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An Extremal Problem for Functions of Positive Real Part with Vanishing Coefficients

Pewien problem ekstremalny dla funkcji o dodatniej części rzeczywistej ze znikającymi współczynnikami

Экстремальная проблема для функции с положительной действительной частью с угасающими коэффициентами

1. Introduction. Let $P(a, n)$ represent the class of functions $p(z) = 1+\sum_{m=n}^{\infty} P_m z^m$, $n \geqslant 1$, which are analytic in $|z|<1$ and satisfy $\mathrm{Re}\{p(z)\} > a$, $0 \leqslant a < 1$, $|z|<1$, and let $P^*(a, n)$ represent the subclass of $P(a, n)$ consisting of n-fold symmetric functions. In a recent article Bernardi [1] determined the sharp upper bound of $\mathrm{Re}\{zp'(z)/p(z)\}$, $p(z) \in P(a, n)$, and used this to obtain some results concerning the partial sums of convex univalent functions. In this paper we determine the sharp lower bound of $\mathrm{Re}\{zp'(z)/p(z)\}$, $p(z) \in P(a, n)$, and apply this to several problems, including one which extends an earlier result due to Sakaguchi [4].

2. The basic inequality. Theorem 1. *If $p(z) \in P(a, n)$ and $|z| = r < 1$, then*

$$(2.1)\quad \mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} \dfrac{-2n(1-a)r^n}{(1+r^n)[1-(1-2a)r^n]}, & 0 \leqslant r \leqslant r_{a,n} \\[2ex] \dfrac{-n[\sqrt{1+(1-2a)r^{2n}} - \sqrt{a(1-r^{2n})}]^2}{(1-a)(1-r^{2n})}, & r_{a,n} < r < 1 \end{cases}$$

where $r_{a,n}$ is the unique root in $(0, 1]$ of the equation

$$(2.2)\qquad 1-3r^n+3(1-2a)r^{2n}-(1-2a)r^{3n} = 0.$$

For each a, $0 \leqslant a < 1$, and each positive integer n, equality is obtained in the first part of (2.1) for

$$p(z) = \frac{1+(2a-1)z^n}{1+z^n} \tag{2.3}$$

and in the second part for

$$p(z) = \frac{1-2a\lambda z^n+(2a-1)z^{2n}}{1-2\lambda z^n+z^{2n}}$$

where λ satisfies the equation

$$\frac{1-r^{2n}}{1-2r^n\lambda+r^{2n}}+\frac{a}{1-a} = \sqrt{\frac{a}{1-a}\left(\frac{1+r^{2n}}{1-r^{2n}}+\frac{a}{1-a}\right)}.$$

Proof. We first consider the case $a = 0$. Lewandowski et al. [2] have shown that solutions to the extremal problem

$$\min_{p(z)\in P(0,n)} \min_{|z|=r} \mathrm{Re}\{\psi[p(z), zp'(z), \ldots, z^N p^{(N)}(z)]\}$$

where $\psi(w_0, w_1, \ldots, w_N)$ is analytic in $\mathrm{Re}\{w_0\} > 0$, $|w_k| < \infty$, $k = 1, 2, \ldots, N$ are always functions in $P^*(0, n)$. Since $q(z) \in P(a, n)$ if and only if $q(z) = (1-a)p(z)+a$ where $p(z) \in P(0, n)$, it is obvious that extremal problems over $P(a, n)$ will also have solutions in $P^*(a, n)$. Furthermore, $q(z)$ is in $P^*(a, n)$ if and only if $q(z) = p(z^n)$ for some $p(z)$ in $P(a, 1)$. Applying these remarks to the extremal problem under consideration here we have

$$\min_{q(z)\in P(a,n)} \min_{|z|=r} \mathrm{Re}\{zq'(z)/q(z)\} = \min_{p(z)\in P(a,1)} \min_{|z|=r} \mathrm{Re}\{nz^n p'(z^n)/p(z^n)\}. \tag{2.5}$$

Zmorovič [6] has shown that if $p(z) \in P(a, 1)$, then

$$\mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} \dfrac{-2r(1-a)}{(1+r)[1-(1-2a)r]}, & 0 \leqslant r \leqslant r_a \\ \dfrac{-[\sqrt{1+(1-2a)r^2}-\sqrt{a(1-r^2)}]^2}{(1-a)(1-r^2)}, & r_a < r < 1 \end{cases} \tag{2.6}$$

where r_a is the unique solution in $(0, 1]$ of the equation

$$1-3r+3(1-2a)r^2-(1-2a)r^3 = 0.$$

Combining (2.5) and (2.6) immediately yields (2.1). The nature of the extremal functions follows from remarks in [6].

3. Applications.

Theorem 2. *Let* $f(z) = z + \sum_{m=n+1}^{\infty} a_m z^m$, $n \geqslant 1$, *be analytic in* $|z| < 1$ *and satisfy* $\operatorname{Re}\{f(z)/z\} > a$. *Let* a_0 *be defined by the equation*

$$\frac{a_0}{1-a_0} = \frac{(\sqrt{3n^2+1}-1)^2}{9n}.$$

Then $f(z)$ *is starlike and univalent in* $|z| < \varrho_{a,n}$ *where* $\varrho_{a,n}$ *is the unique solution in* $(0,1)$ *of the equation*

$$1 + [2a - 2n(1-a)]r^n - (1-2a)r^{2n} = 0 \tag{3.1}$$

when $a \leqslant a_0$ *and* $\varrho_{a,n}$ *is the unique solution in* $(0,1)$ *of the equation*

$$(1-a)(1-r^{2n}) - n[\sqrt{1+(1-2a)r^{2n}} - \sqrt{a(1-r^{2n})}]^2 = 0 \tag{3.2}$$

when $a > a_0$. *This result is sharp for all permissible values of* a *and* n.

Proof. If we let $p(z) = f(z)/z$ then $p(z)$ is in $P(a, n)$ and, applying theorem 1, we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1 + \frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} F_1(r), & 0 \leqslant r \leqslant r_{a,n} \\ F_2(r), & r_{a,n} < r < 1 \end{cases}$$

where

$$F_1(r) = \frac{1 + [2a - 2n(1-a)]r^n - (1-2a)r^{2n}}{(1+r^n)[1-(1-2a)r^n]},$$

$$F_2(r) = \frac{(1-a)(1-r^{2n}) - n[\sqrt{1+(1-2a)r^{2n}} - \sqrt{a(1-r^{2n})}]^2}{(1-a)(1-r^{2n})},$$

and $r_{a,n}$ is the solution of (2.2). If we define $F(r)$ by $F(r) = F_1(r)$ on $[0, r_{a,n}]$ and $F(r) = F_2(r)$ on $(r_{a,n}, 1)$, then the radius of starlikeness of $f(z)$ will be at least as large as the first zero of $F(r)$. $F(r)$ is continuous and decreasing on $[0,1)$, $F(0) = 1$, and $F(r) \to -\infty$ as $r \to 1^-$, hence $F(r)$ has a unique zero in $(0,1)$ which we will denote by $\varrho_{a,n}$. It follows then that $f(z)$ is starlike for $|z| < \varrho_{a,n}$. For a given a and n we must now determine if $\varrho_{a,n}$ is the solution of $F_1(r) = 0$ or of $F_2(r) = 0$. It is always true that $F_1(r_{a,n}) = F_2(r_{a,n})$. If we assume that this common value is also zero, i.e. $F_1(r_{a,n}) = 0$ and $F_2(r_{a,n}) = 0$ where $r_{a,n}$ satifies (2.2), then after eliminating $r_{a,n}$ from these equations we find that a and n satisfy the following equation:

$$\frac{a}{1-a} = \frac{1}{9n}(\sqrt{3n^2+1}-1)^2.$$

For a given n let a_0 be defined by (3.3). An examination of (2.2) shows that $r_{a,n}$ is a decreasing function of a if n is fixed. Hence if $a < a_0$ then $r_{a,n} > r_{a_0,n}$ and, since $F_1(r)$ is a decreasing function of r, it follows that $F_1(r_{a,n}) < F_1(r_{a_0,n}) = 0$. This implies that $\varrho_{a,n} < r_{a,n}$ and therefore $\varrho_{a,n}$ is given by (3.1). Similarly if $a > a_0$ then $\varrho_{a,n} > r_{a,n}$ and $\varrho_{a,n}$ is given by (3.2). Of course when $a = a_0$ then $\varrho_{a,n} = r_{a,n}$ and $\varrho_{a,n}$ is given by either (3.1) or (3.2).

Equality can occur in (3.1) when $f(z) = zp(z)$ and $p(z)$ is defined by (2.3) and in (3.2) when $p(z)$ is defined by (2.4). In either case $f'(z)$ has a zero on $|z| = \varrho_{a,n}$, so $\varrho_{a,n}$ is the radius of starlikeness and radius of univalence. This completes the proof.

Now let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disc and define

$$f_k(z) = z + \sum_{n=1}^{\infty} a_{kn+1} z^{kn+1}, \quad k = 2, 3, \ldots$$

In [4] Sakaguchi showed that if $f(z)$ is convex then $f_2(z)$ is starlike. The following theorem determines the radius of starlikeness for $f_k(z)$, $k \geqslant 4$.

Theorem 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a convex univalent function in $|z| < 1$ and $f_k(z)$ is defined by (3.4), then $f_3(z)$ is starlike for $|z| < (3\sqrt{3}-5)^{1/6}$ and $f_k(z)$ is starlike for $|z| < (k-1)^{-1/k}$, $k = 4, 5, \ldots$ This result is sharp for $k \geqslant 4$.*

Proof. It follows from (3.4) that

$$f_k(z) = (1/k) \sum_{j=0}^{k-1} \bar{\omega}^j f(\omega^j z)$$

where $\omega^k = 1$, $\omega \neq 1$. Using Strohhäcker's well known result [5] that $\mathrm{Re}\{f(z)/z\} > \frac{1}{2}$, we have

$$\mathrm{Re}\{f_k(z)/z\} = (1/k) \sum_{j=0}^{k-1} \mathrm{Re}\{f(\omega^j z)/(\omega^j z)\} > \tfrac{1}{2},$$

hence $f_k(z)/z$ is in $P(1/2, k)$. Applying theorem 2 and noting that $1/2 \leqslant a_0$ if and only if $k \geqslant 4$ yields the desired result. If $f(z) = z/(1-z)$ then $f_k(z) = z/(1-z^k)$ and $f_k'(-(k-1)^{-1/k}) = 0$, hence, for $k \geqslant 4$, $f(z)$ need not be starlike or univalent in any larger disc.

In [3] Robertson showed that if $f(z)$ is convex in the direction of the imaginary axis and has real coefficients or if $f(z)$ is an odd starlike function, then $\mathrm{Re}\{f(z)/z\} > 1/2$, hence we immediately have the following

Theorem 4. *If $f(z)$ is convex in the direction of the imaginary axis and has real coefficients then $f_2(z)$ is starlike for $|z| < (8\sqrt{2}-11)^{1/4}$, $f_3(z)$ is starlike for $|z| < (3\sqrt{3}-5)^{1/6}$, and $f_k(z)$ is starlike for $|z| < (k-1)^{-1/k}$ when $k \geqslant 4$. This result is sharp for $k \geqslant 4$.*

Theorem 5. *If $f(z)$ is an odd starlike function, then $f_k(z)$ is starlike for $|z| < (k-1)^{-1/k}$, $k = 2, 4, 6, \ldots$ This result is sharp.*

Notice that if $f(z)$ is an odd function and k is an odd integer, then $f_k(z) = f_{2k}(z)$, so we need only consider k even. In particular, $f_2(z) = f(z)$, so the radius of starlikeness of $f_2(z)$ is 1. For $k = 4, 6, \ldots$ the result follows from theorem 2 and the extremal function is $f(z) = z/(1-z^2)$.

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STRESZCZENIE

W pracy wyznaczono minimum $\text{Re}\{zp'(z)/p(z)\}$ dla funkcji $p(z)$ klasy Carathéodory'ego postaci $p(z) = 1 + p_n z^n + \ldots$, $\text{Re}\{p(z)\} > a$, $0 \leqslant a < 1$, $|z| < 1$. Wynik ten zastosowano do kilku klas funkcji analitycznych. W szczególności uzykano uogólnienie wyniku Sakaguchiego, dotyczącego k-symetrycznych funkcji wypukłych.

РЕЗЮМЕ

В этой работе получено минимум $\text{Re}\{zp'(z)/p(z)\}$ для функций $p(z)$ класса Каратеодори вида $p(z) = 1 + p_n z^n + \ldots$,

$$\text{Re}\{p(z)\} > a,\ 0 \leqslant a < 1,\ |z| < 1.$$

Этот результат применено к некоторым классам аналитических функций. В частности получено обобщение результата Сакагучи, относящегося к к-симметрическим выпуклым функциям.