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On the Limiting Behaviour of Some Functions of the Average of Independent Random Variables

O granicznym zachowaniu się pewnych funkcji średniej niezależnych zmiennych losowych.

О предельном поведении некоторых функций средних независимых случайных величин.

1. Introduction and preliminaries

Let $\{X_k, k \geqslant 1\}$ be a sequence of random variables. Investigations of the asymptotical behaviour of $\{S_n/n, n \geqslant 1\}$, where $S_n = \sum_{k=1}^{n} X_k$, are important both from the theoretical and the applied point of view. In particular, consideration of the limit distribution of

$$Y_n = \frac{S_n}{b_n} - a_n = d_n\left(\frac{S_n}{n} - c_n\right),$$

where $c_n = a_n b_n/n$ and $d_n = n/b_n > 0$, $n \geqslant 1$, are normalizing constants, constitutes a notable part of probability theory and mathematical statistics. In the case when the limit distribution of $\{Y_n, n \geqslant 1\}$ is normal, then $\{X_k, k \geqslant 1\}$ is sometimes called asymptotically normal. In this note we shall use the following

Definition 1. Let $\{X_k, k \geqslant 1\}$ be a sequence of random variables. The sequence $\{S_n/n, n \geqslant 1\}$, $S_n = \sum_{k=1}^{n} X_k$, is said to have the property of asymptotic normality if there exist normalizing constants c_n and $d_n > 0$, $n \geqslant 1$, such that

$$(1) \qquad d_n\left(\frac{S_n}{n} - c_n\right) \xrightarrow{L} \mathcal{N} \text{ as } n \to \infty \ (L \text{ in law}),$$

where $\mathcal{N}$ is the normal random variable with mean zero and standard deviation 1, i.e.

$$\lim_{n\to\infty} P\left[d_n\left(\frac{S_n}{n}-c_n\right)< x\right] = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Here we are interested in the finding a class $\mathcal{G}$ of real functions g in which the property of asymptotical normality is invariant in the sense that for every function $g \in G$ there exist normalizing constants $c'_n = c_n(g)$, $d'_n = d'_n(g)$, $n \geqslant 1$, such that

$$d'_n(g(S_n/n)-c'_n) \xrightarrow{L} \mathcal{N}_{a,b} \text{ as } n\to\infty \tag{2}$$

where $\mathcal{N}_{a,b}$ is the normal random variable with mean a and standard deviation b.

One can immediately notice, by the central limit theorem, that if $\{X_k,\ k\geqslant 1\}$ is a sequence of independent and identically distributed random variables with $EX_n = \mu$, $\sigma^2 X_n = \sigma^2 < \infty$, $n\geqslant 1$, then $\{S_n/n,\ n\geqslant 1\}$ has the property of asymptotical normality with $c_n = \mu$ and $d_n = \sqrt{n}/\sigma$, $n\geqslant 1$. Hence, we see that statements asserting the condition under which $\{g(S_n/n),\ n\geqslant 1\}$ has the property of asymptotical normality are direct generalizations of the central limit theorem.

The aim of this note is to give some theorems concerning the asymptotic normality of $\{g(S_n/n),\ n\geqslant 1\}$ and to extend these results to the case when n is replaced by an integer-valued random variable N_n.

2. Classes of functions preserving the property of asymptotical normality

We consider here a sequence $\{X_n,\ n\geqslant 1\}$ of independent random variables.

In what follows we shall need the lemmas:

Lemma 1. (see, e.g. [5]). *Let $\{X_n,\ n\geqslant 1\}$ be a sequence of random variables such that $X_n \xrightarrow{P} X$ as $n\to\infty$. If h is a continuous function, then $h(X_n) \xrightarrow{P} h(X)$ as $n\to\infty$.*

Remark. If $X_n \xrightarrow{P} \mu = \text{const.}$, and f is a continuous function at the point μ, then $f(X_n) \xrightarrow{P} f(\mu)$ as $n\to\infty$.

Lemma 2. (see, e.g. [6]). *Let $\{X_n,\ n\geqslant 1\}$ be a sequence of random variables such that $X_n \xrightarrow{L} X$ as $n\to\infty$. If $\{Y_n,\ n\geqslant 1\}$ and $\{Z_n,\ n\geqslant 1\}$ are sequences of random variables such that $Y_n \xrightarrow{P} 1$ as $n\to\infty$ and $Z_n \xrightarrow{P} 0$ as $n\to\infty$, then $U_n = X_nY_n + Z_n \xrightarrow{L} X$ as $n\to\infty$.*

We now give some results concerning the asymptotic behaviour of $\{g(S_n/n), n \geqslant 1\}$.

Theorem 1. [11]. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = \mu$ and $\sigma^2 X_1 = \sigma^2 < \infty$. Suppose that $\mathcal{G}_\mu$ is the class of all functions g differentiable at the point $x = \mu$ with $g'(\mu) \neq 0$.*
Then for every $g \in \mathcal{G}_\mu$

$$(3) \qquad \frac{\sqrt{n}}{g'(\mu)\sigma}\left(g\left(\frac{S_n}{n}\right) - g(\mu)\right) \xrightarrow{L} \mathcal{N} \text{ as } n \to \infty.$$

Proof. Define

$$(4) \qquad h(x) = \begin{cases} \dfrac{g(x)-g(\mu)}{(x-\mu)g'(\mu)} & \text{if } x \neq \mu, \\ 1 & \text{if } x = \mu. \end{cases}$$

Since g is differentiable at $x = \mu$, then h is continuous, so $\lim_{x\to\mu} h(x) = 1 = h(\mu)$. Using the fact that $S_n/n \xrightarrow{P} \mu$ as $n \to \infty$, it then follows from Lemma 1 that $h(S_n/n) \xrightarrow{P} h(\mu)$ as $n \to \infty$.

Moreover, we see that

$$(5) \qquad \frac{\sqrt{n}}{g'(\mu)\sigma}\left(g(S_n/n) - g(\mu)\right) = \frac{\sqrt{n}}{\sigma}\left(\frac{S_n}{n} - \mu\right) h(S_n/n).$$

But, by the central limit theorem

$$\frac{\sqrt{n}}{\sigma}\left(\frac{S_n}{n} - \mu\right) \xrightarrow{L} \mathcal{N} \text{ as } n \to \infty.$$

Therefore, using Lemma 2 and (5), we conclude that (3) holds, i.e. (2) holds with $c_n' = g(\mu)$ and $d_n' = \sqrt{n}/(g'(\mu)\sigma)$.

The proof of Theorem 1 leads to a more general result:

Theorem 1'. *Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables such that $Z_n \xrightarrow{P} \mu$, and $a_n(Z_n - \mu) \xrightarrow{L} Z$ as $n \to \infty$, where μ is a constant and $\{a_n, n \geqslant 1\}$ is a sequence of real numbers. Then for every $g \in \mathcal{G}_\mu$*

$$\frac{a_n}{g'(\mu)}\left(g(Z_n) - g(\mu)\right) \xrightarrow{L} Z \text{ as } n \to \infty.$$

Theorem 1 is also a particular case of the following

Theorem 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = \mu_n$ and $\sigma^2 X_n = \sigma_n^2 < \infty$, $n \geqslant 1$, satisfying the Lindeberg condition. Suppose that $S_n/n \xrightarrow{P} \mu$, and $s_n^{-1}(\sum_{k=1}^{n} \mu_k - n\mu) \to a$ as $n \to \infty$, where $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$. Then for every $g \in \mathscr{G}_\mu$*

$$(6) \qquad \frac{n}{g'(\mu)s_n}\left(g\left(\frac{S_n}{n}\right) - g(\mu)\right) \xrightarrow{L} \mathscr{N}_{a,1} \text{ as } n \to \infty.$$

Proof. As in the proof of Theorem 1, we have

$$(7) \qquad \frac{n}{g'(\mu)s_n}\left(g\left(\frac{S_n}{n}\right) - g(\mu)\right) = \frac{n}{s_n}\left(\frac{S_n}{n} - \mu\right) h\left(\frac{S_n}{n}\right),$$

where h is the function defined by (4).
We see that

$$\frac{n}{s_n}\left(\frac{S_n}{n} - \mu\right) h\left(\frac{S_n}{n}\right) = \frac{S_n - \sum_{k=1}^{n} \mu_k}{s_n} h\left(\frac{S_n}{n}\right) + \frac{\sum_{k=1}^{n} \mu_k - n\mu}{s_n} h\left(\frac{S_n}{n}\right).$$

Hence, by the assumptions of Theorem 2, we obtain (6).
As a consequence of Theorem 1′ or Theorem 2, we have

Corollary. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = \mu$ and $\sigma^2 X_n = \sigma_n^2 < \infty$, $n \geqslant 1$, satisfying the Lindeberg condition. If $S_n/n \xrightarrow{P} \mu$ as $n \to \infty$, then (6) holds with $a = 0$.*

Theorem 3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = \mu_n$, $\sigma^2 X_n = \sigma_n^2 < \infty$ satisfying the Lindeberg condition. Suppose that $S_n/n \xrightarrow{P} \mu$ and $\bar{\mu}_n = \sum_{k=1}^{n} \mu_k/n \to \mu$ as $n \to \infty$, and let $\mathscr{G}_\mu^*$ be the class of all functions g differentiable in some neighbourhood of μ and such that $g'(\mu) \neq 0$,* and g' is continuous at the point μ.
Then for every $g \in \mathscr{G}_\mu^*$

$$(8) \qquad \frac{n}{g'(\mu)s_n}\left(g\left(\frac{S_n}{n}\right) - g(\bar{\mu}_n)\right) \xrightarrow{L} \mathscr{N} \text{ as } n \to \infty.$$

Proof. Using Lagrange's formula we have

$$\frac{n}{g'(\mu)\mathscr{S}_n}\left(g\left(\frac{S_n}{n}\right) - g(\bar{\mu}_n)\right) = \frac{n}{g'(\mu)s_n}\left(\frac{S_n}{n} - \bar{\mu}_n\right) g'\left(\frac{S_n}{n} + \theta\left(\frac{S_n}{n} - \bar{\mu}_n\right)\right)$$

where $0 < \theta < 1$.

But

$$\frac{n}{s_n}\left(\frac{S_n}{n}-\bar{\mu}_n\right)=\frac{S_n-\sum_{k=1}^{n}\mu_k}{s_n}\xrightarrow{L}\mathcal{N}\text{ as } n\to\infty,$$

and, by the assumptions,

$$g'\left(\frac{S_n}{n}+\theta\left(\frac{S_n}{n}-\bar{\mu}_n\right)\right)\xrightarrow{P}g'(\mu)\text{ as } n\to\infty.$$

Corollary. *If under the assumptions of* Theorem 3

$$\frac{n\big(g(\bar{\mu}_n)-g(\mu)\big)}{s_n}\to a \text{ as } n\to\infty,$$

then

$$\frac{n}{g'(\mu)s_n}\left(g\left(\frac{S_n}{n}\right)-g(\mu)\right)\xrightarrow{L}\mathcal{N}^a\text{ as } n\to\infty.$$

We have the following slightly more general result:

Theorem 4. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables with $EX_n=\mu_n$, $\sigma^2X_n=\sigma_n^2<\infty$ satisfying the Lindeberg condition. Suppose that $\{X_n, n\geqslant 1\}$ satisfies the weak law of large numbers and that $\mathcal{G}$, is the class of all functions g with continuous derivative such that $g'(\bar{\mu}_n)\neq 0$, and*

$$\frac{g'(\bar{\mu}_n)+U_n}{g'(\bar{\mu}_n)}\xrightarrow{P}1 \text{ when } U_n\xrightarrow{P}0 \text{ as } n\to\infty,$$

and $\bar{\mu}_n=\sum_{k=1}^{n}\mu_k/n$.

Then, for every $g\in\mathcal{G}'$

$$\frac{ns_n}{g'(\bar{\mu}_n)s_n}\left(g\left(\frac{S_n}{n}\right)-g(\bar{\mu}_n)\right)\xrightarrow{L}\mathcal{N}\text{ as } n\to\infty. \tag{9}$$

Proof. Define

$$h_n(x)=\begin{cases}\dfrac{g(x)-g(\bar{\mu})\bar{\mu}_n}{(x-\bar{\mu}_n)g'(\bar{\mu}_n)} & \text{if } x\neq\bar{\mu}_n,\\ 1 & \text{if } x=\bar{\mu}_n,\ n=1,2,\ldots\end{cases} \tag{10}$$

As before,

$$\frac{n}{g'(\bar{\mu}_n)s_n}\left(g\left(\frac{S_n}{n}\right)-g(\bar{\mu}_n)\right)=\frac{n}{s_n}\left(\frac{S_n-\sum_{k=1}^{n}\mu k}{n}\right)h_n\left(\frac{S_n}{n}\right).$$

But for $S_n/n \neq \bar{\mu}_n$ a.s., we have

$$h_n\left(\frac{S_n}{n}\right) = \frac{g\left(\frac{S_n}{n}\right) - g(\bar{\mu}_n)}{\left(\frac{S_n}{n} - \bar{\mu}_n\right) g'(\bar{\mu}_n)} = \frac{g'\left(\bar{\mu}_n + \theta\left(\frac{S_n}{n} - \bar{\mu}_n\right)\right)}{g'(\bar{\mu}_n)} \xrightarrow{P} 1$$

as $n \to \infty$.

This fact together with (11) proves Theorem 4.

Remark If $\bar{\mu}_n \to \mu$ as $n \to \infty$ and $g'(\mu) \neq 0$, then (9) reduces to (8). Moreover, we have the following theorem

Theorem 5. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = \mu_n$, $\sigma^2 X_n = \sigma_n^2 < \infty$ satisfying the Lindeberg condition and the weak law of large numbers. Suppose that $\mathscr{G}''$ is the class of all functions g with continuous second derivative such that $\sup_x |g''(x)/g'(x)| \leqslant K < \infty$, where K is a positive constant. Then, for every $g \in \mathscr{G}''$ (9) holds.*

Proof. From the definition of h_n in (10), the assumptions concerning g, and Taylor series expansion, we get

$$h_n(S_n/n) - 1 = \frac{g(S_n/n) - g(\bar{\mu}_n)}{(S_n/n - \bar{\mu}_n) g'(\bar{\mu}_n)} - 1 =$$

$$= \frac{g'(\bar{\mu}_n)(S_n/n - \bar{\mu}_n) + \frac{1}{2} g''(\bar{\mu}_n + \theta[S_n/n - \bar{\mu}_n])(S_n/n - \bar{\mu}_n)^2}{(S_n/n - \bar{\mu}_n) g'(\bar{\mu}_n)} - 1 =$$

$$= (S_n/n - \bar{\mu}_n) \frac{g''(\bar{\mu}_n + \theta[S_n/n - \bar{\mu}_n])}{2g'(\bar{\mu}_n)}$$

where $0 < \theta < 1$ Since $S_n/n - \bar{\mu}_n \xrightarrow{P} 0$, as $n \to \infty$ and $\sup_x |g''(x)/g'(x)| < K$, we get $h_n(S_n/n) - 1 \xrightarrow{P} 0$, as $n \to \infty$. Hence, by (11) and Lemma 2, we obtain (9).

3. The behaviour of functions of sums with random indices

Here we extend the previous results to the case of random indexed sums.

Theorem 6. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = \mu$ and $\sigma^2 X_1 = \sigma^2 < \infty$. Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ are independent, and*

$$N_n \xrightarrow{P} \infty \text{ as } n \to \infty. \tag{12}$$

Then for every $g \in \mathscr{G}_\mu$, *where* $\mathscr{G}_\mu$ *is the class of functions defined in* Theorem 1, *we have*

(13) $$\frac{\sqrt{N_n}}{g'(\mu)\sigma}\left(g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right)\xrightarrow{L}\mathscr{N} \text{ as } n\to\infty.$$

Proof. Let

$$p_{nk} = P[N_n = k];\ n, k = 1, 2, \ldots.$$

Using the properties of probability measures and (12), we get

(a) $p_{nk} \geqslant 0;\ n, k = 1, 2, \ldots,$

(b) $\sum_{k=1}^{\infty} p_{nk} = 1;\ n = 1, 2, \ldots,$

(c) $\lim_{n\to\infty} p_{nk} = 0;\ k = 1, 2, \ldots.$

Taking into account the independence of $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ we have

$$P\left[\sqrt{N_n}\,\frac{g(S_{N_n}/N_n)-g(\mu)}{g'(\mu)\sigma} < x\right] = \sum_{k=1}^{\infty} P\left[\frac{\sqrt{k}}{g'(\mu)\sigma}\left(g(S_k/k)-g(\mu)\right) < x\right] P[N_n = k].$$

Using (a)-(c), Toeplitz' Lemma (see, e.g. [3], p. 238) and Theorem 1, we get (13).

Corollary. *Under the assumptions of* Theorem 6. *with* $g(x) \equiv x$, *we get the result of* [6], p. 472.

Under the assumption (12) and the independence of $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ it can be seen that corresponding to Theorems 2-5 we have the following results:

(6′) $$\frac{N_n}{g'(\mu)s_{N_n}}\left(g(S_{N_n}/N_n)-g(\mu)\right)\xrightarrow{L}\mathscr{N}_a \text{ as } n\to\infty,$$

(8′) $$\frac{N_n}{g'(\mu)s_{N_n}}\left(g(S_{N_n}/N_n)-g(\bar{\mu}_{N_n})\right)\xrightarrow{L}\mathscr{N} \text{ as } n\to\infty,$$

(9′) $$\frac{N_n}{g'(\bar{\mu}_{N_n})s_{N_n}}\left(g(S_{N_n}/N_n)-g(\bar{\mu}_{N_n})\right)\xrightarrow{L}\mathscr{N} \text{ as } n\to\infty.$$

Remark. If one uses the fact that $\frac{S_n}{n}\xrightarrow{a.s.}\mu$, (a.s.-almost sure) and $N_n\xrightarrow{P}\infty$ as $n\to\infty$ imply $\frac{S_{N_n}}{N_n}\xrightarrow{P}\mu$ as $n\to\infty$ ([7], p. 148), then the proof of Theorem 6 is similar to that of Theorem 1. Namely, we have

$$\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left(g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right)=\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}\,h\left(\frac{S_{N_n}}{N_n}\right)\xrightarrow{L}\mathcal{N}$$

as $n\to\infty$, where h is the function defined by (4). This follows from Theorem 1 [6], p. 472, the fact mentioned above and Lemma 2.

By the last Remark we have a more general result, namely:

Theorem 6'. *Let $\{Z_n, n\geqslant 1\}$ be a sequence of random variables such that $Z_n\xrightarrow{a.s.}\mu$, and $a_n(Z_n-\mu)\xrightarrow{L}Z$ as $n\to\infty$, where μ is a constant and $\{a_n, n\geqslant 1\}$ is a sequence of real numbers. If $\{N_n, n\geqslant 1\}$ is a sequence of positive integer-valued random variables such that $\{Z_n, n\geqslant 1\}$ and $\{N_n, n\geqslant 1\}$ are independent, and (12) holds, then for every $g\in\mathcal{G}_\mu$*

$$\frac{a_n}{g'(\mu)}\left(g(Z_{N_n})-g(\mu)\right)\xrightarrow{L}Z\ \text{ as } n\to\infty.$$

Proof. The assumptions $Z_n\xrightarrow{a.s.}\mu$ as $n\to\infty$ and (12) imply that $Z_{N_n}\xrightarrow{P}\mu$ as $n\to\infty$ [2]. Moreover, we know that $a_n(Z_n-\mu)\xrightarrow{L}Z$ as $n\to\infty$ and (12) give $a_{N_n}(Z_{N_n}-\mu)\xrightarrow{L}Z$ as $n\to\infty$ [8]. Using the function h defined by (4), and the previous arguments, we have for every $g\in\mathcal{G}_\mu$

$$\frac{a_{N_n}}{g'(\mu)}\,g(Z_{N_n})-g(\mu)) = a_{N_n}(Z_{N_n}-\mu)\,h(Z_{N_n})\xrightarrow{L}Z \text{ as } n\to\infty.$$

As an application of this Theorem, we have the following two theorems:

Theorem 7. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables with $EX_n=\mu$, $\sigma^2X_n=\sigma_n^2<\infty$, $n\geqslant 1$, such that $S_n/n\xrightarrow{a.s.}\mu$ as $n\to\infty$. Suppose that $\{N_n, n\geqslant 1\}$ is a sequence of positive integer-valued random variables such that $\{X_n, n\geqslant 1\}$ and $\{N_n, n\geqslant 1\}$ are independent and (12) holds. If for any given $\varepsilon>0$*

$$\frac{1}{s_{N_n}^2}\sum_{k=1}^{N_n}\int\limits_{|x|>\varepsilon s_{N_n}} x^2\,dF_k(x-\mu)\xrightarrow{P}0\ \text{ as } n\to\infty,$$

or equivalently if the so-called „random Lindeberg condition”

$$\text{(14)}\qquad \lim_{n\to\infty}E\left[\frac{1}{s_{N_n}^2}\sum_{k=1}^{N_n}\int\limits_{|x|\geqslant\varepsilon s_{N_n}} x^2\,dF_k(x-\mu)\right]=0\ [10]$$

is satisfied, then for every $g\in\mathcal{G}_\mu$,

$$\frac{N_n}{g'(\mu)s_{N_n}}\left(g(S_{N_n}/N_n)-g(\mu)\right)\xrightarrow{L}\mathcal{N}\ \text{ as } n\to\infty.$$

Proof. Taking into account Theorem 6′, it is enough to state that the assumptions of Theorem 7 imply that

$$\frac{N_n}{s_{N_n}}\left(\frac{S_{N_n}}{N_n}-\mu\right)\xrightarrow{L}\mathcal{N} \text{ as } n\to\infty.$$

But this assertion follows from Theorem 6.2 of [9] or from Theorem 1 [10].

Theorem 8. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables with $EX_n=\mu$, $\sigma^2 X_n=\sigma_n^2<\infty$, $n\geqslant 1$, such that $S_n/n\xrightarrow{a.s.}\mu$ as $n\to\infty$. Suppose that $\{N_n, n\geqslant 1\}$ is a sequence of positive integer-valued random variables such that $\{X_n, n\geqslant 1\}$ and $\{N_n, n\geqslant 1\}$ are independent and* (12) *holds. If* (14) *holds and*

$$\frac{s_{N_n}^2-Es_{N_n}^2}{\sigma^2 S_{N_n}}\xrightarrow{P}0 \text{ as } n\to\infty,$$

then for every $g\in\mathcal{G}_\mu$

$$\frac{N_n}{g'(\mu)b_n\sigma S_{N_n}}\left(g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right)\xrightarrow{L}\mathcal{N} \text{ as } n\to\infty,$$

where $b_n^2=Es_{N_n}^2/(Es_{N_n}^2+\mu\sigma^2 N_n)$.

Proof. As previously, we have

$$\frac{N_n}{g'(\mu)\sigma S_{N_n}}\left(g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right)=\frac{S_{N_n}-N_n\mu}{\sigma S_{N_n}}h\left(\frac{S_{N_n}}{N_n}\right).$$

But, by Theorem 2 of [10] $\frac{S_{N_n}-N_n\mu}{\sigma S_{N_n}}$ is asymptotically normal $N(0,b_n)$ where $b_n^2=Es_{N_n}^2/(Es_{N_n}^2+\mu\sigma^2 N_n)$, and, moreover, we know that $h(S_{N_n}/N_n)\xrightarrow{P}1$ as $n\to\infty$, which ends the proof.

Now using the considerations mentioned in the previous Remark, together with some facts on the limit behaviour of sums with random indices, one can prove

Theorem 9. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1=\mu$ and $\sigma^2X_1=\sigma^2<\infty$. Suppose that λ is a positive discrete random variable, and put*

$$N_n=[n\lambda],$$

where $[x]$ denotes the integral part of x. Then for every $g\in\mathcal{G}_\mu$ (13) *holds.*

To prove the statement it is enough to use Theorem 1 [4], p. 472 and the considerations given above.

The limit behaviour of functions of sums with random indices when $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ are not assumed to be independent, is given by the following theorem.

Theorem 10. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = \mu$ and $\sigma^2 X_1 = \sigma^2$. If $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that*

$$\frac{N_n}{n} \xrightarrow{P} \lambda \text{ as } n \to \infty, \tag{15}$$

where λ is a positive random variable, then for every $g \in \mathcal{G}_\mu$ (13) holds.

In the case when λ is a discrete, positive-valued random variable, then, under (15), the assertion of Theorem 10 is a consequence of Theorem 4 [6] p. 475 and the facts used earlier. When λ is a positive random variable one can use the results given in [1] or [4].

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STRESZCZENIE

Niech $\{X_k, k \geqslant 1\}$ będzie ciągiem niezależnych zmiennych losowych, a g funkcją rzeczywistą. W pracy określa się warunki asymptotycznej normalności ciągu $\{g(S_n/n), n \geqslant 1\}$, gdzie $S_n = \sum_{k=1}^{n} X_k$, a również asymptotycznej normalności ciągu $\{g(S_{N_n}/N_n), n \geqslant 1\}$, gdzie $\{N_n, n \geqslant 1\}$ jest ciągiem zmiennych losowych o wartościach w zbiorze liczb naturalnych.

РЕЗЮМЕ

Пусть $\{X_k, k \geqslant 1\}$ -последовательность независимых случайных величин и вещественная функция. В работе устанавливается условие асимптотической нормальности последовательности $\{g(S_n/n),\ n = 1\}$, где $S_n = \sum_{k=1}^{n} X_k$, а также асимптотической нормальности последовательности $\{g(S_{N_n}/N_n), n \geqslant 1)$ где $\{N_n, n \geqslant 1\}$-последовательность целочисленных положительных случайных величин.