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University of Delaware, Newark, Delaware 19711, USA

HERB SILVERMAN

# Products of Starlike and Convex Functions

Iloczyny funkcji gwiaździstych i wypukłych
Произведения звёздных и выпуклых функций

## 1. Introduction

Let $S$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z| < 1$. A function $f \in S$ is said to be *starlike of order* $a$, $(0 \leqslant a \leqslant 1)$, denoted $f \in S^*(a)$, if

$$\mathrm{Re}\left\{z\frac{f'}{f}\right\} > a \quad (|z| < 1)$$

and is said to be *convex of order* $a$, denoted $f \in K(a)$, if

$$\mathrm{Re}\left\{1 + z\frac{f''}{f'}\right\} > a \quad (|z| < 1).$$

We determine the order of starlikeness of

$$h(z) = z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{a_i}\prod_{j=1}^{m}\left(g_j'(z)\right)^{b_j},$$

where $f_i \in S^*(a)$, $g_j \in K(\beta)$ and $a_i, b_j \geqslant 0$. We also determine precise restrictions on $a, \beta, a_i, b_j$ for which

$$H(z) = \int_0^z \frac{h(t)}{t}\,dt$$

is univalent and close-to-convex. Finally, we assume $a_i = a_1$ and $b_j = b_1$, and vary the orders of starlikeness of $f_i(z)$ for each $i$ and the orders of convexity of $g_j(z)$ for each $j$ to obtain conditions for which $H(z)$ is close-to-convex. These results generalize some of those of Kim and Merkes [2], Merkes and Wright [3], and Schild [5].

## 2. Orders of starlikeness and convexity theorems

**Theorem 1.** *Suppose* $f_i \in S^*(\alpha)$ $(i = 1, 2, \ldots n)$ and $g_j \in K(\beta)$ $(j = 1, 2, \ldots, m)$. *Let*

$$h(z) = z \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{a_i} \prod_{j=1}^{m} (g_j'(z))^{b_j}, \text{ where } a_i, b_j \geqslant 0,$$

*and set* $\sum_{i=1}^{n} a_i = a$, $\sum_{j=1}^{m} b_j = b$. *Then* $h(z) \in S^*\{1 - a(1-\alpha) - b(1-\beta)\}$. *This result is sharp.*

**Proof.** Forming the derivative of

$$\log h(z) = \log z + \sum_{i=1}^{n} a_i(\log f_i(z) - \log z) + \sum_{j=1}^{m} b_j \log g_j'(z),$$

we obtain

$$(1) \qquad \frac{zh'}{h} = 1 + \sum_{i=1}^{n} a_i\left(z\frac{f_i'}{f_i} - 1\right) + \sum_{j=1}^{m} b_j \frac{zg_j''}{g_j'}$$

$$= 1 - a - b + \sum_{i=1}^{n} a_i \frac{zf_i'}{f_i} + \sum_{j=1}^{m} b_j\left(1 + \frac{zg_j''}{g_j'}\right).$$

Taking real parts in (1) leads to

$$\mathrm{Re}\frac{zh'}{h} \geqslant 1 - a - b + \alpha \sum_{i=1}^{n} a_i + \beta \sum_{j=1}^{m} b_j = 1 - a(1-\alpha) - b(1-\beta).$$

This completes the proof. To show sharpness, set

$$f_i = \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad g_j = \int_0^z \frac{dt}{(1-t)^{2(1-\beta)}}$$

for all $i$ and $j$. Then

$$h(z) = \frac{z}{(1-z)^{2[(1-\alpha)a + (1-\beta)b]}} \in S^*\{1 - a(1-\alpha) - b(1-\beta)\},$$

but is starlike of no greater order. Note that this function is not even univalent when $a(1-\alpha) + b(1-\beta) > 1$.

Setting $a_i = 0$, $b = b_1 = 1$, and $g = g_1$, we obtain the well known

**Corollary 1.** *$g \in K(\beta)$ implies $zg' \in S^*(\beta)$.*

Setting $a_1 = 0$, $b = b_1$, and $g = g_1$, we get

**Corollary 2.** *$g \in K(0)$ implies $z(g')^b \in S^*(1-b)$, a result of* Schild [5].

Setting $a = a_1$, $f = f_1$ and $b_j = 0$, we get

**Corollary 3.** *$f \in S^*(0)$ implies $z\left(\frac{f(z)}{z}\right)^a \in S^*(1-z)$, also a result of* Shild [5].

**Theorem 2.**

$$H(z) = \int_0^z \left(\prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{a_i} \prod_{j=1}^m (g_j'(t))^{b_j}\right) dt \in K\{1-a(1-a)-b(1-\beta)\}.$$

**Proof.** This follows from Theorem 1 and the relationship

$$H \in K\{1-a(1-a)-b(1-\beta)\} \text{ if and only if } zH' = h \in S^*\{1-a(1-a)- -b(1-\beta)\}.$$

**Remark.** When $a_i = 0$ and $b = b_1 + b_2 \leqslant 1$, this reduces to a result of Kim and Merkes [2].

## 3. A close-to-convex theorem

We will need the following:

**Lemma.** *Suppose $P(z)$ is analytic in $|z| < 1$ with $P(0) = 1$ and* $\operatorname{Re} P(z) > \gamma$. *Then for $z = re^{i\theta}$ and $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$, we have*

$$\gamma(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re} P(z)\, d\theta \leqslant 2\pi(1-\gamma) + \gamma(\theta_2 - \theta_1).$$

**Proof.** The left hand inequality is immediate. The right hand inequality follows from

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{P(z)-\gamma}{1-\gamma}\right\} d\theta \leqslant \int_0^{2\pi} \operatorname{Re}\left\{\frac{P(z)-\gamma}{1-\gamma}\right\} d\theta = 2\pi,$$

where this last equality is a consequence of the mean value theorem for harmonic functions.

We may now prove a theorem about $H(z)$ without the restriction that the exponents $a_i, b_j$ be positive.

**Theorem 3.** *Suppose $f_i \in S^*(a)$ $(i = 1, \ldots, n)$ and $g_j \in K(\beta)$ $(j = 1, \ldots, m)$. Let*

$$H(z) = \int_0^z \left(\prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{a_i} \prod_{j=1}^m (g_j'(t))^{b_j}\right) dt,$$

*and set*

$$a = \sum_{i=1}^{n} a_i = \sum_i a_{i+} + \sum_i a_{i-} = a_+ + a_-,$$

$$b = \sum_{j=1}^{m} b_j = \sum_j b_{j+} + \sum_j b_{j-} = b_+ + b_-,$$

*where $\{a_{i+}\}$ and $\{b_{j+}\}$ are, respectively, the subsequences of $\{a_i\}$ and $\{b_j\}$ consisting of the positive terms, and $\{a_{i-}\}$ and $\{b_{j-}\}$ are the subsequences consisting of the negative terms. Then $H(z)$ is close-to-convex if*

$$-\frac{1}{2} \leqslant a_-(1-\alpha) + b_-(1-\beta) \leqslant a_+(1-\alpha) + b_+(1-\beta) \leqslant \frac{3}{2}.$$

*This result is sharp.*

**Proof.** By a criterion of Kaplan [1], $H(z)$ is close-to-convex if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + z\frac{H''(z)}{H'(z)}\right\} d\theta \geqslant -\pi$$

for all $\theta_1, \theta_2$ satisfying $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$ and $0 < r < 1$. We have

$$(2)\qquad 1 + \frac{zH''}{H'} = 1 - a - b + \sum_{i=1}^{n} a_i \frac{zf_i'}{f_i} + \sum_{j=1}^{m} b_j\left(1 + \frac{zg_j''}{g_j}\right).$$

Taking real parts in (2), and integrating from $\theta_1$ to $\theta_2$ we get

$$(3)\qquad \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zH''}{H'}\right\} d\theta = (1-a-b)(\theta_2 - \theta_1)$$
$$+ \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\sum_i a_{i+}\frac{zf_{i+}'}{f_{i+}} + \sum_j b_{j+}\left(1 + \frac{zg_{j+}''}{g_{j+}'}\right)\right\} d\theta$$
$$+ \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\sum_i a_{i-}\frac{zf_i'}{f_{i-}} + \sum_j b_{j-}\left(1 + \frac{zg_{j-}''}{g_{j-}}\right)\right\} d\theta.$$

Using the left hand inequality of the lemma on the first integral on the right side of (3), and the right hand inequality on the second integral, we obtain

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zH''}{H'}\right\} d\theta \geqslant (1-a-b)(\theta_2 - \theta_1) + (a_+\alpha + b_+\beta)(\theta_2 - \theta_1) +$$
$$+ (a_-\alpha + b_-\beta)(\theta_2 - \theta_1) + 2\pi[a_-(1-\alpha) + b_-(1-\beta)]$$
$$= [1 - a(-\alpha) - b(1-\beta)](\theta_2 - \theta_1) + 2\pi[a_-(1-\alpha) + b_-(1-\beta)].$$

This last expression, denoted by $q(\theta_2-\theta_1)$, is a linear function of $(\theta_2-\theta_1)$ and assumes its minimum at either 0 or $2\pi$, depending on whether $1-a(1-a)-b(1-\beta)$ is positive or negative. We have

$$q(0) = 2\pi[a_-(1-a)+b_-(1-\beta)]$$

and

$$q(2\pi) = 2\pi[1-a_+(1-a)-b_+(1-\beta)].$$

Thus $q(0)\geqslant -\pi$ when

$$a_-(1-a)+b_-(1-\beta)\geqslant \frac{1}{2}, \tag{4}$$

and $q(2\pi)\geqslant -\pi$ when

$$a_+(1-a)+b_+(1-\beta)\leqslant \frac{3}{2}. \tag{5}$$

Now $H(z)$ will be close-to-convex whenever $\min\{q(0), q(2\pi)\}\geqslant -\pi$, that is, when both (4) and (5) are satisfied. This completes the proof.

To show sharpness, set $f_i = \dfrac{z}{(1-z)^{2(1-a)}}$ and

$$g_j = \int_0^z \frac{dt}{(1-t)^{2(1-\beta)}}$$

for all $i$ and $j$. Then

$$H(z) = \int_0^z \frac{dt}{(1-t)^{2[(1-a)a+(1-\beta)b]}}.$$

By a Theorem of Royster [4], $H(z)$ is univalent if and only if $2[(1-a)a+(1-\beta)b]\in[-1,3]$. Thus $H(z)$ is not univalent when $a_i, b_j\geqslant 0$ with $(1-a)a+(1-\beta)b>\frac{3}{2}$, or $a_i, b_j<0$ with $(1-a)a+(1-\beta)b<-\frac{1}{2}$.

**Remark.** When $a = a_1$, $b_i = 0$, and $a = 0$ or $a = \frac{1}{2}$, we get results of Merkes and Wright [3]. When $a_i = 0$, $b = b_1+b_2$, and $\beta = 0$, we get a result of Kim and Merkes [2].

## 4. Related classes

By fixing the exponents in our previous classes, we may vary the orders of starlikeness and convexity to obtain results analogous to the previous theorems.

**Theorem 4.** *Suppose* $f_i \in S^*(\alpha_i)$ $(i = 1, \ldots, n)$ *and* $g_j \in K(\beta_j)$ $(j = 1, \ldots, m)$. *Let*

$$h(z) = z \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^a \prod_{j=1}^{m} (g_j'(z))^b,$$

*where* $a, b \geqslant 0$. *Set*

$$\alpha^* = \frac{\sum_{i=1}^{n} \alpha_i}{n} \quad \text{and} \quad \beta^* = \frac{\sum_{j=1}^{m} \beta_j}{m}.$$

*Then* $h(z) \in S^*\{1 - an(1-\alpha^*) - bm(1-\beta^*)\}$. *This result is sharp.*

**Proof.** Forming the logarithmic derivative, we have

$$\frac{zh'}{h} = 1 + a \sum_{i=1}^{n} \left(\frac{zf_i'}{f_i} - 1\right) + b \sum_{j=1}^{m} \frac{zg_j''}{g_j'}$$

$$= 1 - na - mb + a \sum_{i=1}^{n} \frac{zf_i'}{f_i} + b \sum_{j=1}^{m} \left(1 + \frac{zg_j''}{g_j'}\right).$$

Taking real parts leads to

$$\mathrm{Re}\frac{zh'}{h} \geqslant 1 - na - mb + an\alpha^* + bm\beta^*,$$

and the result follows. To show sharpness, set

$$f_i = \frac{z}{(1-z)^{2(1-\alpha_i)}} \quad \text{and} \quad g_j = \int_0^z \frac{dt}{(1-t)^{2(1-\beta_j)}}$$

for all $i$ and $j$.

Just as Theorem 2 followed from Theorem 1, so the next theorem follows from Theorem 4.

**Theorem 5.** *Under the conditions of* Theorem 4,

$$H(z) = \int_0^z \left(\prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^a \prod_{j=1}^{m} (g_j'(t))^b dt \in K\{1 - an(1-\alpha^*) - bm(1-\beta^*)\}.$$

Finally, we prove a theorem analogous to Theorem 3.

**Theorem 6.** *Under the same conditions as* Theorem 5, *except that we allow* $a, b$ *to be any real numbers,* $H(z)$ *is close-to-convex if*

$$-\frac{1}{2} \leqslant an(1-\alpha^*) + bm(1-\beta^*) \leqslant \frac{3}{2} \qquad (ab \geqslant 0)$$

$$\begin{cases} -\frac{1}{2} \leqslant an(1-\alpha^*) \leqslant \frac{3}{2} \\ -\frac{1}{2} \leqslant bm(1-\beta^*) \leqslant \frac{3}{2} \end{cases} \qquad (ab < 0)$$

*This result is sharp.*

**Proof.** We have

$$1 + \frac{zH''}{H'} = 1 - na - mb + a\sum_{i=1}^{n} z\frac{f_i'}{f_i} + b\sum_{j=1}^{m}\left(1 + \frac{zg_j''}{g_j'}\right). \tag{6}$$

In view of the lemma preceding Theorem 3,

$$na^*(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\sum_{i=1}^{n} \frac{zf_i'}{f_i}\right)d\theta \leqslant 2\pi n(1-a^*) + na^*(\theta_2-\theta_1), \tag{7}$$

$$m\beta^*(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\sum_{j=1}^{m}\left(1 + \frac{zg_j''}{g_i'}\right)\right\}d\theta \leqslant 2\pi m(1-\beta^*) + m\beta^*(\theta_2-\theta_1). \tag{8}$$

Starting with the identity in (6), we minimize

$$\int_{\theta_1}^{\theta_2}\left\{\mathrm{Re}1 + \frac{zH''}{H'}\right\}d\theta \qquad \text{over all} \qquad 0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$$

by using either the left or right inequalities in (7) or (8) according as $a$ and $b$ are positive or negative. The result follows, as in Theorem 3, upon determining when the appropriate minimums are $\geqslant -\pi$. In all cases, sharpness is found by setting

$$f_i = \frac{z}{(1-z)^{2(1-\alpha_i)}} \qquad \text{and} \qquad g_j = \int_0^z \frac{dt}{(1-t)^{2(1-\beta_j)}}$$

for all $i$ and $j$.

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## STRESZCZENIE

Przedmiotem noty jest podanie warunków na funkcję holomorficzną $h(z)$, przy których funkcja $\int_0^z t^{-1}h(t)dt$ jest wypukła lub prawie wypukła w kole jednostkowym.

## РЕЗЮМЕ

Предметом данной работы является исследование условий для функций $h(z)$ при которых функция $\int_0^z t^{-1}h(t)dt$ является выпуклой или почти выпуклой в единичном круге.