ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXIX, 16 SECTIO A 1975

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On Second Order Cauchy's Problem in a Hilbert Space with Applications to the Mixed Problems for Hyperbolic Equations, II

O zadaniu Cauchy'ego drugiego rzędu w przestrzeni Hilberta z zastosowaniem do zadań mieszanych dla równań hiperbolicznych, II

О задаче Коши второго порядка в гильбертовом пространстве с приложением к смешанным задачам для уравнений гиперболического типа, II

The purpose of this paper is to give some applications of the results of Theorems 1 and 2 of the previous paper [6].

I. Let Ω be a bounded domain in R^n, and let S be the boundary of Ω. We shall use the notation of [6] and the following ones:

$D(\Omega)$ = the space of all complex-valued tested functions on Ω equipped with the usual topology,

$(D)'\Omega$ = the conjugate space of $D(\Omega)$, i.e. the space of distributions in the sense of L. Schwartz on Ω,

$L^2(\Omega)$ = the Hilbert space of classes of complex-valued measurable and square-integrable functions over Ω with the usual scalar product $((u,v)) = \int_\Omega u(x)\overline{v(x)}dx$ and the norm $\|u\| = ((u,u))^{1/2}$,

$H^k(\Omega)$ = the Hilbert space of elements of $L^2(\Omega)$ having the distributional derivatives of order $\leqslant k$, square-integrable over Ω, with the scalar product $((u,v))_k = \sum_{|\alpha|\leqslant k}((D^\alpha u, D^\alpha v))$ and the norm $\|u\|_k = ((u,u))_k^{1/2}$,

$H_0^k(\Omega)$ = the completion of $D(\Omega)$ in the norm of $H^k(\Omega)$,

$\langle f, u\rangle$ = the value of $f \in D'(\Omega)$ at the point $u \in D(\Omega)$,

$B^k(\Omega)$ = the set of all functions on Ω such that their partial derivatives of order $\leqslant k$ exist and are continuous and bounded.

II. A mixed problem with the boundary condition of the Dirichlet type.

Consider a hyperbolic equation of second order

$$(1)\qquad \frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{n} a_i(t,x)\frac{\partial^2 u}{\partial x_i \partial t}-\sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_k}=0,$$

where the coefficients are real-valued functions belonging to $B^2\big((0,T)\times\Omega\big)$. We assume that $\sum_{i,k=1}^{n} a_{ik}(t,x)\xi_i\xi k \geqslant d\sum_{i=1}^{n}\xi_i^2$ and $a_{ij}(t,x)=a_{ji}(t,x)$ for all $(t,x)\in(0,T)\times\Omega$ and $(\xi_1,\ldots,\xi_n)\in R^n$.

Our problem is to obtain a solution $u(t,x)$ of (1) on $(0,T)\times\Omega$, $u(t,\cdot)\in H_0^-$ for every $t\in\langle 0,T\rangle$, satisfying

$$(2)\qquad \begin{cases} u(0,x)=u_0(x)\\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x)\\ u(t,x)=0 \text{ for } (t,x)\in\langle 0,T\rangle\times S, \end{cases}$$

for any given initial data $u_0(x)$, $u_1(x)$.

The derivatives in (1) in t are taken in the sense of the norm in H_0^- while in $x_1,\ldots,x_n$ in the distributional sense. The second condition of (2) means that $u(t,\cdot)\in H_0^1(\Omega)$ for every $t\in\langle 0,T\rangle$.

The existence and the uniqueness of the solution of (1)–(2) under some assumption on $u_0(x)$, $u_1(x)$ will follow from Theorem 1 of [6]. In order to apply this theorem we set:

$$H=L^2(\Omega),\ H^+=H_0^1(\Omega),\ H_t^+=H_0^1(\Omega)$$

with the scalar product and the norm defined by the formulae

$$\big((u,v)\big)_t^+=\int_\Omega u(x)\overline{v(x)}\,dx+\int_\Omega\sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial u(x)}{\partial x_i}\frac{\overline{\partial v(x)}}{\partial x_k},$$

$$\|u\|_t^+=\Big(\big((u,u)\big)_t^+\Big)^{1/2}.$$

By the definition of the operator $\Lambda_0(t)$ ([6], Lemma 5°) we have:

$$(3)\qquad \begin{cases} D\big(\Lambda_0(t)\big)=\Big\{u\in H_t^+:\ \sup\big\{\big|\big((u,v)\big)_t^+\big|:\ v\in H_t^+,\ \|v\|\leqslant 1\big\}<\infty\Big\},\\ \Big(\big(\Lambda_0(t)u,v\big)\Big)=\big((u,v)\big)_t^+. \end{cases}$$

Now we explain the sense of (3) in the present case. Let $v\in D(\Omega)$ We have:

$$\langle\Lambda_0(t)u,\bar v\rangle=\big((u,v)\big)_t^+=\int_\Omega u(x)\overline{v(x)}\,dx+\int_\Omega\sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial u(x)}{\partial x_i}\frac{\overline{\partial v(x)}}{\partial x_k}\,dx$$

$$=\Big\langle u-\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\Big(\sum_{i=1}^{n} a_{ik}(t,x)\frac{\partial u}{\partial x_i}\Big),\bar v\Big\rangle.$$

Hence the conditions:

$$\begin{cases} u \in D(\Lambda_0(t)) \\ \big((\Lambda_0(t)u, v)\big) = (u, v)_t^+, \quad v \in H_0^1(\Omega) \end{cases}$$

are equivalent to the following ones

$$\left[\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\sum_{i=1}^{n} a_{ik}(t, x)\frac{\partial u(x)}{\partial x_i}\right)\right] \in L^2(\Omega),$$

$$\Lambda_0(t)u = u - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\sum_{i=1}^{n} a_{ik}(t, x)\frac{\partial u}{\partial x_i}\right)$$

in the sense of $D'(\Omega)$ for every $t \in \langle 0, T\rangle$.

From the equality $\big((\Lambda_0(t)u, v)\big)_t^- = \big((u, v)\big)$ for $u \in D(\Lambda_0(t))$, $v \in H^+$, and from the inequality $|((u, v))| \leqslant \|u\|_t^+ \|v\|_t^-$ for $u \in H^+$, $v \in H$ it follows that $\Lambda_0(t)$ is a continuous operator, mapping $D(\Lambda_0(t))$ into H_t^-, and since $D(\Lambda_0(t))$ is a dense subset in H_t^+, $\Lambda(t) =$ the closure of $\Lambda_0(t)$ in H_t^- is an element from $L(H_0^+, H_0^-)$, satisfying

$$\Lambda(t)u = u - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\sum_{i=1}^{n} a_{ik}(t, x)\frac{\partial u}{\partial x_i}\right)$$

in the sense of $D'(\Omega)$. Write (1) in an equivalent form

$$\frac{d^2u}{dt^2} + \left(\sum_{i=1}^{n} a_i(t, x)\frac{\partial}{\partial x_i}\right)\frac{du}{dt} - \sum_{i,k=1}^{n} a_{ik}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_k} = 0 \tag{4}$$

and put:

$$B(t)u = \left(\sum_{i=1}^{n} a_i(t, x)\frac{\partial}{\partial x_i}\right)u$$

$$S(t)u = -\sum_{i,k=1}^{n} a_{ik}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i,k=1}^{n}\left(\frac{\partial}{\partial x_k}(Q_{ik}(t, x)\frac{\partial u}{\partial x_i}\right) - u.$$

Setting $S(t) = P(t) - 1$, we see $P(t)$ is the first order differential operator. Writing (4) in the form

$$\frac{d^2u}{dt^2} + \big(\Lambda(t) + S(t)\big)u + B(t)\frac{du}{dt} = 0 \tag{5}$$

we shall prove that the hypotheses (1.1)–(1.3) of Theorem 1 of paper [6] are fulfilled.

Ad (1.2). Let $u \in D(\Omega)$, $v \in H^{+}$. We have

$$\left|\left(\left(B(t)u, v\right)\right)\right| = \left|\int_{\Omega} \sum_{i=1} a_i(t,x)\frac{\partial u}{\partial x_i}\bar{v}dx\right| \leqslant \sum_{i=1}^{n}\left(\int_{\Omega} a_i^2(t,x)\left|\frac{\partial u}{\partial x_i}\right|^2 dx\right)^{1/2} \times$$

$$\times \left(\int_{\Omega} |v|^2 dx\right)^{1/2} \leqslant c_1 \|u\|_0^+ \|v\|,$$

and

$$\left|\left(\left(B(t)u, v\right)\right)\right| = \left| - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(t,x)\bar{v}\right) u dx\right| \leqslant c_2 \|u\| \|v\|_0^+.$$

Since

$$\|B(t)u\|_0^- = \sup\left\{\left|\left(\left(B(t)u, v\right)\right)\right| : v \in H^{+},\ \|v\|_0^+ \leqslant 1\right\},$$

thus for $u \in D(\Omega)$ we obtain

$$\|B(t)u\|_0^- \leqslant c_2 \|u\|.$$

From the density of $D(\Omega)$ in $H = L^2(\Omega)$, after extension by continuity we come to conclusion that $B(t) \in L(H, H_0^-)$.

Ad (1.3). For every $u \in D(\Omega)$ we have

$$\left(\left(B(t)u, u\right)\right) = - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial}{\partial X_i}\left(a_i(t,x)\bar{u}\right) u dx = - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial a_i(t,x)}{\partial x_i} |u|^2 dx +$$

$$- \int_{\Omega} \sum_{i=1}^{n} a_i(t,x) \frac{\partial \bar{u}}{\partial x_i} u dx = - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial a_i(t,x)}{\partial x_i} |u|^2 dx - \left(\left(u, B(t)u\right)\right).$$

Thus

$$2\mathrm{Re}\left(\left(B(t)u, u\right)\right) = - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial a_i(t,x)}{\partial x_i} |u|^2 dx$$

and

$$\left|\mathrm{Re}\left(\left(B(t)u, u\right)\right)\right| \leqslant c_3 \|u\|^2.$$

Let $u \in H^{+}$. From the density of $D(\Omega)$ in H_0^+ it follows that there exists a sequence $u_n \in D(\Omega)$ such that $\|u_n - u\|_0^+ \to 0$. Of course also $\|u_n - u\| \to 0$. By the inequality $|((B(t)u, v))| \leqslant c_1 \|u\|_0^+ \|v\|$ we see $B(t) \in L(H_0^+, H)$, hence $\|B(t)u_n - B(t)u\| \to 0$. Finally, the passage to the limit when $n \to \infty$ in the inequality $|\mathrm{Re}((B(t)u_n, u_n))| \leqslant c_3 \|u_n\|^2$ gives us the required one:

$$\left|\mathrm{Re}\left(\left(B(t)u\ u\right)\right)\right| \leqslant c_3 \|u\|^2 \text{ for every } u \in H^{+} \text{ and } t \in \langle 0, T\rangle.$$

Ad (1.1). We can write

$$P(t)u = \sum_{i=1}^{n} c_i(t, x)\frac{\partial u}{\partial x_i},$$

where the coefficients $c_i(t, x)$ forms the combination of the derivatives of $a_{ik}(t, x)$. Thus we see $S(t) \in L(H_0^+, H)$.
The weakly continuously differentiability of the functions $t \to S(t)$ and $t \to B(t)$ follows from the forms of the operators $S(t)$ and $B(t)$ and from the properties of the coefficients $a_i(t, x)$ and $a_{ik}(t, x)$.

Hence all hypotheses of Theorem 1 are fulfilled. Moreover, bearing in mind Remark of [6] we see that the solution of problem (1)–(2) has the property:

$$u(t, \cdot) \in C^0(\langle 0, T\rangle; H_0^1(\Omega)) \cap C^1(\langle 0, T\rangle;\ L^2(\Omega)) \cap C^2(\langle 0, T\rangle; H_0^-).$$

Remark 1. Note, that H_0^- one can regard as the antiadjoint space of H^+ (cf. [2], p. 45). In our case when $H^+ = H_0^1(\Omega)$, the corresponding antiadjoint space is denoted by $H^{-1}(\Omega)$ (cf. [5]).

Applying Theorem 1 of [6] we have just proved the following

Theorem A. *For given initial data* $\{u_0(x), u_1(x)\} \in H_0^1(\Omega) \times L^2(\Omega)$ *there exists one and only one solution* $u(t, x)$ *of* (1)–(2) *such that*

$$u(t, \cdot) \in C_0(\langle 0, T\rangle;\ H_0^1(\Omega)) \cap C^1(\langle 0, T\rangle; L^2(\Omega)) \cap C^2(\langle 0, T\rangle; H^{-1}(\Omega)).$$

Remark 2. Theorem 1 can be applied to more general equation, namely to the following one:

$$(1^\circ)\qquad \frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} a_i(t, x)\frac{\partial^2 u}{\partial x_i \partial t} - \sum_{i,k=1}^{n} a_{ik}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_k} + b(t, x)\frac{\partial u}{\partial t} + \\ + \sum_{i=1}^{n} c_i(t, x)\frac{\partial u}{\partial x_i} + c(t, x)u = 0.$$

Putting

$$S_1(t)u = S(t)u + \sum_{i=1}^{n} c_i(t, x)\frac{\partial u}{\partial x_i} + c(t, x)u,$$

$$B_1(t)u = B(t)u + b(t, x)u,$$

one can easy check the conditions (1.1)–(1.3) of Theorem 1.

III. A mixed problem with the boundary condition of the transversal type.

In this section we consider again a mixed problem for hyperbolic equations of second order. The domain Ω and the equation are the same as they were in Section II. But we require now S to be sufficiently smooth, more precisely, Ω is an interior domain of the compact surface S of class C^∞ in R^n.

Our problem is to obtain $u(t, x)$; $u(t, \cdot)\in H^1(\Omega)$ for $t\in\langle 0, T\rangle$, a solution of the equation

$$\text{(i)}\qquad \frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{n} a_i(t, x)\frac{\partial^2 u}{\partial x_i\partial t}-\sum_{i,k=1}^{n} a_{ik}(t, x)\frac{\partial^2 u}{\partial x_i\partial x_k}=0,$$

and for given initial data $\{u_0(x), u_1(x)\}$ satisfying

$$\text{(ii)}\qquad \begin{cases} u(0, x) = u_0(x)\\ \dfrac{\partial u}{\partial t}(0, x) = u_1(x)\end{cases}$$

and

$$\text{(iii)}\qquad \sum_{i,k=1} a_{i,k}(t, x)\frac{\partial u}{\partial x_i}\eta_k = -a(t, x)u, \text{ for } (t, x)\in\langle 0, T\rangle\times S,$$

where $a(t, x)$ is given real-valued, continuous, non-negative function on $\langle 0, T\rangle\times S$ and $\{\eta_k\}_1^n$ denotes the normal exterior vector with respect to surface S.

The derivatives in (i) in t are taken in the sense of the norm in $L^2(\Omega)$ whereas in $x_1, \ldots, x_n$ in the distributional one.

In order to prove the existence and the uniqueness of the solution of (i)–(iii) we shall apply Theorem 2 of [6].

To do this we need the following assumptions:

$$\text{a)}\qquad \sum_{i=1}^{n} a_i(t, x)\eta_i(x) = 0 \text{ for } (t, x)\in\langle 0, T\rangle\times S,$$

$$\text{b)}\qquad \sum_{i,k=1}^{n} a_{ik}(0, x)\frac{\partial u_0}{\partial x_i}\eta_k(x)+a(0, x)u_0 = 0 \text{ for } x\in S.$$

Set: $H = L^2(\Omega)$, $H^+ = H^1(\Omega)$, $H_t^+ = H^1(\Omega)$ with the scalar product and the norm defined by formulae:

$$\big((u,v)\big)_t^+ = \int_\Omega u(x)\overline{v(x)}\,dx + \int_\Omega \sum_{i,k=1}^n a_{ik}(t,x)\frac{\partial u(x)}{\partial x_i}\frac{\partial \overline{v(x)}}{\partial x_k}\,dx + $$

$$+ \int_S a(t,x)u(x)\overline{v(x)}\,d\sigma$$

$$\|u\|_t^+ = \Big(\big((u,u)\big)_t^+\Big)^{1/2}.$$

Let $\Lambda_0(t)$ be as it was in Lemma 5° of [6]. And again it is easy to see that the conditions:

$$\begin{cases} u\in D\big(\Lambda_0(t)\big), \\ \big((\Lambda_0(t)u, v)\big) = \big((u,v)\big)_t^+, \quad u\in D\big(\Lambda_0(t)\big),\ v\in H^1(\Omega) \end{cases}$$

imply the following ones:

$$\text{(iv)}\qquad \begin{cases} \left[\displaystyle\sum_{i,k=1}^n \frac{\partial}{\partial x_k}\left(a_{ik}(t,x)\frac{\partial u}{\partial x_i}\right)\right]\in L^2(\Omega) \\ \Lambda_0(t)u = u - \displaystyle\sum_{i,k=1}^n \frac{\partial}{\partial x_k}\left(a_{ik}(t,x)\frac{\partial u}{\partial x_i}\right) \end{cases}$$

in the sense of $D'(\Omega)$. Let $u, v\in D(\bar\Omega)$. We have

$$\text{(v)}\qquad \int_\Omega -\sum_{i,k=1}^n \frac{\partial}{\partial x_k}\left(a_{ik}(t,x)\frac{\partial u}{\partial x_i}\right)\bar v\,dx = \int_\Omega \sum_{i,k=1}^n a_{ik}(t,x)\frac{\partial u}{\partial x_i}\frac{\partial \bar v}{\partial x_k}\,dx +$$

$$- \int_S \left(a_{ik}(t,x)\frac{\partial u}{\partial x_i}\eta_k\right)\bar v\,d\sigma.$$

Thus for every $u, v\in D(\bar\Omega)$ from the equality $\big((\Lambda_0(t)u, v)\big) = \big((u,v)\big)_t^+$ it follows that

$$\text{(vi)}\qquad \int_S \left[a(t,x)u + \sum_{i,k=1}^n a_{ik}(t,x)\frac{\partial u}{\partial x_i}\eta_k\right]\bar v\,d\sigma = 0$$

Hence, for every $u\in D(\bar\Omega)\cap D\big(\Lambda_0(t)\big)$ we have

$$\text{(vii)}\qquad \left\{\sum_{i,k=1}^n a_{ik}(t,x)\frac{\partial u}{\partial x_i}\eta_k + a(t,x)u\right\}_{/S} = 0 \text{ in the sense of } L^2(S)$$

Putting

$$S(t)u = -\sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i,k=1}^{n} \frac{\partial}{\partial x_k}\left(a_{ik}(t,x)\frac{\partial u}{\partial x_i}\right) - u$$

$$B(t)u = \sum_{i=1}^{n} a_i(t,x)\frac{\partial u}{\partial x_i}$$

the equation (i) takes the form

$$\text{(viii)} \qquad \frac{d^2u}{dt^2} + \big(\Lambda_0(t) + S(t)\big)u + B(t)\frac{du}{dt} = 0.$$

Now we have to prove the conditions (2.1)–(2.3) of Theorem 2 of [6] are fulfilled.

The conditions (2.1) and (2.2) one verifies similarly as in Section II.

Ad (2.3). For any $u \in D(\bar{\Omega})$ we have

$$\big((B(t)u, u)\big) = \int_\Omega \sum_{i=1}^{n} a_i(t,x)\frac{\partial u}{\partial x_i}\bar{u}dx - \int_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(t,x)\bar{u})udx +$$

$$+ \int_S \sum_{i=1}^{n} a_i(t,x)|u|^2\eta_i d\sigma = -\big((u, Bu)\big) - \int_\Omega \sum_{i=1}^{n} \frac{\partial a_i(t,x)}{\partial x_i}|u|^2 dx$$

Thus, for any $u \in D(\bar{\Omega})$ the inequality $\left|\mathrm{Re}\big((B(t)u, u)\big)\right| \leqslant c_1\|u\|^2$ holds. By the density of $D(\bar{\Omega})$ in $H^1(\Omega)$ and by the inequality $\|u\| \leqslant \|u\|_0^+$ for $u \in H^1(\Omega)$, condition (2.3) is proved.

Theorem 2 assures the existence and the uniqueness of the solution $u(t,x)$ of problem (i)–(iii) such that $\big(u(t), u'(t)\big) \in D\big(A_2(t)\big) = D\big(\Lambda_0(t)\big) \times H^1(\Omega)$. Moreover, with the aid theorems of regularity we are able to prove that $u(t)$ (more precisely $u(t,\cdot)$) belongs to $H^2(\Omega)$.

Really, let us put

$$\text{(ix)} \qquad \begin{cases} H_l^2(\Omega) = \left\{u \in H^2(\Omega) : \left[\sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial u}{\partial x_i}\eta_k + a(t,x)u\right]_{/S} = 0\right\}, \\ D\big(\tilde{A}_2(t)\big) = H_l^2(\Omega) \times H^1(\Omega), \\ \tilde{A}_2(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0, & 1 \\ -\Lambda(t) - S(t), & B(t) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \text{ for } \begin{pmatrix} u \\ v \end{pmatrix} \in D\big(\tilde{A}_2(t)\big). \end{cases}$$

The following lemma ([3], p. 345, Lemma 2.3) holds

Lemma. *There exists a constant $\lambda_1 > 0$ such that for any $\lambda > \lambda_1$ the operator $(\lambda - \tilde{A}_2(t))$ is a bijective mapping from $H_l^2(\Omega) \times H^1(\Omega)$ onto $H^1(\Omega) \times L^2(\Omega)$ and the following estimate holds*

$$\left\|\big(\lambda - \tilde{A}_2(t)\big)^{-1}\right\|_{H^1(\Omega)\times L^2(\Omega)} \leqslant \frac{1}{\lambda - \lambda_1}.$$

It has been proved in Lemma 2 of [6] that there exists a constant λ_0 such that for any $|\lambda| > \lambda_0$, $(\lambda - A_2(t))$ is a bijective mapping from $D(A_2(t))$ onto $H^+ \times H$. $(= H^1(\Omega) \times L^2(\Omega)$ in the present case). By (ix), $A_2(t)$ is an extension of $\bar{A}_2(t)$, whereas by Lemma 2 it follows that $A_2(t) = \bar{A}_2(t)$, hence $D(A_2(t)) = H_l^2(\Omega) \times H^1(\Omega)$.

The obtained result permits us to take the equality (iii) in the sense of the norm in $H^{1/2}(S)$ (for the spaces $H^k(S)$ and the tracetheorems cf. for instance [1] and [5].)

By (6°) of Theorem 2 it follows

$$u(t, \cdot) \in C^1(\langle 0, T\rangle;\ H^1(\Omega)) \cap C^2(\langle 0, T\rangle;\ L^2(\Omega)).$$

Furthermore,

$$(u'(t), u''(t)) = A_2(t)(u(t), u'(t)) \in C^0(\langle 0, T\rangle;\ H_0^+ \times L^2(\Omega)),$$

hence, by known estimates (cf. [3], p. 343)

$$\|u\|^2_{H^2(\Omega)} \leqslant c_1 \left(\left\| \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2 + \|u\|^2 + \left\| \sum_{i,j=1} a_{ij}(t, x) \frac{\partial u}{\partial x_j} \eta_i \right\|^2_{H^{1/2}(S)} \right).$$

we see that $u(t, \cdot) \in C^0(\langle 0, T\rangle;\ H^2(\Omega))$.

And, in this way, we have proved the following

Theorem B. *Given* $\{u_0(x), u_1(x)\} \in H^2(\Omega) \times H^1(\Omega)$, *if the conditions* a) *and* b) *are satisfied then there exists one and only one solution* $u(t, x)$ *of the problem* (i)—(iii) *such that*

$$u(t, \cdot) \in C^0(\langle 0, T\rangle;\ H^2(\Omega)) \cap C^1(\langle 0, T\rangle;\ H^1(\Omega)) \cap C^2(\langle 0, T\rangle;\ L^2(\Omega)).$$

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STRESZCZENIE

W pracy tej podaje się niektóre zastosowania twierdzeń uzyskanych w I do zadań mieszanych dla równań cząstkowych typu hiperbolicznego. Uzyskuje się twierdzenia dotyczące istnienia i jednoznaczności rozwiązania zadań brzegowych typu Dirichleta i typu transwersalnego.

РЕЗЮМЕ

В настоящей работе приводятся некоторые применения теорем, полученных в I части к смешанным задачам для гиперболических уравнений с частными производными. Здесь получаются теоремы, касающиеся проблемы существования и единственности решения граничных задач типа Дирихле и трансверсального типа.