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On Some Properties of Integral Moduli of Continuity of Functions of Several Variables Integrable with Mixed Powers

O pewnych własnościach całkowego modułu ciągłości funkcji wielu zmiennych całkowalnych względem mieszanych potęg

О некоторых качествах интегральнного модуля непрерывности многих измснённых функций интегрированных по отношению к смешанным степеням

Investigation of absolute convergence of multiple Fourier series of functions belonging to the space L^p with mixed powers requires application of p – integral moduli od continuity, where $p = (p_1, p_2, \ldots, p_n)$ is a system of powers. This paper gives some properties of such moduli which were applied in [5], analogous to those given in [4] in case of a single power p.

Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in an n-dimensional euclidean space and let $f(x) = f(x_1, \ldots, x_a)$ be a real-valued function periodic with period $b_i - a_i$ in variable x_i, $0 < a_i < b_i$, $i = 1, 2, \ldots, n$. Let $H = \{k_1, k_2, \ldots, k_s\}$, be a nonvoit system of indices, i.e. $H \subset E$, where $E = \{1, 2, \ldots, h\}$. We define a difference $\Delta^H f$ of f with respect to variables whose indices belong to H, in the following manner. If $H = \{k\}$, then

$$\Delta^H(f; x; h) = f(x_1, \ldots, x_{k-1}, x_k + h_k, x_{k+1}, \ldots, x_n) - \\ -f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n),$$

where $h = (h_1, h_2, \ldots, h_n)$. If $H = \{k_1, k_2, \ldots, k_s\}$, where $s > 1$, $k_1 < k_2 < \ldots < k_s$, we define $\Delta^H f$ by induction as follows:

$$\Delta^H(f; x; h) = \Delta^{K_s}[\Delta^{H-\{K_s\}}(f; x; h); x; h].$$

Now, we define integral moduli od continuity corresponding to differences Δ^H in the metric of the space L^p with mixed powers. Let $d = (p_1, p_2, \ldots, p_n)$ be a system of numbers, $1 \leqslant p_i < \infty$, $i = 1, 2, \ldots, n$. Then the space L^p, called the space of functions integrable with mixed powers over the n — dimensional cube $Q = (a_1, b_1)x \ldots x(a_n, b_n)$ (n times), is defined in the following manner. A measurable function $f(x) = f(x_1, \ldots, x_n)$ in the cube Q belongs to L^p, if the value obtained by applying to L^{p_1} — norm to f with respect ot x_1, then the L^{p_2} norm with respect to x_2, etc., the L^{p_n} — norm with respect to x_n, is finite. This value is defined by

$$\|f\|_p = \left[\int_{a_n}^{b_n}\left(\cdots\left[\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1}|f(x_1, \ldots, x_n)|^{p_1}dx_1\right)^{p_2/p_1}dx_2\right]^{p_3/p_2}\cdots\right)^{p_n/p_{n-1}}dx_n\right]^{1/p_n}.$$

Two functions f, g are supposed to be equal in L^p, if $\|f-g\|_p = 0$. In particular, if $p_1 = p_2 = \ldots = p_n = p$ ([2], [3]), then the space L^p is identical with the space L^p of functions integrable with power p in Q. Let $\|\cdot\|_p$ be the norm in L^p, $p = (p_1, p_2, \ldots, p_n)$, and $f \in L^p$ $(b_i - a_i)$ — periodic with respect to the variable x_i, $i = 1, 2, \ldots, n$. Then the p — th integral modulus of continuity corresponding to the set H of indices will be defined in the following way.

$$\omega_p^H(f; h) = \sup \|\Delta^H(f; x; \delta)\|_p,$$

where $h = (h_1, h_2, \ldots, h_n)$, $f \in L^p$.

The following properties of the modulus $\omega_p^H(f; h)$ are well-known in the case of one variable (see e.g. [1]):

1. $\omega_p^H(\lambda f: h) = |\lambda|\,\omega_p^H(f; h)$.
2. $\omega_p^H(f+g; h) \leqslant \omega_p^H(f; h) + \omega_p^H(g; h)$.
3. If $0 < h_i' < h_i''$ for i H, then

$$\omega_p^H(f; h') \leqslant \omega_p^H(f; h''),$$

where $h' = (h_1', h_2', \ldots, h_n')$, $h'' = (h_1'', h_2'', \ldots, h_n'')$.

4. If $m_1, m_2, \ldots, m_n$ are nonnegative integers and $H = \{k_1, k_2, \ldots, k_s\}$, then

$$\omega_p^H(f; mh) \leqslant m_{k_1} \cdot \ldots \cdot m_{k_s}\,\omega_p^H(f; h),$$

where $h = (h_1, \ldots, h_n)$, $mh = (m_1 h_1, \ldots, m_n h_n)$.

5. If $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ and $H = \{k_1, \ldots, k_s\}$, then

$$\omega_p^H(f; \lambda h) \leqslant (\lambda_{k_1}+1)\ldots(\lambda_{k_s}+1)\,\omega_p^H(f; h).$$

If we take the general case $p = (p_1, p_2, \ldots, p_n)$, then properties 1—3 remain obvious and 5 follows from 4 as in case of functions of one variable. Our aim is to prove the property 4 in case $n = 2$, $p = (p_1, p_2)$. According to the three possibilities for H, the property 4 asserts the following:

$$H = \{1\},\ \omega_p^H(f; m_1 h_1, m_2 h_2) \leqslant m_1\,\omega_p^H(f; h_1, h_2).$$

$$H = \{2\},\ \omega_p^H(f; m_1 h_1, m_2 h_2) \leqslant m_2\,\omega_p^H(f; h_1, h_2).$$

$$H = \{1, 2\},\ \omega_p^H(f; m_1 h_1, m_2 h_2) \leqslant m_1 \cdot m_2\,\omega_p^H(f; h_1, h_2).$$

Case $H = \{1\}$. Taking $\frac{\delta_1}{m_1} = \delta_1'$ and applying the same arguments as in the well-known case of a function of one variable we have

$$\omega_p^{\{1\}}(f; m_1 h_1, m_2 h_2)$$

$$= \sup_{|\delta_1| \leqslant m_1 h_1} \left[\int_c^d \left(\int_a^b |f(x_1 + \delta_1, x_2) - f(x_1, x_2)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{1/p_2}$$

$$= \sup_{|\delta_1'| \leqslant h_1} \left[\int_c^d \left(\int_a^b |f(x_1 + m_1 \delta_1', x_2) - f(x_1, x_2)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{1/p_2}$$

$$\leqslant \sup_{|\delta_1'| \leqslant h_1} \left[\int_c^d \left(\int_a^b \left(\sum_{k=1}^{m_1} |f[x_1 + k\delta_1', x_2) - f[x_1 + (k-1)\delta_1, x_2]| \right)^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{1/p_2}$$

$$\leqslant \sum_{k=1}^{m_1} \sup_{|\delta_1'| \leqslant h_1} \left\{ \int_c^d \left(\int_a^b |f(x_1 + k\delta_1', x_2) - f[x_1 + (k-1)\delta_1', x_2]|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right\}^{1/p_2}.$$

Let $t_1 = x_1 + (k-1)\delta_1'$, $t_2 = x_2$. Since f is $(b-a)$ — periodic with respect to x_1, we have

$$\omega_p^{\{1\}}(f; m_1 h_1, m_2 h_2)$$

$$\leqslant \sum_{k=1}^{m_1} \sup_{|\delta_1'| \leqslant h_1} \left\{ \int_c^d \left(\int_{a+(k-1)\delta_1'}^{b+(k+1)\delta_1'} |f(t_1 + \delta_1', t_2) - f(t_1, t_2)|^{p_1} dt_1 \right)^{p_2 p_1} dt_2 \right\}^{1/p_2}$$

$$= m_1 \sup_{|\delta_1'| \leqslant h_1} \left\{ \int_c^d \left(\int_a^b |f(t_1 + \delta_1', t_2) - f(t_1, t_2)|^{p_1} dt_1 \right)^{p_2/p_1} dt_2 \right\}^{1/p_2}$$

$$= m_1 \omega_p^{\{1\}}(f; h_1, h_2).$$

Case $H = \{2\}$. Analogously as in the case $H = \{1\}$ we take $\frac{\delta_2}{m_2} z\, \delta_2'$. Since f is $(d-c)$ — periodic with respect to x_2, so applying the substitution $t_1 = x_1$, $t_2 = x_2 + (k-1)\delta_2'$ and Minkowski inequality, we get

$$\omega_p^{\{2\}}(f; m_1 h_1, m_2 h_2)$$

$$= \sup_{|\delta_2| \leqslant m_2 h_2} \left[\int_c^d \left(\int_a^b |f(x_1, x_2 + \delta_2) - f(x_1, x_2)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{1/p_2}$$

$$\leqslant \sum_{k=1}^{m_2} \sup_{|\delta_2'| \leqslant h_2} \left[\int_c^d \left(\int_a^b |f[x_1, x_2 + k\delta_2'] - f[x_1, x_2 + (k-1)\delta_2']|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{1/p_2}$$

$$= \sum_{k=1}^{m_2} \sup_{|\delta_2'|\leqslant h_2} \Big[\int\limits_{c+(k-1)\delta_2'}^{d+(k-1)\delta_2'} \Big(\int\limits_a^b |f(t_1, t_2+\delta_2')-f(t_1, t_2)|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big]^{1/p_2}$$

$$= m_2 \sup_{|\delta_2'|\leqslant h_2} \Big[\int\limits_c^d \Big(\int\limits_a^b |f(t_1, t_2+\delta_2')-f(t_1, t_2)|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big]^{1/p2}$$

$$= m_2 \omega_p^{(2)}(f; h_1, h_2).$$

Case $H = \{1, 2\}$.

We have

$$\Delta^{(1,2)}(f; x_1, x_2; m_1 \delta_1', m_2 \delta_2') = \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \Delta^{(1,2)}[f; x_1+(k_1-1)\delta_1', x_2+(k_2-1)\delta_2'; \delta_1', \delta_2'].$$

Let $\dfrac{\delta_1}{m_1} = \delta_1'$, $\dfrac{\delta_2}{m_2} = \delta_2'$, $t_1 = x_1+(k_1-1)\delta_1'$, $t_2 = x_2+(k_2-1)\delta_2'$. Since f is $(b-a)$ — periodic with respect to x_1 and $(d-c)$ — periodic with respect to x_2, analogously as in the case of $H = \{1\}$ and $H = \{2\}$, we have

$$\omega_p^{(1,2)}(f; m_1 h_1, m_2 h_2)$$

$$= \sup_{\substack{|\delta_1|\leqslant m_1 h_1 \\ |\delta_2|\leqslant m_2 h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b |\Delta^{(1,2)}(f; x_1, x_2; \delta_1, \delta_2)|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{1/p_2}$$

$$= \sup_{\substack{|\delta_1'|\leqslant h_1 \\ |\delta_2'|\leqslant h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b |\Delta^{(1,2)}(f; x_1, x_2; m_1\delta_1', m_2\delta_2')|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{1/p_2}$$

$$= \sup_{\substack{|\delta_1'|\leqslant h_1 \\ |\delta_2'|\leqslant h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b \Big| \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \Delta^{(1,2)}[f; x_1+(k_1-1)\delta_1', \times$$

$$\times x_2+(k_2-1)\delta_2'; \delta_1', \delta_2'] \Big|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{1/p_2}$$

$$\leqslant \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \sup_{\substack{|\delta_1'|\leqslant h_1 \\ |\delta_2'|\leqslant h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b |\{f[x_1+k_1\delta_1', x_2+k_2\delta_2']-$$

$$-f[x_1+k_1\delta_1', x_2+(k_2-1)\delta_2']-f[x_1+(k_1-1)\delta_1', x_2+k_2\delta_2']+$$

$$+f[x_1+(k_1-1)\delta_1', x_2+(k_2-1)\delta_2']\}|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big]^{1/p_2}$$

$$= \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \sup_{\substack{|\delta_1'|\leq h_1 \\ |\delta_2'|\leq h_2}} \Big\{ \int\limits_{c+(k_2-1)\delta_2'}^{d+(k_2-1)\delta_2'} \Big(\int\limits_{a+(k_1-1)\delta_1'}^{b+(k_1-1)\delta_1'} |[f(t_1+\delta_1', t_2+\delta_2') -$$

$$-f(t_1+\delta_1', t_2) - f(t_1, t_2+\delta_2') + f(t_1, t_2)]|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big\}^{1/p_2}$$

$$= \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \sup_{\substack{|\delta_1'|\leq h_1 \\ |\delta_2'|\leq h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b |\Delta^{\{1,2\}}(f; t_1, t_2; \delta_1', \delta_2')|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big]^{1/p_2}$$

$$= m_1 \cdot m_2 \sup_{\substack{|\delta_1'|\leq h_1 \\ |\delta_2'|\leq h_2}} \Big[\int\limits_c^d \Big(\int\limits_a^b |\Delta^{(1,2)}(f; t_1, t_2; \delta_1', \delta_2')|^{p_1} dt_1\Big)^{p_2/p_1} dt_2\Big]^{1/p_2}$$

$$= m_1 \cdot m_2 \, \omega_p^{(1,2)}(f; h_1, h_2).$$

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STRESZCZENIE

W pracy są określone cząstkowe całkowe moduły ciągłości funkcji wielu zmiennych z mieszanymi potęgami oraz przedstawione podstawowe własności tych modułów.

РЕЗЮМЕ

В работе определены частичные интегральные модули непрерывности функций многих переменных со смешанными степенями, и представлены основные свойства этих модулей.