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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

ZBIGNIEW ŚWIĘTOCHOWSKI

# On Second Order Cauchy's Problem in a Hilbert Space with Applications to the Mixed Problems for Hyperbolic Equations, I

O zadaniu Cauchy'ego drugiego rzędu w przestrzeni Hilberta z zastosowaniem do zadań mieszanych dla równań hiperbolicznych, I

О задаче Коши второго порядка в гильбертовом пространстве с приложеннием к смешанным задачам для уравнений гиперболического типа, I

## I. Preliminaries

This section, unfortunately long, is devoted to the preliminary notions, lemmas and Theorem 1°.

**A.** If $X$ and $Y$ are Banach spaces then by $X^*$, $Y^*$ we denote the conjugate spaces of $X$ and $Y$ respectively and by $L(X, Y)$ — the space of all linear bounded operators from $X$ to $Y$.

**B.** $L(X, Y)$-valued functions. An $L(X, Y)$-valued function $t \to A(t)$, $t \in \langle a, b\rangle$ is called (n times) strongly continuously differentiable on $\langle a, b\rangle$, if the function $t \to A(t)x$ is (n times) strongly continuously differentiable in the sense of the norm in $Y$, for any $x \in X$; it is called (n times) weakly continuously differentiable on $\langle a, b\rangle$, if for any $x \in X$ the function $t \to A(t)x$ is (n times) continuously differentiable in the weak sense.

**C.** Green's operator. Let $X$ be a Banach space and let $A(t)$, $t \in \langle 0, T\rangle$ be a family of linear operators whose domaines $D(A(t))$ and ranges $R(A(t))$ contain in $X$, $D(A(t))$ being dense in $X$ for any $t \in \langle 0, T\rangle$.

Consider the first order Cauchy's problem

$$\text{(I)} \qquad \begin{cases} \dfrac{dx(t)}{dt} = A(t)x(t), & \text{for } t \in \langle 0, T\rangle, \\ x(0) = x_0, \end{cases}$$

for given initial data $x_0$.

An $L(X, X)$-valued function $(t, s)\to G(t, s)$ defined on the triangle $0 \leqslant s \leqslant t \leqslant T$ is called the Green operator of the problem (I) if

(II) $G(s, s) = 1$ for any $s \in \langle 0, T\rangle$;

(III) $G(t, s)G(s, r) = G(t, r)$ for $0 \leqslant r \leqslant s \leqslant t \leqslant T$;

(IV) an $X$-valued function $(t, s)\to G(t, s)x$ is continuous in the sense of the norm in $X$ for any $0 \leqslant s \leqslant t \leqslant T$ and any $x \in X$;

(V) $G(t, s)D(A(s)) \subset D(A(t))$ for $0 \leqslant s \leqslant t \leqslant T$ and, for any $s \in \langle 0, T\rangle$ and $x \in D(A(s))$, the function $t\to G(t, s)x$ is continuously differentiable in the sense of the norm in $X$ on $\langle s, T\rangle$ and satisfies the equation $d/dt G(t, s)x = A(t)G(t, s)x$.

The following theorem (Kisyński, [2], p. 312), playing an important role in our treatment, holds:

**D. Theorem 1°.** *Let $X$ be a Banach space equipped with the norm $\|\cdot\|$ and let $A(t)$, $t \in \langle 0, T\rangle$ be the family of linear operators, $D(A(t)) \subset X$, $R(A(t)) \subset X$. Suppose that the following conditions are satisfied:*

(1°) *$D(A(t))$ is dense in $X$;*

(2°) *there exists a family of norms $\| \,\|_t$, $t \in \langle 0, T\rangle$, equivalent to the given norm $\| \,\|$, such that $|\|x\|_t - \|x\|_s| \leqslant k\|x\|_t|t-s|$, $k =$ const., $0 \leqslant s,\ t \leqslant T$, $x \in X$ and,*

(3°) *there exists a constant $\lambda_0 \geqslant 0$, such that $R(\lambda - \varepsilon A(t)) = X$ and $\|\lambda x - \varepsilon A(t)x\|_t \geqslant (\lambda - \lambda_0)\|x\|_t$ for $\varepsilon = \pm 1$, $\lambda > \lambda_0$, $x \in D(A(t))$;*

(4°) *there exists a family of linear bounded and invertible operators $R(t)$ mapping $X$ onto $X$, such that a function $t\to R(t)$ is twice weakly continuously differentiable on $\langle 0, T\rangle$ and $(R(T))^{-1}D(A(t)) = Y =$ const. for any $t \in \langle 0, T\rangle$;*

(5°) *for any $x \in Y$, the function $t\to (R(t))^{-1}A(t)R(t)x$ is weakly continuously differentiable on $\langle 0, T\rangle$,*

*then there exists one and only one Green operator of problem* (I) *having the following properties:*

(II)° *$(t, s)\to G(t, s)$ is an $L(X, X)$-valued function, strongly continuous on the quadrat $0 \leqslant s,\ t \leqslant T$;*

(III)° *$G(s, s) = 1$ for $s \in \langle 0, T\rangle$,*

(IV)° *$G(t, s)G(s, r) = G(t, r)$ for $0 \leqslant r, s, t \leqslant T$;*

(V)° *$G(t, s)D(A(s)) = D(A(t))$ for $0 \leqslant s,\ t \leqslant T$ and, for any $s \in \langle 0, T\rangle$ and $x \in D(A(s))$, the function $t\to G(t, s)x$ is continuously differentiable in the sense of the norm in $X$ on $\langle 0, T\rangle$ and satisfies $d/dt G(t, s)x = A(t)G(t, s)x$.*

If the conditions (1°)–(5°) of Theorem 1° are satisfied for $R(t) = 1$ and, if the space $Y$ is equipped with the norm $|||\ |||$ under which $Y$ becomes a Banach space and $\|y\| \leqslant k|||y|||$ for any $y \in Y$, then the operator $G(t, s)$ has the following additional properties:

(VI)° an $L(Y, Y)$-valued function $(t, s) \to G(t, s)$ is strongly continuous on the quadrat $0 \leqslant s, t \leqslant T$;

(VII)° an $L(Y, X)$-valued function $(t, s) \to G(t, s)$ is strongly continuously differentiable on the quadrat $0 \leqslant s, t \leqslant T$ and satisfies the equations: $d/dtG(t, s) = A(t)G(t, s)$, $d/dsG(t, s) = -G(t, s)A(s)$, for $0 \leqslant s, t \leqslant T$.

E. Hypotheses (*). Let $H$ be a Hilbert space with the scalar product $((,))$ and let $H^+$ be linear and dense subset of $H$. Futhermore, let $((,))_t^+$ be the scalar product on $H^+$ for $t \in \langle 0, T\rangle$ such that $H^+$ with $((,))_t^+$ constitute a Hilbert space $H_t^+$ with the topology not weaker than the topology induced in $H^+$ by $H$.

Assume moreover that for any $x \in H^+$ and $y \in H^+$ the function $t \to ((x, y))_t^+$ is $n$ times $(n \geqslant 1)$ continuously differentiable on $\langle 0, T\rangle$

F. The following lemmas (cf. [2], pp. 319–322, also [1], p. 45 and [5], pp. 9–14) will be necessary in further considerations.

**Lemma 1°.** *The equality* $((x, y))_t^+ = ((Q(t)x, y))_0^+$, $x, y \in H^+$, $t \in \langle 0, T\rangle$ *defines an* $L(H_0^+, H_0^+)$*-valued function, $n$ times weakly continuously differentiable on* $\langle 0, T\rangle$. *For fixed* $t \in \langle 0, T\rangle$ *the operator* $Q(t)$ *is Hermitian with* $\inf Q(t) > 0$ *in* $H_0^+$.

**Lemma 2°.** *There exists a constant* $0 < a \leqslant 1$, *such that*

$$a^{1/2}\|x\|_0^+ \leqslant \|x\|_t^+ \leqslant a^{-1/2}\|x\|_0^+, \qquad \left|\frac{d}{dt}\|x\|_t^+\right| \leqslant a^{-1/2}\|x\|_t^+,$$

*for any* $x \in H^+$ *and* $t \in \langle 0, T\rangle$.

**Lemma 3°.** *The equality* $((x, y)) = ((J_0(t)x, y))_t^+$, $x \in H$, $y \in H^+$, *defines an invertible, Hermitian operator* $J_0(t) \in L(H, H_t^+)$, *the image* $J_0(t)(H^+)$ *is dense in* $H_t^+$. *Moreover we have:*

$$\|J_0(t)x\|_t^+ = \sup\{|((x, y))|: y \in H^+, \|y\|_t^+ \leqslant 1\}, \text{ for } x \in H, \ t \in \langle 0, T\rangle.$$

**Lemma 4°.** *Setting* $\|x\|_t^- = \|J_0(t)x\|_t^+$ *for* $t \in \langle 0, T\rangle$ *and* $x \in H$ *we define the space* $H_t^-$ *as the completion of* $H$ *in the norm* $\| \ \|_t^-$. *We have:*

(4.1) $H \subset H_t^-$, *the topology of* $H$ *is not weaker than the topology induced in* $H$ *by* $H_t^-$;

(4.2) *if by* $J(t)$ *we denote the extension of* $J_0(t)$ *(by continuity), then* $J(t)$ *is an isometry which maps* $H_t^-$ *onto* $H_t^+$ *and, for any* $t \in \langle 0, T\rangle$ *the equality* $J(t) = (Q(t))^{-1}J(0)$ *holds;*

(4.3) *for any* $t \in \langle 0, T\rangle$ *the space* $H_t^-$ *has the structure of Hilbert space under the scalar product*:

$$\big((x, y)\big)_t^- = \big((J(t)x, J(t)y)\big)_t^+ = \big((Q(t)^{-1}J(0)x, J(0)y)\big)_0^+;$$

(4.4) *there exists a constant* $0 < \beta \leqslant 1$, *such that the estimates*

$$\beta^{1/2}\|x\|_0^- \leqslant \|x\|_t^- \leqslant \beta^{-1/2}\|x\|_0^-, \quad \left|\frac{d}{dt}\|x\|_t^-\right| \leqslant \beta^{-1/2}\|x\|_t^-,$$

*for any* $x \in H_0^-$ *and* $t \in \langle 0, T\rangle$ *hold*;

(4.5) *the inequality* $|((x, y))| \leqslant \|x\|_t^+\|x\|_t^-$ *holds for* $x \in H^+$, $y \in H$, $t \in \langle 0, T\rangle$. *Thus the form* $(x, y) \to ((x, y))$ *has the extension by continuity on the set* $(H \times H) \cup (H^+ \times H_t^-) \cup (H_t^- \times H^+)$. *We have* $((x, y)) = ((x, J(t)y))_t^+ = ((J(t)^{-1}x, y))_t^-$, *for* $x \in H^+$, $y \in H_t^-$, $t \in \langle 0, T\rangle$.

**Lemma 5°.** *The conditions*

$$\begin{cases} D(\Lambda_0(t) = \{x \in H^+ : \sup\{|((x, y))| : y \in H^+, \|y\| \leqslant 1\} < \infty\} \\ ((\Lambda_0(t)x, y)) = ((x, y))_t^+, \text{ for } x \in D(\Lambda_0(t)),\ y \in H^+ \end{cases}$$

*define in the space* $H$ *an invertible, self-adjoint, positive operator* $\Lambda_0(t)$. *We have* $D(\Lambda_0(t)) = (Q(t))^{-1}D(\Lambda_0(0))$ *and* $\Lambda_0(t) = (J_0(t))^{-1} = \Lambda_0(0)Q(t)$ *for* $t \in \langle 0, T\rangle$.

**Lemma 6°.** *Denote by* $\Lambda(t)$ *the closure of* $\Lambda_0(t)$ *in* $H_t^-$. $\Lambda(t)$ *is an invertible, self-adjoint, positive operator in* $H_t^-$. $D(\Lambda(t)) = H^+$, $\Lambda(t) = (J(t))^{-1} = \Lambda(0)Q(t)$, *for any* $t \in \langle 0, T\rangle$.

## II. Second order Cauchy's problem in a Hilbert space

Suppose that the hypotheses (*) of Section I are fulfilled and the following conditions:

(1.1) $t \to S(t)$ is an $L(H_0^+, H)$-valued, weakly continuously differentiable function on $\langle 0, T\rangle$,

(1.2) $t \to B(t)$ is an $L(H, H_0^-)$-valued, weakly continuously differentiable function on $\langle 0, T\rangle$,

(1.3) there exists a constant $b \geqslant 0$, such that an inequality $|\mathrm{Re}((B(t)x, x))| \leqslant b\|x\|^2$ holds, for any $x \in H^+$ and $t \in \langle 0, T\rangle$

Consider second order Cauchy's problem

$$(1.4)\quad \begin{cases} \dfrac{d^2x(t)}{dt^2} + (\Lambda(t) + S(t))x(t) + B(t)\dfrac{dx(t)}{dt} = 0, \ t \in \langle 0, T\rangle, \\ x(0) = x_0, \ \dfrac{dx}{dt}(0) = x_1. \end{cases}$$

We shall treat it as first order problem in $t$ in the space $H\times H_0^-$. To this end we put

$$(1.5)\quad \begin{cases} D(A_1(t)) = H^+\times H, \\ A_1(t)(x_0, x_1) = (x_1, -(A(t)+S(t))x_0 - B(t)x_1), \text{ for } (x_0, x_1)\in D(A_1(t)), \end{cases}$$

and we consider the problem

$$(1.6)\quad \begin{cases} \dfrac{dX(t)}{dt} = A_1(t)X(t) \quad \text{for } t\in\langle 0, T\rangle, \\ X(0) = X_0,\ X_0 = (x_0, x_1) \end{cases}$$

in the space $H\times H_0^-$.

We can state

**Theorem 1.** *If the hypotheses* $(*)$ $(n\geqslant 1)$ *and* (1.1)–(1.3) *are satisfied then there exists one and only one Green operator of problem* (1.6) *having the following properties:*

(1°) $(t, s)\to G(t, s)$ *is an* $L(H\times H_0^-, H\times H_0^-)$*-valued, strongly continuous function on the quadrat* $0\leqslant s,\ t\leqslant T$;

(2°) $G(s, s) = 1$ *for* $s\in\langle 0, T\rangle$;

(3°) $G(t, s)G(s, r) = G(t, r)$ *for* $0\leqslant s, r, t\leqslant T$;

(4°) $G(t, s)(H^+\times H) = H^+\times H$, *for* $0\leqslant s,\ t\leqslant T$ *and,* $(t, s)\to G(t, s)$ *is an* $L(H_0^+\times H, H_0^+\times H)$*-valued, strongly continuous function on the quadrat* $0\leqslant s,\ t\leqslant T$;

(5°) $(t, s)\to G(t, s)$ *is an* $L(H_0^+\times H, H\times H_0^-)$*-valued, strongly continuously differentiable on the quadrat* $0\leqslant s,\ t\leqslant T$ *function, satisfying the equations*

$$\frac{d}{dt}G(t, s) = A_1(t)G(t, s),\quad \frac{d}{ds}G(t, s) = -G(t, s)A_1(s), \text{ for } 0\leqslant s, t\leqslant T.$$

Before we prove Theorem 1, we will state Theorem 2, which is connected with the same problem under some modified assumptions. Namely now we assume:

(2.1) $t\to S(t)$ is an $L(H_0^+, H)$-valued, weakly continuously differentiable function on $\langle 0, T\rangle$,

(2.2) $t\to B(t)$ is an $L(H_0^+, H)$-valued, weakly continuously differentiable function on $\langle 0, T\rangle$,

(2.3) there exists a constant $b$, such that the inequality $|\mathrm{Re}((B(t)x, x))|\leqslant b\|x\|^2$ holds for any $x\in H^+$ and $t\in\langle 0, T\rangle$.

As before, we consider second order Cauchy's problem (1.4) and by setting

$$(2.4)\quad \begin{cases} D\big(A_2(t)\big) = \{(x_0, x_1)\colon x_0 \in H^+,\ x_1 \in H^+,\ \big[\big(A(t)+S(t)\big)x_0 + B(t)x_1\big] \in H,\\ A_2(t)(x_0, x_1) = \big(x_1, -\big(A(t)+S(t)\big)x_0 - B(t)x_1\big), \text{ for } (x_0, x_1) \in D\big(A_2(t)\big), \end{cases}$$

we obtain the first order problem equivalent to

$$(2.5)\quad \begin{cases} \dfrac{dX(t)}{dt} = A_2(t)X(t),\ t \in \langle 0, T\rangle, \\ X(0) = X_0, \end{cases}$$

which is treated in the space $H_0^+ \times H$.

**Theorem 2.** *If we assume that the hypotheses* $(*)$ $(n \geqslant 2)$ *and* (2.1)–(2.3) *are satisfied, then there exists one and only one Green operator of problem* (2.5) *having the properties* (2°)–(4°) *of Theorem* 1 *and the following one*: (6°) $G(t, s)D\big(A_2(s)\big) = D\big(A_2(t)\big)$ for $0 \leqslant s, t \leqslant T$ *and, for any* $x \in D\big(A_2(t)\big)$ *and* $s \in \langle 0, T\rangle$, $t \to G(t, s)x$ *is continuously differentiable in the sense of the norm in* $H_0^+ \times H$ *function, satisfying* $d/dt G(t, s)x = A_2(t)G(t, s)x$.

Theorems 1 and 2 are suggested by professor J. Kisyński operator formulations which strenghten the theorems of Lions on weak solutions of some differential equations in a Hilbert space expressing by means of bilinear forms (cf. [5], pp. 150–159). The strengthening is that here we get solutions with strong continuous derivatives (belonging to $H$, $H^-$ and so on) while Lions has analogous derivatives but in the distributional sense. Both cases of equations with constant (independent of t) operators were given in Lions' paper [4].

The proofs of Theorems 1 and 2 will be based on the following lemmas.

**Lemma 1.** *Assume that hypotheses* $(^*)$, (1.1) *and* (1.3), *and either* (1.2) *or* (2.2) *are satisfied. Then for every* $t \in \langle 0, T\rangle$ *and real* $\lambda$, $|\lambda| > \lambda_0$ (*where*

$$\lambda_0 = \frac{1}{2}\left(b + \left(\frac{s^2}{a} + b^2\right)^{1/2}\right),$$

*a being a constant as in* **Lemma 2°**, *s being a constant not less than the norm of* $S(t)$ *in the space* $L(H_0^+, H)$), *the operator* $P(t, \lambda) = A(t) + S(t) + \lambda B(t) + \lambda^2$ *belongs to the space* $L(H_0^+, H_0^-)$, *is invertible and* $R\big(P(t, \lambda)\big) = H^-$.

**Proof.** From Lemmas 4° and 6° of Section I it follows

$$\big((P(t, \lambda)x, x)\big) = \big((x, x)\big)_t^+ + \big(S(t)x, x\big) + \lambda\big((B(t)x, x)\big) + \lambda^2\big((x, x)\big).$$

Thus for every $\lambda$, $|\lambda| > \lambda_0$ we have

$$\mathrm{Re}\big((P(t,\lambda)x, x)\big) \geqslant a(\|x\|_0^+)^2 - s\|x\|_0^+\|x\| + |\lambda|(|\lambda|-b)\|x\|^2$$

$$= \epsilon(\|x\|_0^+)^2 + \left[\frac{s}{2\big(|\lambda|(|\lambda|-b)\big)^{1/2}}\|x\|_0^+ - \big(|\lambda|(|\lambda|-b)\big)^{1/2}\|x\|\right]^2$$

$$\geqslant \varepsilon(\|x\|_0^+)^2, \quad \text{for } t\in\langle 0, T\rangle,\ x\in H^+ \quad \text{and } \varepsilon = a - \frac{s^2}{4|\lambda|(|\lambda|-b)} > 0$$

Consequently

(7°)

$$\left|\begin{array}{l}\text{for every rel } \lambda,\ |\lambda| > \lambda_0, \text{ there exists a constant } \varepsilon_\lambda > 0 \\ \text{such that } \mathrm{Re}\big((P(t,\ )x, x)\big) \geqslant \varepsilon_\lambda(\|x\|_0^+)^2, \text{ for every } t\in\langle 0,T\rangle \text{ and } x\in H^+.\end{array}\right.$$

Fix $t\in\langle 0, T\rangle$ and $\lambda\in R$, $|\lambda| > \lambda_0$. By (7°) and Lemma 4° we have

$$\|P(t,\lambda)x\|_0^-\,\|x\|_0^+ \geqslant \big|\big((P(t,\lambda)x, x)\big)\big| \geqslant \varepsilon_\lambda(\|x\|_0^+)^2,$$

hence

$$\|P(t,\lambda)x\|_0^- \geqslant \varepsilon_\lambda\|x\|_0^+ \quad \text{for } x\in H^+.$$

Since $P(t,\lambda)\in L(H_0^+, H_0^-)$, thus $R\big(P(t,\lambda)\big)$ is closed in $H_0^-$. It remains to prove the density of $R\big(P(t,\lambda)\big)$ in the space $H_0^-$. Suppose that $R\big(P(t,\lambda)\big)$ is not dense in $H_0^-$, then there exists $x_0\in H_0^-$, $x_0\neq 0$, such that $\big((P(t,\lambda)x, x_0)\big)_0^- = 0$ for every $x\in H^+$ and, by Lemma 4° we have $\big((P(t,\lambda)y_0, y_0)\big) = \big((P(t,\lambda)y_0, x_0)\big)_0^- = 0$, where $0\neq y_0 = J(0)x_0\in H^+$, what is contradiction of (7°). Lemma is proved.

**Lemma 2.** *Assuming that the hypotheses of* **Theorem 1** *are fulfilled then for every* $t\in\langle 0,T\rangle$ *and real* $\lambda$,

$$|\lambda| > \lambda_0 = \frac{1}{2}\left(b + \left(\frac{s^2}{a} + b^2\right)^{1/2}\right)$$

*the operators* $\big(\lambda - A_1(t)\big)$ *and* $\big(\lambda - A_2(t)\big)$ *are invertible and*

$$R\big(\lambda - A_1(t)\big) = H\times H^-,\ R\big(\lambda - A_2(t)\big) = H^+\times H.$$

**Proof.** Consider the equation

$$(8°)\qquad \big(\lambda - A_1(t)\big)(x_0, x_1) = (y_0, y_1),$$

where $t\in\langle 0, T\rangle$ and $\lambda\in R$, $|\lambda| > \lambda_0$ are fixed, $(y_0, y_1)$ is a given element from $H\times H^-$, $(x_0, x_1)\in D(A_1(t)) = H^+\times H$ being the unknown. Since $B(t)\in L(H, H_0^-)$, then by (1.5) the equation (8°) is equivalent to the following system

$$(9°)\qquad \left|\begin{array}{l} P(t,\lambda)x_0 = y_1 + B(t)y_0 + \lambda y_0 \\ x_1 = \lambda x_0 - y_0.\end{array}\right.$$

Lemma 1 assures the existence and the uniqueness of the solution of (9°). Thus $R(\lambda - A_1(t)) = H \times H^-$ and the operator $(\lambda - A_1(t))$ is invertible. In view of (1.5) and (2.4) we have:

$$D(A_2(t)) = \{x\colon x \in D(A_1(t)),\ A_1(t)x \in H^+ \times H\} \quad \text{and } A_2(t) \subset A_1(t),$$

and from this it follows that the operator $(\lambda - A_2(t))$ is invertible and $R(\lambda - A_2(t)) = H^+ \times H$.

**Lemma 3.** *Under the hypotheses of* **Theorem 2,** *the operator* $(\lambda - A_2(t))$ *is invertible and* $R(\lambda - A_2(t)) = H^+ \times H$, *for every* $t \in \langle 0, T\rangle$ *and* $\lambda \in R$ *with* $|\lambda| > \lambda_0$.

**Proof.** Fix $t \in \langle 0, T\rangle$ and $\lambda \in R$, $|\lambda| > \lambda_0$. Since $B(t) \in L(H_0^+, H)$, thus $(y_1 + B(t)y_0 + \lambda y_0) \in H$ and, by Lemma 1, the system (9°) has a unique solution $(x_0, x_1) \in H^+ \times H^+$. Therefore the condition (1.5) assures that $(x_0, x_1)$ is the unique solution of (8°). From (9°) it follows that $(A(t) + S(t))x_0 + B(t)x_1 = (y_1 + \lambda y_0 - \lambda^2 x_0) \in H$, hence $(x_0, x_1) \in D(A_2(t))$. This fact jointly with the inclusion $A_2(t) \subset A_1(t)$ complete the proof of the lemma.

**Lemma 4.** *Assume that hypotheses* (*), (1.1), (1.3) *and either* (1.2) *or* (2.2) *are fulfilled. Then the condition* (3°) *of Theorem 1° of Section* I *is fulfilled for* $X = H_0^+ \times H$, $\|(x_0, x_1)\|_t = \big((\|x_0\|_t^+)^2 + \|x_1\|^2\big)^{1/2}$, $A(t) = A_2(t)$ and $\lambda_0 = \dfrac{1}{2}\left(b + \left(\dfrac{s^2}{a} + b^2\right)^{1/2}\right)$.

**Proof.** Put $((x, y))_t = ((x_0, y_0))_t^+ + ((x_1, y_1))$ for $x = (x_0, x_1)$ and $y = (y_0, y_1)$, $x, y \in H^+ \times H$ and $\|x\|_t = ((x, x))_t^{1/2}$. By (2.4) and by Lemmas 4° and 6° of Section I, we have $((A_2(t)x, x))_t = ((x_1, x_0))_t^+ - ((A(t)x_0 + S(t)x_0 + B(t)x_1, x_1)) = ((x_1, x_0))_t^+ - ((x_0, x_1))_t^+ - ((S(t)x_0 + B(t)x_1, x_1))$. Hence $\mathrm{Re}((A_2(t)x, x))_t = -\mathrm{Re}((S(t)x_0 + B(t)x_1, x_1))$, for every $t \in \langle 0, T\rangle$ and $x = (x_0, x_1) \in D(A_2(t))$.

From (1.1) and (1.3) and one of (1.2), (2.2), making use of the inequality $2ab \leqslant \mu a^2 + \dfrac{1}{\mu} b^2$, $\mu > 0$, $a, b \in R$, and putting in it $a = \|x_0\|_t^+$, $b = \|x_1\|$, $\mu = \dfrac{2\sqrt{a}}{s}\lambda_0$, we obtain

$$|\mathrm{Re}((A_2(t)x, x))_t \leqslant (s\|x_0\|_0^+ + b\|x_1\|)\|x_1\| \leqslant \left(\frac{s}{\sqrt{a}}\|x\|_t^+ + b\|x_1\|\right)\|x_1\|$$

$$\leqslant \frac{s}{2\sqrt{a}}\left(\mu(\|x_0\|_t^+)^2 + \frac{1}{\mu}\|x_1\|^2\right) + b\|x_1\|^2$$

$$= \lambda_0(\|x_0\|_t^+)^2 + \left(\frac{s^2}{4a\lambda_0} + b\right)\|x_1\|^2 = \lambda_0\|x\|_t^2.$$

As a consequence of the latter, for every $t\in\langle 0,T\rangle$, $x\in D(A_2(t))$, $\lambda>\lambda_0$ and $\varepsilon=\pm 1$, we get

$$(10^\circ)\quad \left|\begin{array}{l} \|\lambda x-\varepsilon A_2(t)x\|_t^2=\|(\lambda-\lambda_0)x+(\lambda_0-\varepsilon A_2(t))x\|_t^2 \\ =(\lambda-\lambda_0)^2\|x\|_t^2+\|(\lambda_0-\varepsilon A_2(t)x\|_t^2+2(\lambda-\lambda_0)(\lambda_0\|x\|_t^2+ \\ +\varepsilon\mathrm{Re}\big(((A_2(t)x,x))_t\big)\geqslant(\lambda-\lambda_0)^2\|x\|_t^2. \end{array}\right.$$

From Lemmas 2 and 3 we have

(11°) $R(\lambda-A_2(t))=H^+\times H$, for every $t\in\langle 0,T\rangle, \lambda>\lambda.$ and $\varepsilon=\pm 1$, and the proof of (3°) of Theorem 1° follows from (10°) and (11°).

**Lemma 5.** *Under the hypotheses* (*) *the condition* (2°) *of Theorem 1 is satisfied for* $X=H_0^+\times H$ *and* $\|\ \|_t=\|\ \|_{H_t^+\times H}$, *where*

$$\|(x_0,x_1)\|_{H_t^+\times H}=\big((\|x_0\|_t^+)^2+\|x_1\|^2\big)^{1/2}.$$

**Proof.** It follows from Lemma 2° of Section I.

**Lemma 6.** *If the hypotheses of Theorem 1 are satisfied, then* $t\to A_1(t)$ *is an* $L(H_0^+\times H, H\times H_0^-)$*-valued function, weakly continuously differentiable on* $\langle 0,T\rangle$, *and the conditions* (2°) *and* 3°) *of* Theorem 1° *are fulfilled for* $X=H\times H_0^-$, $A(t)=A_1(t)$ *and*

$$\|\ \|_t=\|\ \|_{H\times H_t^-},\ \textit{where}\ \|x\|_{H\times H_t^-}=\big\|(\lambda_0+1-A_1(t))^{-1}x\big\|_{H_t^+\times H}$$

$$\lambda_0=\frac{1}{2}\left(b+\left(\frac{s^2}{a}+b^2\right)^{1/2}\right).$$

**Proof.** For every $x=(x_0,x_1)\in H^+\times H$ and $y=(y_0,y_1)\in H\times H^-$ from Lemmas 4° and 6° of Section I it follows

$$\big((A_1(t)x,y)\big)_{H\times H_0^-}=((x_1,y_0))-\big((A(t)x_0-B(t)x_1,y_1)\big)_0^-$$
$$=((x_1,y_0))-\big((Q(t)x_0,J(0)y_1)\big)_0^+-\big((S(t)x_0+B(t)x_1,y_1)\big)_0^-.$$

Thus, by (1.1) and (1.2) and Lemma 1°, $t\to A_1(t)$ is an $L(H_0^+\times H,H_0^-)$-valued, weakly continuously differentiable on $\langle 0,T\rangle$ function. The function $t\to(\lambda_0+1-A_1(t))$ is the same. Moreover, by Lemma 2 it follows that for every $t\in\langle 0,T\rangle$, the operator $(\lambda_0+1-A_1(t))$ is invertible and maps $H_0^+\times H$ onto $H\times H_0^-$. Hence $t\to(\lambda_0+1-A_1(t))^{-1}$ is an $L(H\times H_0^-,H_0^+\times H)$-valued, weakly continuously differentiable function on $\langle 0,T\rangle$.

To prove (2°), we put

$$C(t)=(\lambda_0+1-A_1(t))^{-1}.$$

We have $\|x\|_{H\times H_t^-} = \|C(t)x\|_{H_t^+\times H}$, for $x\in H\times H_0^-$. From the weak differentiability of $C(t)$ it follows that there exists a constant $k_1\geqslant 0$ such that $\left\|\frac{d}{dt}C(t)x\right\|_{H_s^+\times H}\leqslant k_1\|x\|_{H\times H_s^-}$. From the equivalence of the norms $\|\ \|_{H\times H_s^-}$, $s\in\langle 0,T\rangle$ and from the equality

$$\left|\frac{d}{dt}\|C(t)x\|^2_{H_s^+\times H}\right| = 2\|C(t)x\|_{H_s^+\times H}\left|\frac{d}{dt}\|C(t)x\|_{H_s^+\times H}\right|$$

$$= 2\left|\left(\left(\frac{d}{dt}C(t)x,\, C(t)x\right)\right)_{H_s^+\times H}\right|$$

we have

$$\left|\frac{d}{dt}\|C(t)x\|_{H_s^+\times H}\right|\leqslant k_2\|x\|_{H\times H_s^-}.$$

Hence there exists a constant $k_3$ such that

$$\big|\|C(s)x\|_{H_s^+\times H}-\|C(t)x\|_{H_s^+\times H}\big|\leqslant k_3\|C(t)x\|_{H_t^+\times H}|t-s|.$$

From the latter, by Lemma 5 and, by the inequality

$$\big|\|C(t)x\|_{H_t^+\times H}-\|C(s)x\|_{H_s^+\times H}\big|\leqslant\big|\|C(t)x\|_{H_t^+\times H}-\|C(t)x\|_{H_s^+\times H}\big|+$$

$$+\big|\|C(t)x\|_{H_s^+\times H}-\|C(s)x\|_{H_s^+\times H}\big|$$

it follows that

$$\big|\|C(t)x\|_{H_t^+\times H}-\|C(s)x\|_{H_s^+\times H}\big|\leqslant k_4\|C(t)x\|_{H_t^+\times H}|t-s|.$$

Hence

$$\big|\|x\|_{H\times H_t^-}-\|x\|_{H\times H_s^-}\big|\leqslant k_4\|x\|_{H\times H_t^-}|t-s|,$$

and the condition (2°) is satisfied.
From the inclusion $A_2(t)\subset A_1(t)$, by Lemma 4 we have

$$\left\|\big(\lambda-\varepsilon A_1(t)\big)^{-1}x\right\|_{H_t^+\times H} = \left\|\big(\lambda-\varepsilon A_2(t)\big)^{-1}x\right\|_{H_t^+\times H}\leqslant(\lambda-\lambda_0)^{-1}\|x\|_{H_t^+\times H},$$

$$\text{for}\quad t\in\langle 0,T\rangle,\ \lambda>\lambda_0,\ \varepsilon=\pm 1\ \text{and}\ x\in H_0^+\times H.$$

Thus

$$\|(\lambda-\varepsilon A_1(t))^{-1}x\|_{H\times H_t^-} = \|(\lambda_0+1-A_1(t))^{-1}(\lambda-\varepsilon A_1(t))^{-1}x\|_{H_t^+\times H}$$

$$= \|(\lambda-\varepsilon A_1(t))^{-1}(\lambda_0+1-A_1(t))^{-1}x\|_{H_t^++H}$$

$$\leqslant (\lambda-\lambda_0)^{-1}\|(\lambda_0+1-A_1(t))^{-1}x\|_{H_t^+\times H} = (\lambda-\lambda_0)^{-1}\|x\|_{H\times H_t^-},$$

$$\text{for } t\in\langle 0, T\rangle,\ \lambda>\lambda_0,\ \varepsilon=\pm 1,\ x\in H\times H_0^-,$$

what proves the condition (3°).

**Proof of Theorem 1.** Put $X = H\times H_0^-$, $Y = H^+\times H$, $\|\ \|_t = \|\ \|_{H\times H_t^-}$, $A(t) = A_1(t)$, and $R(t) = 1$, then from Lemma 6 it follows that the conditions (1°)–(5°) of Theorem 1° of Section I are fulfilled. This proves the theorem.

**Proof of Theorem 2.** Set $X = H_0^+\times H$, $Y = D(\Lambda_0(0))\times H^+$, $A(t) = A_2(t)$, $\|\ \|_t = \|\ \|_{H_t^+\times H}$ and

(12°) $\quad R(t)(x_0, x_1) = ((Q(t))^{-1}x_0, x_1)$, for $(x_0, x_1)\in H_0^+\times H$, $t\in\langle 0, T\rangle$

one can see that the conditions (1°)–(5°) of Theorem 1° are fulfilled.

Indeed, from Lemmas 4 and 5 it follows that the conditions (2°) and (3°) of Theorem 1° are fulfilled. Since $S(t)\in L(H_0^+, H)$ and $B(t)\in L(H_0^+, H)$, from (2.4) it follows

$D(A_2(t)) = \{x_0 : x_0\in H^+,\quad (\Lambda(t)+S(t))x_0\in H\}\times H^+ = D(\Lambda_0(t))\times H^+$, for every $t\in\langle 0, T\rangle$.

By Lemma 3°, $R(J_0(t))$ is a dense subspace of $H_0^+$ and, by Lemma 5° $(\Lambda_0(t))^{-1} = J_0(t)$, so $D(\Lambda_0(t))$ is dense in $H_0^+$. Thus $D(A_2(t))$ is dense in $H_0^+\times H$ for $t\in\langle 0, T\rangle$, what proves the condition (1°) of Theorem 1°.

By (12°) and Lemma 5°, and Lemma 1°, the operators $R(t)$, $t\in\langle 0, T\rangle$ are invertible and map $H_0^+\times H$ onto itself, $t\to R(t)$ is an $L(H_0^+\times H, H_0^+\times H)$-valued, twice weakly continuously differentiable function on $\langle 0, T\rangle$, satisfying

$$(R(t))^{-1}D(A_2(t)) = Q(t)D(\Lambda_0(t))\times H^+ = D(\Lambda_0(0))\times H^+, \text{for every } t\in\langle 0, T\rangle.$$

Hence (4°) is satisfied.

Finally, from (12°) and Lemma 5° we obtain

$$(R(T))^{-1}A(t)R(t)x = (R(t))^{-1}A_2(t)((Q(t))^{-1}x_0, x_1)$$

$$= (R(t))^{-1}(x_1, -(\Lambda_0(t)+S(t)(Q(t))^{-1}x_0-B(t)x_1)$$

$$= (Q(t)x_1, -\Lambda_0(0)X_0-S(t)(Q(t))^{-1}x_0-B(t)x_1),$$

for every

$$x = (x_0, x_1) \in Y = D(\Lambda_0(0)) \times H^+ \text{ and } t \in \langle 0, T \rangle.$$

Taking account (2.1), (2.2) and Lemma 1° we see that the condition (5°) of Theorem 1° is fulfilled. This completes the proof.

**Remark.** The existence and the uniqueness of the Green operator $G(t, s)$ of problem (1.6) assures the existence and the uniqueness of the solution $X(t)$ of the following problem:

$$\begin{cases} \dfrac{dX(t)}{dt} = A_1(t)X(t), \ t \in \langle 0, T \rangle, \\ X(0) = X_0, \ X_0 \in H^+ \times H. \end{cases}$$

The solution of this problem takes a form: $X(t) = G(t, 0)X_0$. By (5°) of Theorem 1, we have $X(t) = \left(x(t), \dfrac{dx(t)}{dt}\right) \in C^1(\langle 0, T \rangle; H \times H_0^-)$, thus $x(t) \in C^1(\langle 0, T \rangle; H) \cap C^2(\langle 0, T \rangle; H_0^-)$ and, by (4°) of Theorem 1, $x(t) \in C^0(\langle 0, T \rangle; H_0^+)$.

Consequently $x(t) \in C^0(\langle 0, T \rangle; H_0^+) \cap C^1(\langle 0, T \rangle; H) \cap C^2(\langle 0, T \rangle; H_0^-)$.

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## STRESZCZENIE

Opierając się na wynikach [2], w pracy tej dowodzi się dwóch twierdzeń dotyczących problemu istnienia i jednoznaczności rozwiązania pewnego zadania Cauchy'ego drugiego rzędu w przestrzeni Hilberta.

## РЕЗЮМЕ

Пользуясь результатами [2] в работе доказываются две теоремы касающиеся проблемы существования и единственности решения некоторой задачи Коши второго порядка в гильбертовом пространстве.