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An Almost Convergence and its Applications

O prawie zbieżności i jej zastosowaniach

Об законе сходимости и ее приложениях

It is the purpose of the present paper to describe a class of sets which have a fixed point property for nonexpansive compact set valued mappings. We shall use asymptotic center technique [4].

Let X be a Banach space witch a norm $\| \ \|$ and let K be a nonempty subset of X. We choose an arbitrary boundet sequence $\{x_i\}$ in X and a point x. A number

$$r(x, \{x_i\}) = \limsup_i \|x - x_i\|$$

is called an asymptotic radius of $\{x_i\}$ at x and a number

$$r_k(\{x_i\}) = \inf_{x \in K} r(x, \{x_i\})$$

is an asymptotic radius of $\{x_i\}$ with respect to K (or in K). The set

$$A(K, \{x_i\}) = \{x \in K : r(x, \{x_i\}) = r_K(\{x_i\})\}$$

is called an asymptotic center of $\{x_i\}$ in K. It is obvious that if $\{x_{i_n}\}$ is a subsequence of $\{x_i\}$ then $r_K(\{x_{i_n}\}) \leqslant r_K(\{x_i\})$. We will call the sequence $\{x_i\}$ regular with respect to K (shortly regular) if all its subsequences have the same asymptotic radius in K and almost convergent with respect to K (shortly almost convergent) if all its subsequences have the same asymptotic center consisting of exactly one point x. Then we will write $x = A_K - \lim_i x_i$. In [9] K. Goebel proved the following very useful theorem.

Theorem 1. *Any bounded sequence* $[x_i]$ *contains a regular (with respect to K) subsequence.*

The almost convergence and regularity of sequences have the following properties which we will collect in a few lemmas.

Lemma 1. a) [19] *If* $x_i = x$ *for every i and* $x \in K$, *then* $A_K - \lim_i x_i = x$.

b) [19] *If* $A_K - \lim_i x_i = x$, *then for every subsequence* $[x_{i_n}]$ $A_K - \lim_n x_{i_n} = x$.

c) [19] *If* $[x_i]$ *does not almost converge to* $x \in K$, *then there exists a subsequence of which every sequence does not almost converge to x.*

d) *If* $[x_i]$ *is regular in K, then every subsequence* $[x_{i_n}]$ *is also regular in K and* $A(K, [x_i]) \subset A(K, [x_{i_n}])$.

e) *If* $[x_i]$ *is regular with respect to K and* $x \in A(K, [x_i])$ *then there exists* $\lim_i \|x - x_i\|$ *and* $\lim_i \|x - x_i\| = r(x, [x_i]) = r_K([x_i])$.

f) *If* $[x_i]$ *is regular in K and* $x = A_K - \lim_i x_i$, *then for each* $y \in K \setminus [x]$ *we have.*

$$r_K([x_i]) = r(x, [x_i]) < \liminf_i \|y - x_i\|$$

Lemma 2. *Let x belong to K. If for each regular subsequence* $[x_{i_n}]$ *of the sequence* $[x_i]$ *there exists subsequence* $[x_{i_{n_k}}]$ *which has* $[x]$ *as its asymptotic center, then* $x = A_K - \lim_i x_i$.

In some Banach spaces we may say something more about almost convergent sequences. We call a Banach space X an Opial space with respect to a weak (weak$-*$) convergence, if for every sequence $[x_i]$ in X, which converges weakly (weakly$-*$) to $x \in X$ and for all $y \neq x$ we have $\limsup_i \|y - x_i\| > \limsup_i \|x - x_i\|$ Z. Opial introduced this condition in [20]. For connections between the Opial's condition and other important properties of normed spaces cf. [11], [14], [17]. We say that X is uniformly convex in every direction (u.c.e.d.) if for all $0 < \epsilon \leqslant 2$ and $z \in X$ with $\|z\| = 1$ we have

$$\delta(\epsilon, z) = \inf \left[1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leqslant 1, \quad \|y\| \leqslant 1, \quad \|x-y\| \geqslant \epsilon, \right.$$

$$\left. \bigvee_{t \in R} x - y = tz \right] > 0$$

(see [1], [2], [8], [23]).

Lemma 3. *Let X be an Opial space with respect to a weak (weak$-^*$) convergence and $\emptyset \neq K \subset X$ weak (sequentially weak$-^*$) compact. Then a sequence $[x_i] \in K^N$ is almost convergent iff it is weak (weak$-^*$) convergent.*

Lemma 4. *X is an u.c.e.d. Banach space and $\emptyset \neq K \subset X$ is convex. Then each regular sequence w th nonempty its asymptotic center is almost convergent in K.*

Corollary 1 [9]. *If X is an uniformly convex Banach space and K is a nonempty closed convex subset of X, then each regular (in K) sequence is almost convergent in K.*

The next lemma answers the following question: are asymptotic centers of sequences are 'similar'?

Lemma 5. *Let X be a Banach space, $\emptyset \neq K \subset X$, $[x_i]$ – bounded and $x \in A(K, [x_i])$. For a sequence of nonnegative real numbers $[\alpha_i]$ we define $y_i = \alpha_i x_i + (1 - \alpha_i) x$ ($i = 1, 2, \ldots$). Then the following statements are fulfiled:*

a) *If $\alpha_i = \alpha$ for each i and $0 \leqslant \alpha \leqslant 1$, then $\alpha r_K([x_i]) = r_K([y_i])$ and $x \in A(K, [y_i]) \subset \subset A(K, [x_i])$.*

b) *If $x = A_K - \lim_i x_i$ and $\limsup_i \alpha_i \leqslant 1$, then $x = A_K - \lim_i y_i$..*

c) *If $x = A_K - \lim_i x_i$, $1 < \limsup_i \alpha_i < +\infty$ and K is convex, then $x = A_K - \lim_i y_i$.*

The last properties of asymptotic centers are related to an notion of 'inwardness' ([12]). Let X be a Banach space, $\emptyset \neq K \subset X$ and $x \in K$. Then we define the inward set of x relative to K, denoted $I_K(x)$, as follows

$$I_K(x) = [(1-\alpha)x + \alpha y : y \in K, \alpha \geqslant 0].$$

Lemma 6. *For X, K as above and for a bounded sequence $[x_i]$ such that $[x] = A(K, [x_i])$. we have $r_K([x_i]) < r(y, [x_i]) = \limsup_i \|y - x_i\|$ for each $y \in I_K(x) \setminus [x]$.*

Lemma 7. *If X is u.c.e.d., K is a nonemty convex subset of X, $[x_i]$ is bounded and $[x] = A(K, [x_i])$, then for every $y \in \overline{I_K(x)} \setminus [x]$ we have $r_K([x_i]) < r(y, [x_i])$.*

Now we may introduce a new topology in a nonempty subset K of a Banach space X.

Definition 1. A subset M of K is said to be A_K – closed if for each bounded sequence $[x_i]$ of its elements, which is almost convergent to x in K, have $x \in M$. A family of all A_K – closed subsets of K is a closed sets family of a new topology in K, that we will call an A_K – topology. K will be called sequentially A-compact iff from each sequence $[x_i]$ of its elements we may choose an almost convergent in K subsequence $[x_{i_n}]$.

Theorem 2. *A sequence $[x_i]$ of elements of set K is convergent to x in an A_K – topology iff $x = A_K - \lim_i x_i$.*

Proof. It follows from well known facts relative to S^* – spaces and L^* – spaces (see [5], [15]).

Remark 1. We notice that if $K_1 \subset K$, then an A_K – topology in K_1 may be different from an A_K – topology relativized to K_1 (examples may be constructed in the same way as in [10], see also examples which are shown later in this paper).

Remark 2. If K_1 and K_2 are sequentially A – compact and $K = K_1 \cap K_2 \neq \emptyset$, then K needn't to be sequentially A – compact.

Remark 3. If K is sequentially A – compact, then K is bounded and closed in $(X, \| \|)$.

Remark 4. It may happen, that an A_K – topology is a topology induced by a norm topology of X even, if K is not compact.

Remark 5. It is known, that K is sequentially A – compact if:

a) K is compact,

b) K is weakly compact and X is an Opial space with respect to a weak convergence (then the A_K – topology is the weak – topology relativized to K),

c) K is weakly – $*$ compact, X is a Banach space that is adjoint to a separable Banach space and X is an Opial space with respect to a weak – $*$ convergence (then the A_K – topology is the weak – $*$ topology relativized to K),

d) K is a bounded convex closed subset of an uniformly convex Banach space X.

Remark 6. In many cases A_K – topologies are not the relativized weak or weak – $*$ topologies in K. For example see a unit ball in $L^p(0,1)$ for $p > 1$ and $p \neq 2$ ([20]). It isn't even known whether an A_K – topology is a T_2 – topology (Hausdorff topology).

In this part of the paper we shall give some fixed point theorems for nonexpansive mappings.

Theorem 3. *Let K be a nonempty sequentia ly A – compact subset of a Banach space X and let $T: K \to \pi$ be a nonexpansive mapping, where π denotes the family of nonempty compact subsets of X, equipped w th the Hausdorff metric. If T is an inward mapping, i.e. $Tx \subset I_K x$ for $x \in K$, and there exists a sequence $[x_i] \in K^N$ such that dist $(x_i, Tx_i) \underset{i}{\to} 0$, then*

$\mathrm{Fix}\, T = [x \in K : x \in Tx] \neq \emptyset$. *If addditionally X is* u.c.e.d. *the condition '$Tx \subset I_K x$ for each $x \in K$' may be replaced by the condition '$Tx \subset \overline{I_K(x)}$ for each $x \in K$'.*

This result may be proved by an approach due to K. Goebel [9] (see also [3], [17], [18], [19]).

Exemple 1. Suppose $[a_i]$, $[b_i]$ are bounded sequence of positive numbers, $\inf_i a_i > 0$ and $a_i \leqslant b_i$ for $i = 1,2,\dots$. Chose two points $f_i = a_i e_i$, $g_i = b_i e_i$ in each 'axis' and define $K = \mathrm{conv}[f_i, g_i]_{i \in N}$. This K is not weak $-*$ compact. Simply calculations show that for weak $-*$ convergent $[x_i]$ to x

$$A(K, [x_i]) = \mathrm{Proj}_K x \quad \text{where } \mathrm{Proj}_K x = [y \in K : \| y - x \| = \inf_{z \in K} \| z - x \|]$$

Then $\mathrm{Proj}_K x$ consists of exactly one point iff there exists exactly one index j, such that $a_j = \min_i a_i$.

Choosing properly four sequences $[a_i]$, $[a_i']$, $[b_i]$, $[b_i']$ such that $a_i \leqslant a_i' \leqslant b_i' \leqslant b_i$ $(i = 1,2,\dots)$, we may construct, in a similar way, two sets K', K that $K' \subset K$ and K' is sequentially $A-$ compact, while K is not or vice versa. It explains statements given in remark 1, 2, 4. Repeated this constructions infinitely many times we may also construct the sequence of sets $[K_i]([L_i])$ with the following properties:

a) each $K_i(L_i)$ is nonempty closed and convex,

b) $K_i \subset K_{i+1}$ $(L_i \supset L_{i+1})$ for $i = 1,2,\dots$,

c) $\overline{\bigcup_i K_i} = K$ $(\bigcap_i L_i = L \neq \emptyset)$.

d) $K_i(L_i)$ has a fixed point property for nonexpansive mappings (f.p.p.) for $i = 1,3,5,\dots$, and $K_i(L_i)$ has not f.p.p. for $i = 2,4,\dots$,

e) for each $\epsilon > 0$ there exists and such that $H(K, K_i) < \epsilon$ $(H(L, L_i) < \epsilon)$, where H denotes the Hausdorff metric,

f) $K(L)$ has f.p.p. (the point $f/$ may be replaced by $f'/K(L)$ has not f.p.p.).

Definition 2 [21]. Let K be a nonempty subset of a Banach space X and $T: K \to K$. The mapping T is called asymptotically regular if

$$\lim_i \| T^{i+1} x - T^i x \| = 0$$

for each $x \in K$.

Theorem 4. *If K is a nonempty sequentially $A-$compact subset of a Banach space X and $T: K \to K$ is a nonexpansive asymptotically regular mapping, then for each x $[T^i x]$ almost converges to some fixed point.*

Corollary 2. *If K is a nonempty convex sequentially $A-$compact subset of a Banach space X and $T: K \to K$ is nonexpansive, then for each $0 < \alpha < 1$, and each $x \in K$ a sequence $[S^i_\alpha x]$ $(S_\alpha = \alpha T + (1-\alpha)\mathrm{Id})$ is almost convergent to a fixed point of T.*

Proof. S_α is nonexpansive and asymptotically regular [13].

Example 2. Let $X=l^1$, $K=[x=[\xi_k]\in l^1 : \|x\|\leq 1,\ \xi_k\geq 0$ for $k=1,2,\ldots]$, $Tx=T([\xi_k])= =[0,\xi_1,\xi_2,\ldots]$ and $S_{1/2}=1/2(T+\mathrm{Id})$. Then $[S^i_{1/2}\,e_1]$ is $w-*$ convergent to 0 and $\|S^i_{1/2}\,e_1\|=1$ for $i=1,2,\ldots$.

Example 3. Let $X=l^1$, $a>0$, $f_1=e_1$, $f_i=(1+a)e_i$ for $i\geq 2$ and $K=\overline{\mathrm{conv}}[f_i]$. If $Tx= =T([\xi_k])=[\xi_1,0,\xi_2,\xi_3,\ldots]$ for $x\in K$ and $S_{1/2}=1/2(T+\mathrm{Id})$, then $[S^i_{1/2}f_2]$ almost convergens to f_1 and $\|S^i_{1/2}f_2\|=1+a$ for each i.

Remark 7. In theorem 4 the assumption 'K is sequentially A – compact' may be replaced by the following assumptions: $[T^i x]$ is bounded for some $x\in K$ and from each bounded sequence $[x_i]$ in K we may choose an almost convergent in K subsequence $[x_{i_n}]$. Some other generalizations are related to so called normal Mann iteration process for T (see [6], [7], [13], [16], [20], [21], [22]).

Finally we will be concerned with methods of constructions of some sequentially A – compact sets. If we have a countably family of Banach spaces $[(X_l,\|\;\|_l)]$ and $p>1$, $l^p(X_l)=X$ will signify a Banach space of all sequences $x=(x^l)$ such that x^l belongs to X_l for each l and

$$\|x\|=[\sum_{l=1}^{\infty}(\|x^l\|_l)^p]^{1/p}<+\infty.$$

Let K_l be a nonempty subset of x_l $(l=1,2,\ldots)$. We define $K=\prod_{l=1}^{\infty}K_l\cap X$ and we always assume that $K\neq\emptyset$. Let $[x_i]=[(x^l_i)]$ be a bounded sequence in X. Then we can obtain the following lemma.

Lemma 8. *$[x_i]$ is almost convergent in K iff each $[x^l_i]_{i\in N}$ is almost convergent in K_l.*

A proof depends upon two facts:

1) if each $[x^l_i]_{i\in N}$ is almost convergent to x^l in K^l, then $x=(x^l)\in X$,

2) if each of $[x^l_i]_{i\in N}$ $(l=1,2,\ldots)$ and $[x_i]$ is regular in K_l or K (respectively), $y=(Y^l)\in \in K\setminus[x]$ and there exist the following limits: $\lim_i\|x-x_i\|$, $\lim_i\|y-x_i\|$, $\lim_i\|x^l-x^l_i\|_l$, $\lim_i\|y^l-x^l_i\|_l$ $(l=1,2,\ldots)$ and

$$\lim_i\sum_{l=k}^{\infty}(\|x^l_i\|_l)^p\ (k=1,2,\ldots),\ \text{ then } \lim_i\|y-x\|=[\sum_{l=1}^{\infty}r^p(y^l,[x^l_i]_{i\in N})+$$

$$+\lim_k\lim_i\sum_{l=k}^{\infty}(\|x^l_i\|_l)^p]^{1/p}>[\sum_{l=1}^{\infty}r^p(x^l,[x^l_i]_{i\in N})+$$

$$+\lim_k\lim_i\sum_{l=k}^{\infty}(\|x^l_i\|_l)^p]^{1/p}=\lim_i\|x-x_i\|.$$

Corollary 3. *K is sequentially A-compact iff every K_l is sequentially A-compact and* $\sum_{l=1}^{\infty} (\operatorname{diam} K_l)^p < +\infty$.

Let us choose a function $G : R_+^m \to R_+$ ($R_+ = [t \in R : t \geq 0]$) such that

1. $$\bigwedge_{(t^1,\dots,t^m)\in R_+^m} [G(t^1,\dots,t^m)=0 \Leftrightarrow \bigwedge_{1\leq l\leq m} t^l = 0]$$

2. $$\bigwedge_{\alpha\in R_+} \bigwedge_{(t^1,\dots,t^m)\in R_+^m} G(\alpha t^1,\dots,\alpha t^m) = \alpha G(t^1,\dots,t^m)$$

3. $$\bigwedge_{\substack{(t^1,\dots,t^m)\in R_+^m\\ (s^1,\dots,s^m)\in R_+^m}} [(s^1 \leq t^1,\dots,s^m \leq t^m) \Rightarrow G(s^1,\dots,s^m) \leq G(t^1,\dots,t^m)]$$

4. $$\bigwedge_{\substack{(t^1,\dots,t^m)\in R_+^m\\ (s^1,\dots,s^m)\in R_+^m}} G(s^1+t^1,\dots,s^m+t^m) \leq G(s^1,\dots,s^m) + G(t^1,\dots,t^m)$$

Then in a product of a Banach spaces $(X_1, \| \|_1), \dots, (X_m, \| \|_m)$ we may introduce a new norm

$$\|x\|_G = G(\|x^1\|_1, \dots, \|x^m\|_m)$$

for $x = (x^1, \dots, x^m) \in X = \prod_{l=1}^{m} X_l$.

Lemma 9. *If each K_l ($l = 1, \dots, m$) is a nonempty subset of X_l and each sequence $[x_i^l]_{i\in N}$ ($l = 1, \dots, m$) is regular in K_l, then a sequence $[x_i] = [(x_i^1, \dots, x_i^m)]$ is regular in $K = \prod_{l=1}^{m} K_l \subset X$, $r_K([x_i]) = G(r_{K_1}([x_i^1]), \dots, r_{K_m}([x_i^m]))$ and $\prod_{l=1}^{m} A(K_l, [x_i^l]) \subset A(K, [x_i])$.*

If in place of 3 there is a condition

$$3' \quad \bigwedge_{\substack{(t^1,\dots,t^m)\in R_+^m \\ (s^1,\dots,s^m)\in R_+^m}} [[(s^1 \leqslant t^1, \dots, s^m \leqslant t^m) \wedge (s^1, \dots, s^m) \neq (t^1, \dots, t^m)] \Rightarrow$$

$$\Rightarrow G(s^1, \dots, s^m) < G(t^1, \dots, t^m)],$$

then $\prod_{l=1}^{m} A(K_l, [x_i^l]) = A(K, [x_i])$.

If G satisfies conditions 2, 3', 4, then we have the following corollaries.

Corollary 4. *A sequence* $[x_i]$ *is almost convergent to* $x = (x^1, \dots, x^m)$ *in* K *iff each* $[x_i^l]_{i\in N}$ *is almost convergent to* x^l *in* K_l $(l = 1, 2, \dots, m)$.

Corollary 5. *K is sequentially A – compact iff K_l is sequentially A – compact for* 1, 2, …, *m.*

Using the last corollary we may construct an example of a sequentially A – compact set K with nonempty its interior in a Banach space X, which is neither u.c.e.d. nor Opial space with respect to weak – * convergence.

Example 4. Let $X = L^p(0,1) \times l^1$ $(p > 1, p \neq 2)$ with the norm $\|(x,y)\| = (\|x\|^2_{L^p} + \|y\|^2_{l^1})^{1/2}$. Let $K = K_{L^p} \times K_{l^1}$, where K_{L^p}, K_{l^1} are closed unit balls in $L^p(0,1)$ and l^1 respectively. Then K is sequentially A – compact.

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STRESZCZENIE

W pracy zdefiniowano nową rodzinę zbiorów mających własność punktu stałego dla wielowartościowych operacji nieoddalających.

РЕЗЮМЕ

В работе определено новое семейство множеств имеющих принцип неподвижной точки для многозначных слабосжимающих отображений.