline{a_1F_1}) - \right.$$

$$-\left[\frac{a_2}{f(z_0)} + \frac{a_1^2}{(f(z_0))^2}\right]F_2 + a_1^2 F_2 + \qquad (2.8)$$

$$\left. + \frac{f(z_0)}{(z_0 f'(z_0))^2}\left[(a_2 + a_1 z_0)F_2 + \overline{F}_2\left(a_2 + \frac{a_1}{z_0}\right)\right]\right\} + o(\epsilon)$$

where $F_1 = \dfrac{\partial F}{\partial a_1}$, $F_2 = \dfrac{\partial F}{\partial a_2}$.

Since F is a real valued function defined on the set of pairs (a_1, a_2) of real numbers the constants F_1, F_2 are real. In view of arbitrariness of $e^{i\alpha}$ the condition (2. 8) leads to the equation

$$\left(\frac{zf'(z)}{f(z)}\right)^2\left[a_1F_1 + a_2F_2 + a_1^2F_2\left(\frac{1}{f(z)} - f(z)\right)\right] = \qquad (2.9)$$

$$= a_1F_1 + 2a_2F_2 + a_1F_2(z + z^{-1})$$

where we have put $f(z_0) \equiv f(z)$, $z_0 \equiv z$.

Take

$$P(z) = a_1F_1 + 2a_2F_2 + a_1F_2(z + z^{-1}).$$

It is easy to notice that $P(e^{i\theta})$ is real. We now want to prove more, namely

$$P(e^{i\theta}) \geqslant 0.$$

For we construct a function f^{**} according to (2. 6) and we obtain

$$\Delta F = 2\mathrm{Re}\left\{-\epsilon(a_1F_1 + 2a_2F_2 + 2a_1F_2e^{i\theta})\right\} + o(\epsilon) \leqslant 0$$

which is equivalent to

$$0 \leqslant \mathrm{Re}(a_1F_1 + 2a_2F_2 + 2a_1F_2e^{i\theta}) \equiv \mathrm{P}(e^{i\theta}).$$

The equation (2. 9) has a solution $f(z)$, $f(z)$ being a function with real coefficients. It results from the fact that all coefficients of this equation are real and from its form. Hence, if $W = w^{-1} - w$, then W is univalent in $f(D)$.

In fact, suppose that there exist points $z_1, z_2 \in D$ such that $f(z_1)\, f(z_2) = -1$. Then $f(z_1)\overline{f(\overline{z_2})} = -1$ which is contradictory to the definition of G.

Moreover, in a similar way as in the proof of Theorem 2 one can show that if the points $w_0, -\overline{w_0}^{-1}$ do not belong to $f(D)$, then at least one of them belongs to $\partial f(D)$. Hence, the mapping $W = w^{-1} - w$ is univalent in $f(D) = \Delta$ and it maps Δ onto slit-domain D' and such that $\pm 2i \notin D'$.

By making in (2. 9) the substitutions

$$2W = i(w - w^{-1})$$

$$2Z = z + z^{-1}$$

we end up with the equation

$$\frac{(a_1F_1 + 2a_2F_2 + 2a_1F_2Z)\,dZ^2}{1 - Z^2} = \frac{(a_1F_1 + a_2F_2 + 2a_1^2F_2iW)\,dW^2}{1 - W^2} \qquad (2.10)$$

The l.h.s. of (2. 10) takes on zero at, say $Z_0 = -\tau^{-1}$, $0 < \tau \leqslant 1$, while the r.h.s. of (2. 10) takes on zero at $W_0' = i/\mu$. We may assume that $\mu > 0$. Denote

$$\rho = \frac{a_1F_1 + 2a_2F_2}{a_1F_1 + a_2F_2} > 1 .$$

A differential equation of the type (2. 10) has been obtained by J. Hummel and M. Schiffer and it has been extensively discussed [4].

Since our equation may be treated in almost exactly the same manner, we restrict ourselves to the conclusions. We get following relations

$$a_2 = 2a_1\left(1 - \frac{a_1}{o}\right),\quad a_1 = \frac{\mu_0}{\rho},\quad \rho \geqslant \frac{q_0^2}{8}$$

$$a_2 = 2a_1\left(1 - \frac{a_1}{\mu_1}\right),\quad a_1 = \frac{\mu_1}{\rho},\quad \rho < \frac{q_0^2}{8},\quad \mu_1 = \frac{\mu}{\tau}$$

where μ, ρ and τ satisfy the conditions

$$q(\mu) = \sqrt{\rho}\, p(\tau),$$

$$q(\mu) = \int_{-1}^{1}\left(\frac{1 + i\mu W}{1 - W^2}\right)^{1/2} dW, \quad p(\tau) = \int_{-1}^{1}\left(\frac{1 + \tau Z}{1 - Z^2}\right)^{1/2} dZ$$

$$r(\mu) = \sqrt{\rho}\, s(\tau), \quad s(\tau) = \int_1^{\tau^{-1}} \left(\frac{1 - \tau t}{t^2 - 1}\right)^{1/2} dt,$$

$$r(\mu) = \int_0^1 \left(\frac{1 - t}{\mu^2 + t^2}\right)^{1/2} dt - \frac{1}{\sqrt{2}} \int_0^{\pi/2} [(1 + \mu^2 \sin^2\theta)^{1/2} - 1]^{1/2} d\theta,$$

μ_0 satisfies the equation $r(\mu_0) = 0$, $\mu_0 \approx 1{,}162205...$ and $q(\mu_0) = q_0 \approx 3{,}3519319...$ These conditions define the boundary of V_2 implicitely.

The method presented here may be successfully applied to other extremal problems within the class G.

Results presented in this paper were obtained within the research supported by Polish Academy of Sciences (MR. I. 1, 11/1/3) in 1979. We have learned from the referee that the variational formulas for the class G have been obtained independently by H. Jondro ([8]), however, without examples of their applications.

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STRESZCZENIE

Niech G oznacza klasę funkcji analitycznych i jednolistnych postaci $f(z) = a_1 z + a_2 z^2 + \dots$ w kole jednostkowym $D(|z| < 1)$ spełniających warunek: $f(z_1)\overline{f(z_2)} \neq -1$ dla $z_1, z_2 \in D$.

W pracy tej zostały podane wzory wariacyjne dla klasy G i ich zastosowania do wyznaczenia obszaru zmienności $f(z)$ i obszaru zmienności współczynników (a_1, a_2), $f \in G$.

РЕЗЮМЕ

Пусть G обозначает класс функций вида $f(z) = a_1 z + a_2 z^2 + \dots$ аналитических и однолистных в единичном круге $D(|z| < 1)$ выполняющих условие: $f(z_1)\,\overline{f(z_2)} \neq -1$ для $z_1, z_2 \in D$.

В этой работе дается вариационные формулы в класс G и их приложения к определению области изменения $f(z)$ и области изменения коэффициентов (a_1, a_2), $f \in G$.