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The Podkovyrin's Connections with a Torsion

Koneksje Podkowyrina ze skręceniem

Связности Подковырина с кручением

We consider the structure (M, e, b, a) where: M is differentiable manifold of dimension $n = 2m$, e is the tensor field of the type $(1, 1)$ such that:

$$e\colon TM \to TM$$

with

$$(1) \qquad e \cdot e = \omega I,$$

where $\omega = +1$, or $\omega = -1$, $I = \mathrm{id}_{TM}$, b is the field of symmetric corelations (i.e. a tensor field of type $(0, 2)$ which satisfies the condition:

$$(2) \qquad b(u, e(v)) = b(v, e(u)), \qquad u, v \in F^1;$$

a is a covector field. Moreover, we assume that $u \to b(u, -)$ is an inversible function.

Theorem 1. *Given a point of the manifold M then there exists a frame R, such that the matrix $e^i_{\cdot j}$ of the components of the tensor e takes the form*

$$(e^i_{\cdot j}) = \left(\begin{array}{c|c} 0 & E \\ \hline \omega E & 0 \end{array}\right)$$

Proof. In fact, at the point $x_0 \in M$ this frame may be defined in the following way. Let x_1 be an arbitrary vector in x_0. We set $ex_1 = x_{m+1}$. The vectors x_1, x_{m+1} are linearly independent and they spanned a 2-dimensional space P_2. The next step: at the point x_0 we choose a pair of vectors x_2 and $x_{m+2} = ex_2$ where x_2 is b-orthogonal to P_2. Thus we obtain four linearly independent vectors $x_1, x_2, x_{m+1}, x_{m+2}$ which have spanned a 4-dimensional space P^1_4. Then, we choose next pair of vectors x_3, x_{m+3}, where $x_3 \notin P_4$ and $x_{m+3} = ex_3 \notin P_4$. Now we have six linearly independent vectors $x_1, x_2, x_3, x_{m+1}, x_{m+2}, x_{m+3}$ generalizing the space P_6. By a prolongation this process step-by-step we obtain the frame $R(x_0, x_1, \ldots \ldots x_m, x_{m+1}, \ldots x_{2m})$ in which the components $e^{i}_{.j}$ of the tensor e have the form:

$$(e^{i}_{.j}) = \left(\begin{array}{c|c} e^{\lambda}_{\mu} & e^{\bar\lambda}_{\mu} \\ \hline e^{\lambda}_{\bar\mu} & e^{\bar\lambda}_{\bar\mu} \end{array}\right),$$

$$\lambda, \mu = 1, 2, \ldots, m, \bar\lambda, \bar\mu = 1, 2, \ldots, m,$$

where

(3) $$e^{\lambda}_{\mu} = 0, e^{\bar\lambda}_{\mu} = \delta^{\bar\lambda}_{\mu}, e^{\lambda}_{\bar\mu} = \omega\,\delta^{\lambda}_{\bar\mu}, e^{\bar\lambda}_{\bar\mu} = 0.$$

(The symbol δ^{α}_{β} also denotes Kronecker delta, with

$$\delta^{\bar\lambda}_{\mu} = \delta^{\lambda}_{\bar\mu} = 0, \ \delta^{\lambda}_{\mu} = \delta^{\bar\lambda}_{\mu}, \ \delta^{\bar\lambda}_{\bar\mu} = \delta^{\bar\lambda}_{\mu}).$$

Then we introduce the operators

(4)

(a) $$\Omega = \frac{1}{2}(I \otimes I + b \otimes \check{b}),$$

(b) $$\Omega^* = \frac{1}{2}(I \otimes I - b \otimes \check{b}),$$

where $\check{b}$ is the inverse corelation with respect to b. These operators were introduced by M. Obata [3]. By a direct computation we obtain the following:

Lemma 1.

(5) $$\Omega \cdot \Omega = \Omega, \ \Omega^* \cdot \Omega^* = \Omega^*, \ \Omega \cdot \Omega^* = \Omega^* \cdot \Omega = 0.$$

Corollary.

$$\ker \Omega = \operatorname{im} \Omega^*, \ \ker \Omega^* = \operatorname{im} \Omega, \ \ker \Omega^* \cap \ker \Omega = \{0\}.$$

Denote by F^1_1 the moduli of tensor fields of type $(1, 1)$ on M.

Proposition 1.

$$F^1_1 = \ker \Omega \oplus \ker \Omega^*.$$

Proof.

Let $v \in F^1_1$. We may assume, that $v = x + y$, where $x = \Omega^* v$, $y = v - \Omega^* v \in \ker \Omega$. It follows that $v \in \ker \Omega \oplus \ker \Omega^*$ by corollary.

Denote by L the Lie algebra of $GL(n, R)$.

Proposition 2. *Let V be an L-valued 1-form. Then the tensor equation of the form*

$$\Omega X = V, \tag{6}$$

in which X is an unknown tensor of the same type as V, has a solution if and only if

$$\Omega^* V = 0. \tag{7}$$

A general solution is of the form

$$X = V + \Omega^* U, \tag{8}$$

where U is an arbitrary linear L-valued form.

In virtue of (2) we have $c_{ij} = c_{ji}$, and moreover for the matrix of the components c^{is} of the inverse tensor $\check{c}$ we have $c^{is} := \omega b^{ki} e^{s}_{.k}$.

Then we look for a most general connection ∇ on the manifold M, which satisfies the conditions

$$\nabla e = 0 \tag{9}$$

and

$$(\nabla_v b)(u, w) = a(v)\, b(u, e(w)), \quad u, v, w \in F^1. \tag{10}$$

We call them Podkovyrin connections.

Theorem 2. *Local components ω^s_i of a Podkovyrin connections are of the form*

$$\omega^s_i = \frac{1}{4}\left(e^s_{.r}\, de^r_{.i} + c^{qs}\, db_{iq} - 2A e^s_{.i} + b^{rs}\, db_{ri} + (e^s_{.r} e^p_{.i} - c_{ri} c^{ps}) A^r_p\right), \tag{11}$$

where $s, i, r, \ldots = 1, 2, \ldots, n$, b_{ij} and b^{is} are the local components of the tensors b and $\check{b}$ respectively, and $c_{ij} := e^k_{.j} b_{ik}$.

We assume $A = a_k dx^k$, where a_k — are components of a vector field and A^r_p are components of an arbitrary linear form valued in a Lie algebra.

Proof.

The formulas (9) and (10) in a holonomic field of frames take the form:

$$\nabla_k e^i_{.j} = 0, \tag{9'}$$

$$\nabla_k b_{ij} = a_k b_{is} e^s_{.j}. \tag{10'}$$

If we write the left hand member of (10′) in the expanded form and we pass to forms we have:

$$db_{ij} = b_{is}\omega^s_j + b_{sj}\omega^s_i + A b_{is} e^s_{.j}.$$

Multiply this equality by b^{ir} and divide by 2, we got

$$\tag{12} \frac{1}{2}(\delta^r_s \delta^i_j + b_{sj} b^{ir})\omega^s_i = \frac{1}{2}(b^{ir} db_{ij} - Ae^r_{\cdot j}).$$

In the bracket on the left hand side of (12) there are just the components Ω^{ri}_{sj} of the Obata operator (4a). Thus the components Ω^{*ri}_{sj} of the operator (4b) take the form:

$$\Omega^{*ri}_{sj} = \frac{1}{2}(\delta^r_s \delta^i_j - b_{sj} b^{ir}).$$

It is easy to verify that the formulas (5) hold well. This means that the equation (12) is the tensor equation. A solution of the equation of (12) is the following:

$$\tag{13} \omega^s_i = \frac{1}{2}\left(b^{ks} db_{ki} - Ae^s_{\cdot i} + (\delta^s_k \delta^l_i - b_{ki} b^{ls})\tilde{\omega}^k_l\right),$$

where $\tilde{\omega}^k_l$ is an arbitrary linear L-valued form. Let's turn to the equation (9′). It is equivalent to the following:

$$e^k_{\cdot j}\omega^i_k - e^i_{\cdot k}\omega^k_j = -de^i_{\cdot j}.$$

We contract both members of this equation by $e^j_{\cdot h}$. In view of $\omega^2 = 1$ we have:

$$\tag{14} (\delta^s_h \delta^i_k - \omega e^i_{\cdot k} e^s_{\cdot h})\omega^k_s = -\omega e^j_{\cdot h} de^i_{\cdot j}.$$

As (9′) and (10′) are to be satisfied simultanously, so the right hand member of (13) should satisfy (14). Then we have:

$$\frac{1}{2}(\delta^s_h \delta^i_k - \omega e^i_{\cdot k} e^s_{\cdot h})\left(b^{rk} db_{rs} - Ae^k_{\cdot s} + (\delta^k_p \delta^l_s - b_{ps} b^{lk})\tilde{\omega}^p_l\right) = \omega e^i_{\cdot j} de^j_{\cdot h}.$$

Thus we have to solve the following equation:

$$\tag{15} \frac{1}{2}(\delta^l_h \delta^i_p - b_{ph} b^{li} - \omega e^i_{\cdot p} e^l_{\cdot h} + c^{li} c_{ph})\tilde{\omega}^p_l = -\frac{1}{2} b^{ri} db_{rh} + \frac{1}{2} c^{ri} e^s_{\cdot h} db_{rs} + \omega e^i_{\cdot j} de^j_{\cdot h}.$$

We shall show that if $\omega = 1$ then the expression in the bracket on the left hand member of (15) is an Obata operator i.e. if we denote it by $\tilde{\Omega}$, then it may be expressed in the form (4a) or (4b) in the following way:

$$\tilde{\Omega} = \frac{1}{2}(I \otimes I - B \otimes \check{B}),$$

where the components of the product $B\otimes\check{B}$ are of the form:

$$B^{li}_{ph} = b_{ph}b^{li} + \omega e^{i}_{.p}e^{l}_{.h} - c_{li}c_{ph}.$$

Denote by $\tilde{\Omega}^*$ the operator

$$\tilde{\Omega}^* = \frac{1}{2}(I\otimes I + B\otimes\check{B}).$$

Lemma 2. *In a case* $\omega = 1$ *the operators* $\tilde{\Omega}$ *and* $\tilde{\Omega}^*$ *satisfy* (5).

Proof.

Let's find a mapping

$$P\colon F^1_1 \to F^1_1$$
$$[X^l_s] \mapsto [\tilde{\Omega}^{li}_{hs}\tilde{\Omega}^{*hr}_{qi}X^q_r].$$

P is of the form:

$$P^{lr}_{pq}X^p_r = \frac{1}{2}(\delta^r_p\delta^l_q - \omega e^l_{.q}e^r_{.p} + c_{pq}c^{lr} - b_{pq}b^{lr})X^p_r.$$

Let us find the kernel of this mapping. Thus it suffices to find a solution of the following system:

$$(\delta^r_p\delta^l_q - \omega e^l_{.q}e^r_{.p} + c_{pq}c^{lr} - b_{pq}b^{lr})X^q_r = 0. \tag{16}$$

Making use of the theorem 1 and of formulas (3) we may write the system (16) as four groups of systems of equations:

$$\begin{aligned}
&\text{(a)}\ (\delta^\alpha_\mu\delta^\lambda_\beta - \omega e^\lambda_{.\beta}e^\alpha_{.\mu} + c_{\mu\beta}c^{\alpha\lambda} - b_{\mu\beta}b^{\alpha\lambda})X^\beta_\alpha +\\
&+(\delta^\alpha_\mu\delta^\lambda_{\bar\beta} - \omega e^\alpha_{.\mu}e^\lambda_{.\bar\beta} + c_{\mu\bar\beta}c^{\alpha\lambda} - b_{\mu\bar\beta}b^{\alpha\lambda})X^{\bar\beta}_\alpha +\\
&+(\delta^{\bar\alpha}_\mu\delta^\lambda_\beta - \omega e^{\bar\alpha}_{.\mu}e^\lambda_{.\beta} + c_{\mu\beta}c^{\bar\alpha\lambda} - b_{\mu\beta}b^{\bar\alpha\lambda})X^\beta_{\bar\alpha} +\\
&+(\delta^{\bar\alpha}_\mu\delta^\lambda_{\bar\beta} - \omega e^{\bar\alpha}_{.\mu}e^\lambda_{\bar\beta} + c_{\mu\bar\beta}c^{\bar\alpha\lambda} - b_{\mu\bar\beta}b^{\bar\alpha\lambda})X^{\bar\beta}_{\bar\alpha} = 0\\
&\text{(b)}\ (\delta^\alpha_\mu\delta^{\bar\lambda}_\beta - \omega e^\alpha_{.\mu}e^{\bar\lambda}_{.\beta} + c_{\mu\beta}c^{\alpha\bar\lambda} - b_{\mu\beta}b^{\alpha\bar\lambda})X^\beta_\alpha +\\
&+(\delta^\alpha_\mu\delta^{\bar\lambda}_{\bar\beta} - \omega e^\alpha_{.\mu}e^{\bar\lambda}_{.\bar\beta} + c_{\mu\bar\beta}c^{\alpha\bar\lambda} - b_{\mu\bar\beta}b^{\alpha\bar\lambda})X^{\bar\beta}_\alpha +\\
&+(\delta^{\bar\alpha}_\mu\delta^{\bar\lambda}_\beta - \omega e^{\bar\alpha}_{.\mu}e^{\bar\lambda}_{.\beta} + c_{\mu\beta}c^{\bar\alpha\bar\lambda} - b_{\mu\beta}b^{\bar\alpha\bar\lambda})X^\beta_{\bar\alpha} +\\
&+(\delta^{\bar\alpha}_\mu\delta^\lambda_{\bar\beta} - \omega e^{\bar\alpha}_{.\mu}e^{\bar\lambda}_{.\bar\beta} + c_{\mu\bar\beta}c^{\bar\alpha\bar\lambda} - b_{\mu\bar\beta}b^{\bar\alpha\lambda})X^{\bar\beta}_{\bar\alpha} = 0\\
&\text{(c)}\ (\delta^\alpha_{\bar\mu}\delta^\lambda_\beta - \omega e^\alpha_{.\bar\mu}e^\lambda_{.\beta} + c_{\bar\mu\beta}c^{\alpha\lambda} - b_{\bar\mu\beta}b^{\alpha\lambda})X^\beta_\alpha +\\
&+(\delta^\alpha_{\bar\mu}\delta^\lambda_{\bar\beta} - \omega e^\alpha_{.\bar\mu}e^\lambda_{.\bar\beta} + c_{\bar\mu\bar\beta}c^{\alpha\lambda} - b_{\bar\mu\bar\beta}b^{\alpha\lambda})X^{\bar\beta}_\alpha +\\
&+(\delta^{\bar\alpha}_{\bar\beta}\delta^\lambda_{\bar\beta} - \omega e^{\bar\alpha}_{.\bar\mu}e^\lambda_{.\bar\beta} + c_{\bar\mu\bar\beta}c^{\bar\alpha\lambda} - b_{\bar\mu\bar\beta}b^{\bar\alpha\lambda})X^{\bar\beta}_{\bar\alpha} +\\
&+(\delta^{\bar\alpha}_{\bar\mu}\delta^\lambda_\beta - \omega e^{\bar\alpha}_{.\bar\mu}e^\lambda_{.\beta} + c_{\bar\mu\beta}c^{\bar\alpha\lambda} - b_{\bar\mu\beta}b^{\bar\alpha\lambda})X^\beta_{\bar\alpha} = 0\\
&\text{(d)}\ (\delta^\alpha_{\bar\mu}\delta^{\bar\lambda}_\beta - \omega e^\alpha_{.\bar\mu}e^{\bar\lambda}_\beta + c_{\bar\mu\beta}c^{\alpha\bar\lambda} - b_{\bar\mu\beta}b^{\alpha\bar\lambda})X^\beta_\alpha +\\
&+(\delta^\alpha_{\bar\mu}\delta^{\bar\lambda}_{\bar\beta} - \omega e^\alpha_{.\bar\mu}e^\lambda_{.\bar\beta} + c_{\bar\mu\bar\beta}c^{\alpha\lambda} - b_{\bar\mu\bar\beta}b^{\alpha\bar\lambda})X^{\bar\beta}_\alpha +\\
&+(\delta^{\bar\alpha}_{\bar\mu}\delta^{\bar\lambda}_\beta - \omega e^{\bar\alpha}_{.\bar\mu}e^\lambda_{.\beta} + c_{\bar\mu\beta}c^{\bar\alpha\bar\lambda} - b_{\bar\mu\beta}b^{\bar\alpha\bar\lambda})X^\beta_{\bar\alpha} +\\
&+(\delta^{\bar\alpha}_{\bar\mu}\delta^\lambda_\beta - \omega e^{\bar\alpha}_{.\bar\mu}e^{\bar\lambda}_{.\beta} + c_{\bar\mu\bar\beta}c^{\bar\alpha\bar\lambda} - b_{\bar\mu\bar\beta}b^{\alpha\bar\lambda})X^{\bar\beta}_\alpha = 0
\end{aligned}\tag{17}$$

which yields

$$
\begin{aligned}
&\text{(a)}\ (\omega-1)(b_{\mu\bar\beta}b^{\alpha\lambda}X^{\bar\beta}_{\alpha}+b_{\mu\bar\beta}b^{\bar\alpha\lambda}X^{\bar\beta}_{\bar\alpha})=0,\\
&\text{(b)}\ (\omega-1)(b_{\mu\beta}b^{\alpha\bar\lambda}X^{\beta}_{\alpha}-X^{\bar\lambda}_{\mu}+b_{\mu\beta}b^{\bar\alpha\bar\lambda}X^{\beta}_{\bar\alpha})=0,\\
(18)\quad&\text{(c)}\ (\omega-1)(b_{\mu\beta}b^{\alpha\lambda}X^{\bar\beta}_{\alpha}-X^{\lambda}_{\bar\mu}+b_{\bar\mu\bar\beta}b^{\bar\alpha\lambda}X^{\bar\beta}_{\bar\alpha})=0,\\
&\text{(d)}\ (\omega-1)(b_{\bar\mu\beta}b^{\alpha\bar\lambda}X^{\beta}_{\alpha}+b_{\bar\mu\beta}b^{\bar\alpha\bar\lambda}X^{\beta}_{\bar\alpha})=0.
\end{aligned}
$$

If $\omega=1$ then these equation are satisfied identically, now then $\tilde\Omega\cdot\tilde\Omega^*=\tilde\Omega^*\cdot\tilde\Omega=0$ holds. Thus (5) is satisfied. In the case $\omega=1$, for the equation (15), the condition (7) holds well. In fact, we have:

$$\tilde\Omega^{*li}_{hr}\left(-\frac{1}{2}b^{rh}db_{ri}+\frac{1}{2}c^{rh}e^{s}_{\cdot i}db_{rs}+e^{h}_{\cdot r}de^{r}_{\cdot i}\right)=\frac{1}{2}(-b^{rl}db_{rp}+c^{rl}e^{s}_{\cdot p}db_{rs}).$$

If we split this expression into four groups of indices and we make use of (3) then we obtain the identity:

$$-b^{rl}db_{rp}+c^{rl}e^{s}_{\cdot p}db_{rs}=0.$$

In power of the proposition 2 a solution of (15) for $\omega=1$ is

$$
\begin{aligned}
(19)\qquad \tilde\omega^{p}_{q}=\frac{1}{2}\big(&-b^{rp}db_{rq}+c^{rp}e^{s}_{\cdot q}db_{rs}+e^{p}_{\cdot r}de^{r}_{\cdot q}+\\
&+(\delta^{s}_{q}\delta^{p}_{r}+b^{sp}b_{rq}+e^{p}_{\cdot r}e^{s}_{\cdot q}-c_{rq}c^{sp})A^{r}_{s}\big),
\end{aligned}
$$

where A^r_s is an arbitrary linear L-valued form. By substituting (19) into (13) we get (11). This is a most general connection, satisfying (9) and (10).

A torsion tensor T^{s}_{ji}, expresses by means of the following formulas in a holonomic field of frames

$$
\begin{aligned}
T^{s}_{ji}=\tfrac{1}{2}\big(&e^{s}_{\cdot r}(\partial_j e^{r}_{\cdot i}-\partial_i e^{r}_{\cdot j})+c^{rs}(\partial_j c_{ri}-\partial_i c_{rj})+\\
&+b^{rs}(\partial_j b_{ri}-\partial_i b_{rj})+2(a_i e^{s}_{\cdot j}-a_j e^{s}_{\cdot i})+\\
&+(e^{s}_{\cdot r}e^{p}_{\cdot i}-c_{ri}c^{rs})A^{r}_{jp}-(e^{s}_{\cdot r}e^{p}_{\cdot j}-c_{rj}c^{ps})A^{r}_{ip}\big).
\end{aligned}
$$

Theorem 3. *If A^r_{jp} is any skew-symmetric tensor satisfying the conditions*

$$(20)\qquad c_{ri}c^{ps}A^{r}_{jp}=-c_{rj}c^{ps}A^{r}_{ip}$$

and

$$(21)\qquad A^{t}_{[ji]}=b^{rt}\partial_{[i}b_{j]r}+a_{[j}\delta^{t}_{i]}+\partial_{[i}e^{t}_{j]},$$

then connection which is expressed by (11) *is a torsionless connection.*

Proof. Let us introduce the tensor A^r_{ip} by means of the torsion tensor T^s_{ji} provided that A^r_{kp} satisfies (20). Then, we have

$$
\begin{aligned}
e^{s}_{\cdot r}(e^{p}_{\cdot i}A^{r}_{jp}-e^{p}_{\cdot j}A^{r}_{ip})=e^{s}_{\cdot r}(\partial_j e^{r}_{\cdot i}-\partial_i e^{r}_{\cdot j})-&\\
-c^{rs}(b_{iq}\partial_j e^{q}_{\cdot r}-b_{jq}\partial_i e^{q}_{\cdot r})-b^{rs}(\partial_j b_{ri}-\partial_i b_{rj})-&\\
-a_i e^{s}_{\cdot j}+a_j e^{s}_{\cdot i}-2T^{s}_{ji}.&
\end{aligned}
$$

By a contraction of both members by $e^{t}_{\cdot s}$ and writing the obtained equalitions in four groups of indices and making use of (3) we get

$$A^{t}_{[ji]} = b^{rt}\partial_{[i}b_{j]r} + a_{[j}\delta^{t}_{i]} + \partial_{[i}e^{t}_{\cdot j]} - 2T^{t}_{ji}.$$

Then it suffices to put any skew-symmetric tensor satisfying (20) and (21) instead of A^{t}_{ji}. Thus we obtain a connection which is torsionless.

Remark 1. Podkovyrin considers some special surfaces in a biplanar space [7] of even dimension. He gives a construction of a connection for which the given tensor e is parallel (9′). Then the two components of the corresponding immersion tensor b, c, are non-degenerated and they satisfy the relations

$$c_{ij} = b_{ik}e^{k}_{\cdot j},$$

(10′) $$\nabla_k b_{ij} = a_k c_{ij},$$

(*) $$\nabla_k c_{ij} = \omega a_k b_{ij}.$$

There is also introduced a complex tensor B, where

(22) $$B_{ij} = b_{ij} + \varkappa c_{ij}$$

and $\varkappa = \sqrt{\omega}$. B is of rank ($\frac{1}{2}$ rank b). The formulas (*) are in a formal analogy with the conditions for a connection to be a Weyl one. But there is no angle-like invariant so that B would be used for a parallel transport of this invariant.

If a connection satisfies (10′) and, simultanously

(23) $$\nabla_k(\lambda b_{ij}) = \check{a}_k(\lambda c_{ij}),$$

λ being a real scalar function, then λ must be a constant. In fact, we have from (23)

$$\nabla_k(\lambda b_{ij}) = \lambda\check{a}_k c_{ij}$$

or

$$(\nabla_k\lambda)b_{ij} + \lambda\nabla_k b_{ij} = \lambda\check{a}_k c_{ij}.$$

From (10′) we have:

$$(\partial_k\lambda)b_{ij} + \lambda a_k c_{ij} = \lambda\check{a}_k c_{ij}.$$

Hence

$$\lambda\big((\partial_k \ln\lambda)b_{ij} + a_k c_{ij}\big) = \lambda\check{a}_k c_{ij},$$

or

$$\lambda\big((\partial_k \ln\lambda)e^{s}_{\cdot j} + a_k\delta^{s}_{j}\big)c_{is} = \lambda\check{a}_k c_{ij}.$$

A contraction of this equality by c^{ip}, yields

$$\lambda\big((\partial_k \ln\lambda)e^{p}_{\cdot j} + a_k\delta^{p}_{j}\big) = \lambda\check{a}_k\delta^{p}_{j}.$$

Hence

$$\delta_j^p \check{a}_k = (\partial_k \ln\lambda) e^p_{.j} + a_k \delta_j^p .$$

Because we have $e^p_{.p} = 0$ then

$$n\cdot\check{a}_k = na_k$$

Hence

$$\check{a}_k = a_k$$

Hence we conclude that $\lambda = \text{const.}$ By similar reason the tensor $B_{ij} = b_{ij} + \sqrt{\omega} c_{ij}$ considered in [7] can not be used for measuring angles of tangent vectors.

Remark 2. In the paper [7] there is defined a connection by its coefficients

$$\tilde{\Gamma}^k_{ij} = G^k_{ij} - \tfrac{1}{2}(a_i e^k_{.j} + a_j e^k_{.i} - a_s b^{sk} c_{ij}), \tag{24}$$

where G^k_{ij} are Christoffels of b. These coefficients do not satisfy (9′). There is considered a special case, namely, if the components a_k satisfy the condition

$$\tilde{a}_i = \omega e^k_i a_k = \partial_i \theta . \tag{**}$$

Such a field is called a solenoid one. (**) implies a posibility of finding certain new tensors h and $\tilde{h}$ such that it holds

$$\tilde{h}_{ij} = h_{ik} e^k_{.j}$$
$$h_{ij} = \tilde{h}_{ik} e^k_{.j}$$

and

$$b_{ij} = e^{-\theta} h_{ij}$$
$$c_{ij} = e^{-\theta} \tilde{h}_{ij} \tag{25}$$

and

$$\nabla_k h_{ij} = \tilde{a}_k h_{ij} + a_k \tilde{h}_{ij}$$
$$\nabla_k \tilde{h}_{ij} = \tilde{a}_k \tilde{h}_{ij} + \omega a_k h_{ij} \tag{26}$$

where $\tilde{a}_k = e^p_{.k} a_p$. Thus there may be computed the coefficients of a connection $\mathring{\Gamma}^k_{ij}$:

$$\mathring{\Gamma}^k_{ij} = \mathring{G}^k_{ij} - \tfrac{1}{2}(\tilde{a}_i \delta^k_j + \tilde{a}_j \delta^k_i + a_i e^k_{.j} + a_j e^k_{.i}) + \tfrac{1}{2}\tilde{a}_s (h^{sk} h_{ij} + \tilde{h}^{sk} \tilde{h}_{ij}), \tag{27}$$

where $\mathring{G}^k_{ij}$ are Christoffels of h. These satisfy (9′) with h in a place of b.

Now there arises the following question: what conditions are to be satisfied, that the connection determined by (11) is the canonical

Podkovyrin connection (27). By substituting (**) and (25) into (27) we obtain

$$\mathring{\Gamma}^k_{ij} = G^k_{ij} - \tfrac{1}{2}(a_i e^k_{\cdot j} + a_j e^k_{\cdot i} - a_p b^{pk} c_{ij}), \tag{28}$$

where G^k_{ij} are of Christoffels with respect to b_{ij}. In virtue of (11), we have

$$\Gamma^k_{ij} = \tfrac{1}{2}(e^k_{\cdot r}\partial_i e^r_{\cdot j} + c^{rk} b_{jq}\partial_i e^q_{\cdot r} - a_i e^k_{\cdot j} + b^{rk}\partial_i b_{rj} + (e^k_{\cdot r} e^r_{\cdot j} - c_{rj} c^{pk}) A^r_{ip}). \tag{29}$$

By comparing right nunbers (28) and (29), we get

$$(e^k_{\cdot r} e^p_{\cdot j} - c_{rj} c^{pk}) A^r_{ip} = + a_p b^{pk} c_{ij} - e^k_{\cdot r}\partial_i e^r_{\cdot j} - c^{rk} b_{jq}\partial_i e^q_{\cdot r}. \tag{30}$$

By a contraction these equations by $\frac{1}{2} e^t_{\cdot k} e^j_{\cdot s}$ we obtain

$$\tfrac{1}{2}(\delta^t_r \delta^p_s - b_{rs} b^{pt}) A^r_{ip} = \tfrac{1}{2}(c^{pt} e^j_{\cdot s}\partial_j b_{ip} - c^{pt} e^j_{\cdot s}\partial_p b_{ij} - a_j e^j_{\cdot s}\delta^t_i + a_p c^{pt} b_{is} - e^j_{\cdot s}\partial_i e^t_{\cdot j} - b^{rt} c_{qs}\partial_i e^q_{\cdot r}). \tag{31}$$

This is a tensor equation of the type (6), which satisfies (7). Then a solution of (31) is of the form

$$A^t_{is} = \tfrac{1}{2}(c^{pt} e^j_{\cdot s}\partial_j b_{ip} - c^{pt} e^j_{\cdot s}\partial_p b_{ij} - a_j e^j_{\cdot s}\delta^t_i + a_p c^{pt} b_{is} - e^j_{\cdot s}\partial_i e^t_{\cdot j} - b^{rt} c_{qs}\partial_i e^q_{\cdot r} + (\delta^t_r \delta^p_s - b_{rs} b^{pt}) U^r_{ip}), \tag{32}$$

where U^r_{ip} is an arbitrary tensor of the type (1, 2).

Proposition 3. *If A^t_{is} is defined by* (32), *then the connection* (11) *is the canonical Podkovyrin connection.*

Remark 3. In a case $\omega = 1$, or in a complex case, equalities (18) imply directly $X^t_s = 0$. Also in this case there exists a unique connection which is consistent with our structure.

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STRESZCZENIE

Rozpatrzmy strukturę postaci (M, e, b, a), gdzie M jest rozmaitością różniczkowalną wymiaru $2n$, e jest polem tensorowym typu (1,1), takim że $e \cdot e = \varepsilon I$, przy czym $\varepsilon^2 = 1$, a I jest tensorem jednostkowym, b jest polem symetrycznych korelacji spełniających warunek $b(u, e(v)) = b(v, e(u))$, a jest polem kowektorów. Zakładamy ponadto, że korelacja $u \to b(u, -)$ jest odwracalna i korelację odwrotną oznaczamy symbolem $\check{b}$.

W pracy tej znajdujemy ogólną postać koneksji Podkowyrina, oraz wyliczamy ich skręcenia. Lokalne współrzędne ω_i^s otrzymanej koneksji są postaci

$$\omega_i^s = \tfrac{1}{4}[e_r^s de_i^r + c^{qs} db_{iq} - 2Ae_i^s + b^{rs} db_{ri} + (e_r^s e_i^p - c_{ri} c^{ps}) A_p^r]$$

gdzie $a = a_k dx^k$, A_p^r są współrzędnymi dowolnej formy liniowej o wartościach w algebrze Lie'go liniowej grupy L^n.

РЕЗЮМЕ

Рассмотрим структуру вида $(M, e, в, a)$, где M является дифференциальным многообразием размерности $2n$, e является тензоровым полем типа (1,1), таким что $e \cdot e = \varepsilon I$, при чём $\varepsilon^2 = 1$, а I единичным тензором, $в$ — является полем симметрических корреляций совершающих условие $в(u,e(v)) = в(v,e(u))$, а является полем ковекторов. Кроме того, предполагаем что корреляция $u — в(u, -)$ оборотная и эту оборотную корреляцию обозначаем символом $\check{в}$.

В данной работе находим общий вид связности Подковырина и подсчитываем их кручения. Местные координаты ω_i^s полученной связности имеют вид:

$$\omega_i^s = \tfrac{1}{4}[e_r^s de_i^r + c^{qs} db_{iq} - 2Ae_i^s + в^{rs} db_{ri} + e_r^s c_i^p - c_{ri} c^{ps} A_p^r]$$

где $a = a_k dx^k$, Ap_p^r являются координатами любой формы со значениями в алгебре Ли линейной группы L^n.