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Recurrence Relations for Moments of Inflated Modified Power Series Distribution

Wzory rekurencyjne na momenty dla zmodyfikowanego rozkładu szeregowo-potęgowego typu inflated

Рекуррентные формулы на моменты для модифицированного распределения

1. Introduction. Gupta [2] has defined a so-called modified power series distribution (MPSD) as a general class of random variables with the discrete probability density function

$$P(X=x)=\frac{a_x\,[g(\theta)]^x}{f(\theta)},\quad x\in T \tag{1}$$

where T is a sub-set of the set of non-negative integers, $a_x \geqslant 0$, and there exists such $x \in T$ that $a_x > 0$. $g(\theta)$ and $f(\theta)$ are positive, finite and differentiable. For $g(\theta)$ to be invertible, it reduces to Patil's [5] generalized power series distribution (GPSD) and to power series distribution (PSD) defined by Noack [3] if in addition, T is the entire set of non-negative integers.

Now we define an inflated modified power series distribution and its truncation case. A random variable X is said to have the inflated (at the point $x = s$) modified power series distribution (IMPSD), if

$$P(X=x)=\begin{cases} 1-\alpha+\alpha a_x\,[g(\theta)]^{x/f(\theta)} & \text{for } x=s\\ \alpha a^x\,[g(\theta)]^{x/f(\theta)} & \text{for } x\neq s,\ x\in T,\end{cases} \tag{2}$$

where $0<\alpha\leqslant 1$, $a_x \geqslant 0$, $\bigvee_{x\in T} a_x>0$, $f(\theta)=\sum_x a_x\,[g(\theta)]^x$ for $\theta\in\Omega=[\,\theta{:}0<\theta<R\,]$

the parameter space and R the radius of convergence of the power series $f(\theta)$, T is the sub-set of the set of non-negative integers, and $s \in T$.

A random variable X is said to have the IMPSD truncated at the point $x = x_0$, if

$$P(X = x) = \begin{cases} \left[1-\alpha+\alpha \dfrac{a_x [g(\theta)]^x}{f(\theta)}\right] \Big/ \left[1-\alpha a_{x_0} \dfrac{[g(\theta)]^{x_0}}{f(\theta)}\right] & \text{for } x = s \neq x_0 , \\ \dfrac{\alpha a_x\, g(\theta)^x}{f(\theta)} \Big/ \left[1-\alpha a_{x_0} \dfrac{[g(\theta)]^{x_0}}{f(\theta)}\right] & \text{for } x \in T,\ x \neq s,\ x \neq x_0 \end{cases} \tag{3}$$

where α, θ and a_x are as defined earlier and $x_0 = \min_{x \in T} x$.

We see that when $g(\theta) = \theta$, the IMPSD (2) reduces to an inflated GPSD (IGPSD) defined by Patel and Shah [4] and Sobich [6]. Estimation problem is dealt with in [4], where as in [6], we find recurrence relations for moments. In this note, we give recurrence relations for the moments, the central moments of (2) and (3), including the recurrence relations between moments of (1) and (2). Moreover, the factorial moment relations of (2) are obtained. It is interesting to note that the moment relations obtained by Sobich [6] are easily reduced from those estabilished here. Formulaes for the recurrence relations for moments of the inflated generalized negative binomial (GNB) distribution as a particular case of (2) are also found out.

2. Moments of the IMPSD. 2.1. Mean of the IMPSD.

By definition, for the mean of the IMPSD, we have

$$E(X) = m_1 = s(1-\alpha) + \alpha \sum_{x \in T} X\, a_x [g(\theta)]^x / f(\theta)$$

That is,

$$m_1 = \beta s + \alpha m_1' \tag{4}$$

where $\beta = 1 - \alpha$, for bravity, and $m_1' = \dfrac{g(\theta) f'(\theta)}{g'(\theta) f(\theta)} = (g/g')(f'/f)$, the mean of the simple MPSD in [1].

2.2.Recurrence relation between moments. For the r-th moment of the IMPSD, we write

$$m_r = \beta s^r + \alpha \sum_{x \in T} x^r a_x g^x / f$$

Differentiating with respect to θ, we get

$$\frac{dm_r}{d\theta} = \alpha \sum_{x\in T} x^r a_x [x g^{x-1} g'/f - g^x f'/f^2] = (g'/g)(m_{r+1} - \beta s^{r+1} - m_1'(m_r - \beta s^r))$$

In view of (4), we have

$$m_{r+1} = (g/g')\frac{dm_r}{d\theta} + m_1'(m_r - \beta s^r) + \beta s^{r+1} \tag{6}$$

which, for $\alpha = 1$, reduces to that of recurrence relation established by Gupta [1].

2.3 **Recurrence relation between central moments.** For the r-th central moment of the distribution (2), by definition, we write

$$\mu_r = \beta(s - m_1)^r + \alpha \sum_{x\in T} (x - m_1)^r a_x g^x/f. \tag{7}$$

Differentiating with respect to θ, we get

$$\frac{d\mu_r}{d\theta} = \beta r(s - m_1)^{r-1}\left(-\frac{dm_1}{d\theta}\right) + \alpha \sum_{x\in T} a_x r(x - m_1)^{r-1}\left(-\frac{dm_1}{d\theta}\right) g^x/f +$$

$$+ \alpha \sum_{x\in T} x(x - m_1)^r a_x g^{x-1} g'/f - \alpha(f'/f) \sum_{x\in T} (x - m_1)^r a_x g^x/f$$

Hence, we have

$$(g/g')\frac{d\mu_r}{d\theta} = -r(g/g')\frac{dm_1}{d\theta}[\beta(s - m_1)^{r-1} + \alpha \sum_{x\in T} (x - m_1)^{r-1} a_x g^x/f +$$

$$+ \alpha \sum_{x\in T} (x - m_1)^{r+1} a_x g^x/f + \alpha(m_1 - m_1') \sum_{x\in T} (x - m_1)^r a_x g^x/f.$$

Since from (4), we have

$$m_1 - (g/g')(f'/f) = \beta(s - m_1)/\alpha$$

therefore, in view of (7), we write

$$(g/g')\frac{d\mu_r}{d\theta} = -r(g/g')\frac{dm_1}{d\theta}\mu_{r-1} + \mu_{r+1} - \beta(s - m_1)^{r+1} +$$

$$+\frac{\beta}{\alpha}(s-m_1)[\mu_r-\beta(s-m_1)^r].$$

This will yield

(8) $$\mu_{r+1}=(g/g')\left[\frac{d\mu_r}{d\theta}+r\frac{dm_1}{d\theta}\mu_{r-1}\right]-(\beta/\alpha)(s-m_1)\mu_r+(\beta/\alpha)(s-m_1)^{r+1}.$$

Taking $r=1$, in this result and noting $\mu_1=0$, $\mu_0=1$, weobtain the expression for the variance of (2) as

$$\mu_2=(g/g')\frac{dm_1}{d\theta}+\frac{\beta}{\alpha}(s-m_1)^2$$

from which, for $\alpha=1$, we get $\mu_2=(g/g')\dfrac{dm_1}{d\theta}$.

Hence, we write

(9) $$\mu_{r+1}=(g/g')\frac{d\mu_r}{d\theta}+r\mu_2\mu_{r-1}-\frac{\beta}{\alpha}(s-m_1)[\mu_r-r(s-m_1)\mu_{r-1}-(s-m_1)^r].$$

Note that both (8) and (9) for $\alpha=1$ will give the moment relations for simple MPSD described in [1].

Again when $g=\theta$ with $s=0$ and T as the entire set of non-negative integers, (8) will yield

$$\mu_{r+1}=\theta\left[\frac{d\mu_r}{d\theta}+r\frac{dm_1}{d\theta}\mu_{r-1}\right]+\frac{\beta}{\alpha}m_1\mu_r+(-1)^{r+1}\frac{\beta}{\alpha}m_1^{r+1}$$

as the recurrence formula for moments in inflated PSD (IPSD) inflated at the cell zero.

Moreover, for $g=\theta$ with $\alpha=1$, both (8) and (9) describe the recurrence relations for the central moments of the PSD defined by Noack [3]

$$\mu_{r+1}=\theta\left[\frac{d\mu_r}{d\theta}+r\frac{dm_1}{d\theta}\mu_{r-1}\right].$$

Also, in this case of inflation, we could derive a relation in central moments as follows:

Differentiating (7) again with respect to α (the inflation parameter), we get

$$\frac{d\mu_r}{d\alpha} = -(s-m_1)^r - r\beta(s-m_1)^{r-1}\frac{dm_1}{d\alpha} +$$

$$+ \sum_{x\in T}(x-m_1)^r a_x g^x/f + \alpha\sum_{x\in T}(x-m_1)^{r-1}\left(-\frac{dm_1}{d\alpha}\right)a_x g^x/f =$$

$$= -(s-m_1)^r - r\frac{dm_1}{d\alpha}\left[\beta(s-m_1)^{r-1} + \sum_{x\in T}(x-m_1)^{r-1}a_x g^x/f +\right.$$

$$+ [\mu_r - \beta(s-m_1)^r]/\alpha.$$

Which gives

(9a) $$\alpha\frac{d\mu_r}{d\alpha} = \mu_r + r\alpha(s-m_1')\mu_{r-1} - \alpha^r(s-m_1')^r.$$

2.4. **Recurrence relation for factorial moments.** For the r-th factorial moment, we have

$$E(X^{[r]}) = m^{[r]} = \beta s^{[r]} + \alpha\sum_x x^{[r]}a_x g^x/f$$

Differentiating with respect to θ, we obtain

$$\frac{dm^{[r]}}{d\theta} = \alpha\sum_x x^{[r]}a_x[xg^{x-1}g'/f - g^x f'/f^2] = \alpha(g'/g)\sum_x x^{[r+1]}a_x g^x/f +$$

$$+ r\alpha(g'/g)\sum_x x^{[r]}a_x g^x/f - \alpha(f'/f)\sum_x x^{[r]}a_x g^x/f =$$

$$= (g'/g)[m^{[r+1]} - \beta s^{[r+1]}] + (rg'/g - f'/f)(m^{[r]} - \beta s^{[r]})$$

This will give, again,

(10) $$m^{[r+1]} = (g/g')\frac{dm^{[r]}}{d\theta} + \beta s^{[r+1]} - (r-m_1')(m^{[r]} - \beta s^{[r]})$$

which, for $\alpha = 1$, describes the recurrence relation between the factorial moments of the MPSD

$$m^{[r+1]} = (g/g')\frac{dm^{[r]}}{d\theta} + m_1'm^{[r]} - rm^{[r]}$$

given by Gupta [1].

2.5. Recurrence relation for negatiąe moments. By definition, the r-th negative moment of X in (2) is given by

$$m_{-r}=\beta s^{-r}+\alpha \sum_x x^{-r}a_x g^x/f.$$

differentiating with respect to θ, we get

$$\frac{dm_{-r}}{d\theta}=\alpha \sum_x x^{-r}a_x\,[xg^{x-1}g'/f-g^x f'/f^2].$$

Hence, we have

$$(g'/g)\frac{dm_{-r}}{d\theta}=m_{-r+1}-\beta s^{-r+1}-m_1'(m_{-r}-\beta s^{-r})$$

which will yield

$$m_{-r+1}=(g/g')\frac{dm_{-r}}{d\theta}+m_1'm_{-r}+\beta s^{-r}(s-m_1') \tag{11}$$

This, for $g=\theta$ with $\alpha=1$, will reduce to

$$m_{-r+1}=\theta\left[\frac{dm_{-r}}{d\theta}+\left(\frac{f'}{f}\right)^{m}-r\right].$$

3. Recurrence relation between moments of MPSD and IMPSD. Again, we have

$$m_1=\beta s+\alpha m_1'$$

Substituting for m_1 in (7) and using the fact that

$$\sum_x (x-m_1)^r a_x g^x/f=\sum_{x\in T}\sum_{j=0}^{r}\binom{r}{j}(x-m_1')^j(\beta m_1'-\beta s)^{r-j}a_x g^x/f=$$

$$=\sum_{j=0}^{r}\binom{r}{j}\beta^{r-j}(m_1'-s)^{r-j}\mu_j'$$

we get, where $\mu_j'=\sum_{x\in T}(x-m_1')^j a_x g^x/f$,

$$\mu_r=\beta\alpha^r(s-m_1')^r+\alpha\sum_{j=0}^{r}\binom{r}{j}\beta^{r-j}(m_1'-s)^{r-j}\mu_j'.$$

Noting $\mu_0'=1$ and $\mu_1'=0$, this, after further algebraic simplification, gets reduced to

$$\mu^r = \beta\alpha(s-m_1')^r[\alpha^{r-1}+(-1)^r\beta^{r-1}] + \alpha\sum_{j=2}^{r}\binom{r}{j}\beta^{r-j}(m_1'-s)^{r-j}\mu_j'.$$

Hence, using the formula for the r-th central moment of the simple MPSD

$$\mu_r' = (g/g')\left[\frac{d\mu_{r-1}'}{d\theta} + (r-1)\frac{dm_1'}{d\theta}\mu_{r-2}'\right], \quad r = 2, 3, \dots .$$

we get

$$\mu_r = \alpha\beta(s-m_1')^r[\alpha^{r-1}+(-1)^r\beta^{r-1}] +$$

$$+\alpha(g/g')\sum_{j=2}^{r}\binom{r}{j}\beta^{r-j}(m_1'-s)^r\left[\frac{d\mu_{j-1}'}{d\theta} + (j-1)\frac{dm_1'}{d\theta}\mu_{j-2}'\right]$$

the recurrence relation between moments of simple MPSD (1) and an inflated MPSD (2).

4. Truncation. 4.1 Recurrence relation for positive and negative moments.

The r-th moment of the distribution (3), truncated at the point $x = x_0$, we have

(13) $$m_r = [\beta s^r + \alpha\sum_{x\neq x_0} x^r a_x g^x/f]/Q$$

where $Q = 1 - \alpha a_x g^{x_0}/f$.

Differentiating with respect to θ, and after simplification in view of (13), we arrive at

(14) $$m_{r+1} = (g/g')\frac{dm_r}{d\theta} - [x_0(1-Q) - m_1']m_r/Q + \beta s^r(s-m_1')/Q$$

which, for $\alpha = 1$ and $g = \theta$, reduces to

(15) $$m_{r+1} = \theta\frac{dm_r}{d\theta} - \left[x_0 a_{x_0}\frac{\theta^{x_0}}{f} - \theta\frac{f'}{f}\right]m_r/\left[1 - a_{x_0}\frac{\theta^{x_0}}{f}\right].$$

Again, for the r-th negative moment under trucation, we have

$$m_{-r} = [\beta s^{-r} + \alpha\sum_x x^{-r}a_x g^x/f]/Q.$$

Following the same lines described in (2.5), the recurrence realation for negative moments of truncated IMPSD (3), we get

$$(16)\qquad m_{-r+1} = (g/g')\frac{dm_{-r}}{d\theta} - [x_0(1-Q) - m_1']m_{-r}/Q + \beta s^{-r}(s-m_1')/Q.$$

It is interesting to note from above that for $g=\theta$, the expressions (8), (11), (12), (14) and (16) reduce to those recurrence relations for moments established by Sobich [6]. In a way Sobich's case becomes a particular of the present paper.

5. Example. An inflated GNB distribution.

A random variable X is said to have an inflated (at the point $x=s$) GNB distribution, if

$$(17)\qquad P(X=x) = \begin{cases} 1-\alpha+\alpha G(x) & \text{for } x=s \\ \alpha G(x) & \text{for } x\in T,\ x\neq s \end{cases}$$

where $G(x) = n\binom{n+\gamma x}{x}[\theta(1-\theta)^{\gamma-1}]^{x}/(1-\theta)^{-n}$, $0<\theta<1$, $|\theta^\gamma|<1$, $n>0$.

Were $g=\theta(1-\theta)^{\gamma-1}$ and $f=(1-\theta)^{-n}$. Then we have $g/g' = \theta(1-\theta)/(1-\gamma\theta)$ and $f'/f = n/(1-\theta)$ giving $m_1' = n\theta/(1-\gamma\theta)$.

Thus on substitution, particularly in (4), (6) and (8), we get the recurrence relations for moments of the inflated GNB (IGNB) distribution (17) as follows:

$$(18a)\qquad m_1 = \beta s + \alpha n\theta/(1-\gamma\theta),$$

$$(18b)\qquad m_{r+1} = \frac{\theta(1-\theta)}{(1-\gamma\theta)}\frac{dm_r}{d\theta} + \frac{n\theta}{1-\gamma\theta}[m_r - \beta s^r]$$

$$(18c)\qquad \mu_{r+1} = \frac{\theta(1-\theta)}{1-\gamma\theta}\left[\frac{d\mu_r}{d\theta} + r\frac{dm_1}{d\theta}\mu_{r-1}\right] - (\beta/\alpha)(s-m_1)\mu_r + (\beta/\alpha)(s-m_1)^{r+1}.$$

This, for $r=1$, will yield the variance of the IGNB distribution (17)

$$\mu_2 = \frac{n\alpha\theta(1-\theta)}{(1-\gamma\theta)^3} + \beta\left(s - \frac{n\theta}{1-\gamma\theta}\right)^2,$$

which ultimately, for $\alpha=1$, reduces to

$$\mu_2 = \frac{n\theta(1-\theta)}{(1-\gamma\theta)^3}.$$

Likewise, the other moment relations can be obtained for the IGNB distribution (17).

A Truncated IGNB distribution. A random variable X is said to have a truncated (at the point $x = x_0$) IGNB distribution, if

$$(19)\quad P(X = x) = \begin{cases} [1-\alpha+\alpha G(x)] \Big/ [1-\alpha G(x_0)] & \text{for } x = s,\ s \neq x_0 \\ \alpha G(x) \Big/ [1-\alpha G(x_0)] & \text{for } x \in T, x \neq s, x \neq x_0 \end{cases}$$

where $G(x)$ is as before.

In this case we have

$$(19a)\qquad m_r = [\beta s^r + \alpha \sum_{x \neq x_0} x^r G(x)]/Q$$

$$(19b)\ m_{r+1} = \frac{\theta(1-\theta)}{1-\gamma\theta}\frac{dm_r}{d\theta} - [x_0(1-Q) - \frac{n\theta}{1-\gamma\theta}]\, m_r/Q + \beta s^r(s - \frac{n\theta}{1-\gamma\theta})/Q$$

$$(19c)\ m_{-r+1} = \frac{\theta(1-\theta)}{1-\gamma\theta}\frac{dm_{-r}}{d\theta} - [x_0(1-Q) - \frac{n\theta}{1-\gamma\theta}]\, m_{-r}/Q + \beta s^{-r}(s - \frac{n\theta}{1-\gamma\theta})/Q$$

where $Q = 1 - \alpha G(x_0)$.

It may be noted that for $\gamma = 1$ and truncation at $x = x_0 = 0$, (19b) and (19c) agree with the recurrence relations for the truncated inflated binamial distribution established by Sobich [6]. Furthermore, the particular case $\gamma = 1$ for $\alpha = 1$ conforms with the truncated GNB distribution (truncated at $x = 0$) in Gupta [1].

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STRESZCZENIE

W pracy podano wzory na momenty zwykłe i centralne zniekształconego zmodyfikowanego rozkładu zadanego przez szeregi potęgowe, który zawiera jako szczególne przypadki wcześniej rozważane w literaturze rozkłady dyskretne.

РЕЗЮМЕ

В работе представлено формулы на обыкновенные и центральные моменты исказенного модифицированного распределения, вызванного степенными рядами. Главным результатом работы являются рекуррентные формулы для вышеупомянутых моментов.