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Convolutions of Univalent Functions with Negative Coefficients

Sploty funkcji jednolistnych o współczynnikach ujemnych

Свертки однолистных функций с отрицательными коэффициентами

1. Introduction

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in $|z| < 1$. $f \in S$ is said to be ***starlike of order*** a, $0 \leqslant a < 1$, if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > a$, $|z| < 1$. We denote this class by $S^*(a)$. Similarly, $f \in S$ is said to be ***convex of order*** a, $0 \leqslant a < 1$, if $\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > a$, $|z| < 1$. This class of functions is denoted by $K(a)$.

We denote by T the subclass of functions of S of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqslant 0$. We denote by $T^*(a)$ and $C(a)$ the subclasses of T which are, respectively, starlike of order a and convex of order a.

Functions of this type have been investigated, among others, by: Schild [6], Gray and Schild [1], Lewandowski [2, 3], Pilat [4] and Silverman [7, 8]. Recently, Ruscheweyh and Sheil-Small [5] proved the Polya-Schoenberg conjecture that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K$, then $h(z) = f(z)^* g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K$. We investigate some properties of $h(z) = f(z)^* g(z)$ where $f(z), g(z) \in T^*(a)$ or $C(a)$. The following two theorems proved in (7) will be used repeatedly in this paper. They are:

(A) *A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqslant 0$, *is in* $T^*(a)$ *if and only if* $\sum_{n=2}^{\infty} (n-a) a_n \leqslant 1 - a$.

(**B**) *A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqslant 0$, *is in* $C(a)$ *if and only if* $\sum_{n=2}^{\infty} n(n-a)a_n \leqslant 1-a$.

2. Convolutions of functions from subclasses of $T(a)$

We investigate now the nature of $h(z) = f(z)^* g(z)$, given that $f(z)$ and $g(z)$ are members of $T^*(a)$ and $C(a)$.

Theorem 1. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geqslant 0$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geqslant 0$ *are elements of* $T^*(a)$, *then* $h(z) = f(z)^* g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ is an element of $T^*\left(\frac{2-a^2}{3-2a}\right)$. *The result is best possible.*

Proof. From (**A**) we know that $\sum_{n=2}^{\infty} (n-a)a_n \leqslant 1-a$ and $\sum_{n=2}^{\infty} (n-a)b_n \leqslant 1-a$. We wish to find the largest $\beta = \beta(a)$ such that $\sum_{n=2}^{\infty} (n-\beta)a_n b_n \leqslant 1-\beta$. Equivalently, we want to show that

$$\sum_{n=2}^{\infty} \left(\frac{n-a}{1-a}\right) a_n \leqslant 1 \tag{1}$$

and

$$\sum_{n=2}^{\infty} \left(\frac{n-a}{1-a}\right) b_n \leqslant 1 \tag{2}$$

imply that

$$\sum_{n=2}^{\infty} \left(\frac{n-\beta}{1-\beta}\right) a_n b_n \leqslant 1 \text{ for all } \beta = \beta(a) \leqslant \frac{2-a^2}{3-2a}. \tag{3}$$

From (1) and (2) we get by means of the Cauchy-Schwarz inequality:

$$\sum_{n=2}^{\infty} \left(\frac{n-a}{1-a}\right) \sqrt{a_n b_n} \leqslant 1. \tag{4}$$

It will be, therefore, sufficient to prove that

$$\left(\frac{n-\beta}{1-\beta}\right) a_n b_n \leqslant \left(\frac{n-a}{1-a}\right) \sqrt{a_n b_n}, \; \beta \leqslant \beta(a), \; n = 2, 3, \ldots$$

or:

$$\sqrt{a_n b_n} \leqslant \left(\frac{n-a}{1-a}\right)\left(\frac{1-\beta}{n-\beta}\right).$$

From (4) it follows that $\sqrt{a_n b_n} \leqslant \dfrac{1-a}{n-a}$ for each n. Hence, it will be sufficient to show that

$$\frac{1-a}{n-a} \leqslant \left(\frac{n-a}{1-a}\right)\left(\frac{1-\beta}{n-\beta}\right) \text{ for all } n. \tag{5}$$

Inequality (5) is equivalent to:

$$\beta \leqslant \frac{1-n\left(\dfrac{1-a}{n-a}\right)^2}{1-\left(\dfrac{1-a}{n-a}\right)^2}. \tag{6}$$

The right hand side of (6) is an increasing function of $n (n = 2, 3, \ldots)$. Therefore, setting $n = 2$ in (6) we get:

$$\beta \leqslant \frac{1-2\left(\dfrac{1-a}{2-a}\right)^2}{1-\left(\dfrac{1-a}{2-a}\right)^2} = \frac{2-a^2}{3-2a}.$$

The result is sharp, with equality when $f(z) = g(z) = z - \dfrac{1-a}{2-a} z^2 \in T^*(a)$.

Remark. Clearly, $\beta = \beta(a) = \dfrac{2-a^2}{3-2a} > a,\ 0 \leqslant a < 1$. It is rather surprising, though, that if $f, g \in T^*(0)$ then $h \in T^*\left(\dfrac{2}{3}\right)$.

Corollary. *For* $f(z)$ *and* $g(z)$ *as in* Theorem 1, *we have:* $h(z) = z - \sum\limits_{n=2}^{\infty} \sqrt{a_n b_n} \cdot z^n \in T^*(a)$. *This result follows from the Cauchy-Schwarz inequality* (4). *It is sharp for the same functions as in* Theorem 1.

Theorem 2. *If* $f \in T^*(a)$ *and* $g \in T^*(\gamma)$, *then* $f^* g \in T^*\left(\dfrac{2-a\gamma}{3-a-\gamma}\right)$.

Proof. Proceeding as in the proof of Theorem 1, we get:

$$\beta \leqslant \frac{1-n\left(\dfrac{1-a}{n-a}\right)\left(\dfrac{1-\gamma}{n-\gamma}\right)}{1-\left(\dfrac{1-a}{n-a}\right)\left(\dfrac{1-\gamma}{n-\gamma}\right)} \tag{7}$$

As in Theorem 1, β is an increasing function of n. Since $n \geqslant 2$, setting $n = 2$ in (7) we get:

$$\beta \leqslant \frac{1-2\left(\frac{1-a}{2-a}\right)\left(\frac{1-\gamma}{2-\gamma}\right)}{1-\left(\frac{1-a}{2-a}\right)\left(\frac{1-\gamma}{2-\gamma}\right)} = \frac{2-a\gamma}{3-a-\gamma}.$$

Corollary. *If* $f(z), g(z), h(z) \in T^*(a)$, *then* $f^* g^* h \in T^*\left(\frac{6-6a+a^3}{7-9a+3a^2}\right)$.

Proof. From Theorem 1 we have: $f^* g \in T^*\left(\frac{2-a^2}{3-2a}\right)$

We use now Theorem 2 and get:

$$(f^* g)^* h \in T^*\left(\frac{2-a\cdot\frac{2-a^2}{3-2a}}{3-a-\frac{2-a^2}{3-2a}}\right) = T^*\left(\frac{6-6a+a^3}{7-9a+3a^2}\right).$$

For functions of class $C(a)$ we have similar results. We have:

Theorem 3. *If* $f \in C(a)$ *and* $g \in C(\gamma)$ *then* $f^* g \in C\left(\frac{2(3-a-\gamma)}{7-3a-3\gamma+a\gamma}\right)$

Proof. From (B) we know that $\sum_{n=2}^{\infty} n(n-a)a_n \leqslant 1-a$ and $n(n-\gamma)b_n \leqslant 1-\gamma$. We wish to find the largest $\beta = \beta(a, \gamma)$ such that $\sum_{n=2}^{\infty} n(n-\gamma)a_n b_n \leqslant 1-\beta$. Equivalently, we want to show that

$$\sum_{n=2}^{\infty} \frac{n(n-a)}{1-a} a_n \leqslant 1 \text{ and}$$

$$\sum_{n=2}^{\infty} \frac{n(n-\gamma)}{1-\gamma} b_n \leqslant 1 \text{ imply}$$

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta} a_n b_n \leqslant 1 \text{ for all } \beta = \beta(a, \gamma) = \frac{2(3-a-\gamma)}{7-3a-3\gamma+a\gamma}.$$

Proceeding similarly as in the proofs of Theorems 1 and 2, we get:

$$\frac{n-\beta}{1-\beta} \leqslant \frac{n(n-a)(n-\gamma)}{(1-a)(1-\gamma)}$$

or

$$\beta \leqslant \frac{1-\dfrac{(1-a)(1-\gamma)}{(n-a)(n-\gamma)}}{1-\dfrac{1-a)(1-\gamma)}{n(n-a)(n-\gamma)}}. \tag{8}$$

The R.H.S. of (8) is an increasing function of n. Hence we replace n by 2 and we get our result.

Remark. As was pointed out earlier, it follows from Theorem 1 that if $f, g \epsilon T^*(0)$ then $f^* g \epsilon T^*\left(\frac{2}{3}\right)$. In general, if $h(z) \epsilon T^*(a)$, $0 \leqslant a < 1$, it does not follow that $h(z) \epsilon C(\beta)$ for any $0 \leqslant \beta < 1$. The following theorem is, therefore, rather surprising:

Theorem 4. *If* $f, g \epsilon T^*(0)$, *then* $f^* g \epsilon C(0)$, i.e. *the convolution of any two functions of* $T^*(0)$ *is convex.*

Proof. If $f, g \epsilon T^*(0)$, then $\sum_{n=2}^{\infty} n a_n \leqslant 1$ and $\sum_{n=2}^{\infty} n b_n \leqslant 1$. But these two inequalities imply $\sum_{n=2}^{\infty} n^2 a_n b_n \leqslant 1$, i.e. $h(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \epsilon C(0)$ by (**B**). More generally we have:

Theorem 5. *If* $f \epsilon T^*(a)$ *and* $g \epsilon T^*(\gamma)$ *then* $f^* g \epsilon C\left(\dfrac{2a+2\gamma-3a\gamma}{2-a\gamma}\right)$.

Proof. Since $f \epsilon T^*(a)$ and $g \epsilon T^*(\gamma)$, therefore

$$\sum_{n=2}^{\infty} (n-a) a_n \leqslant 1-a \quad \text{and} \quad \sum_{n=2}^{\infty} (n-\gamma) b_n \leqslant 1-\gamma.$$

It follows that $\sum_{n=2}^{\infty} (n-a)(n-\gamma) a_n b_n \leqslant (1-a)(1-\gamma)$. We want to find the largest $\beta = \beta(a, \gamma)$ such that

$$\sum_{n=2}^{\infty} n(n-\beta) a_n b_n \leqslant 1-\beta.$$

This will certainly be satisfied if

$$\frac{n(n-\beta)}{1-\beta} \leqslant \frac{(n-a)(n-\gamma)}{(1-a)(1-\gamma)}, \text{ i.e. for:}$$

$$\beta \leqslant \frac{1-\dfrac{n^2(1-a)(1-\gamma)}{(n-a)(n-\gamma)}}{1-\dfrac{n(1-a)(1-\gamma)}{(n-a)(n-\gamma)}}$$

The RHS is an increasing function of n. Replacing n by 2 we get our result. The result is sharp. Equality is attained for

$$f(z) = z - \frac{1-a}{(2-a)} z^2 \epsilon T^*(a) \text{ and } g(z) = z - \frac{1-\gamma}{(2-\gamma)} z^2 \epsilon T^*(\gamma).$$

Theorem 6. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \epsilon T^*(a)$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, *with* $|b_i| \leqslant 1$, $i = 2, 3, \ldots$, *then* $f^* g \epsilon S^*(a)$.

Proof. $\sum_{n=2}^{\infty} (n-a)|a_n b_n| = \sum_{n=2}^{\infty} (n-a)|a_n||b_n| \leqslant \sum_{n=2}^{\infty} (n-a) a_n = 1-a$. Note that $g(z)$ need not be schlicht.

Corollary. *If* $f(z) \epsilon T^*(a)$ *and* $g(z) = z - \sum_{n=2} b_n z^n$, *with* $0 \leqslant b_i \leqslant 1$, $i = 2, 3, \ldots$, *then* $f^* g \epsilon T^*(a)$.

Theorem 7. *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \epsilon C(a)$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ *with* $|b_i| \leqslant 1$, $i = 2, 3, \ldots$, *then* $f^* g \epsilon K(a)$.

Proof. $\sum_{n=2}^{\infty} n(n-a)|a^n b^n| \leqslant \sum_{n=2}^{\infty} n(n-a) a_n \leqslant 1-a$.

Corollary. *If* $f(z) \epsilon C(a)$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, *with* $0 \leqslant b_i \leqslant 1$, $i = 2, 3, \ldots$, *then* $f^* g \epsilon C(a)$.

The functions $f(z) = z - \frac{1-a}{2-a} z^2$ and $g(z) = z - \frac{1-a}{3-a} z^3$ are both $\epsilon T^*(a)$. However, $h(z) = z - \frac{1-a}{2-a} z^2 - \frac{1-a}{3-a} z^3 \notin T^*(0)$ for some a. This shows that if $f, g \epsilon T^*(a)$, we need not necessarily have that $h(z) = z - \sum_{n=2}^{\infty} (a_n + b_n) z^n \epsilon T(\beta)$ for any $\beta \geqslant 0$:

However, we have:

Theorem 8. *If* $f, g \epsilon T^*(a)$, *then* $h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \epsilon T^* \left(\frac{4a - 3a^2}{2 - a^2}\right)$.

Proof. Since $\sum_{n=2}^{\infty} (n-a) a_n \leqslant 1-a$, therefore:

$$\sum_{n=2}^{\infty} \left(\frac{n-a}{1-a}\right)^2 a_n^2 \leqslant \left\{\sum_{n=2}^{\infty} \frac{n-a}{1-a} a_n\right\}^2 \leqslant 1.$$

Similarly:

$$\sum_{n=2}^{\infty} \left(\frac{n-a}{1-a}\right)^2 b_n^2 \leqslant 1$$

and hence:

$$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{n-a}{1-a}\right)^2 (a_n^2+b_n^2) \leqslant 1. \tag{9}$$

We want to find the largest $\beta = \beta(a)$ such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} (a_n^2+b_n^2) \leqslant 1. \tag{10}$$

Comparing this with (9), we see that (10) will be satisfied if

$$\frac{n-\beta}{1-\beta} \leqslant \frac{1}{2}\left(\frac{n-a}{1-a}\right)^2$$

or:

$$\beta \leqslant \frac{\left(\frac{n-a}{1-a}\right)^2 - 2n}{\left(\frac{n-a}{1-a}\right)^2 - 2} \tag{11}$$

The RHS of (11) is an increasing function of n. Since $n \geqslant 2$,

$$\beta \leqslant \frac{\left(\frac{2-a}{1-a}\right)^2 - 4}{\left(\frac{2-a}{1-a}\right)^2 - 2} = \frac{4a-3a^2}{2-a^2}.$$

The result is sharp for the functions $f(z) = g(z) = z - \frac{1-a}{2-a} z^2$. Note that if in Theorem 5 we let $\gamma = a$, we get the same value for β as here.

In Theorem 1 we showed that if $f, g \in T^*(a)$ then $f^* g \in T^*\left(\frac{2-a^2}{3-a}\right)$. One is tempted to ask the following question: Given $h \in T^*\left(\frac{2-a^2}{3-a}\right)$ do there exist functions $f, g \in T^*(a)$, such that $h = f^* g$? The following example shows that the answer is no:

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, with $f, g \in T^*(a)$. Clearly

$$a_n \leqslant \frac{1-a}{n-a}, \quad b_n \leqslant \frac{1-a}{n-a}$$

and hence:

$$f^*g = z - \sum_{n=2}^{\infty} a_n b_n z^n \in T^*\left(\frac{2-a^2}{3-2a}\right)$$

by Theorem 1.

Note that $a_n b_n \leqslant \left(\frac{1-a}{n-a}\right)^2$, i.e. for the convolution of any two functions from $T^*(a)$ we have $a_n b_n \leqslant \left(\frac{1-a}{n-a}\right)^2$. Now consider:

$$h(z) = z - \frac{1 - \frac{2-a^2}{3-2a}}{n - \frac{2-a^2}{3-2a}} z^n \in T^*\left(\frac{2-a^2}{3-2a}\right).$$

For this function we have:

$$\frac{1 - \frac{2-a^2}{3-2a}}{n - \frac{2-a^2}{3-2a}} = \frac{(1-a)^2}{(3-2a)n + a^2 - 2} > \frac{(1-a)^2}{(n-a)^2} \text{ for } n \geqslant 3,$$

i.e. there is no f and $g \in T^*(a)$ such that $f^*g = h \in T^*\left(\frac{2-a^2}{3-2a}\right)$.

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STRESZCZENIE

W pracy tej rozważane są funkcje postaci

$$h(z) = f(z)*g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

gdzie $f(z) = \left(z - \sum_{n=2}^{\infty} a_n z^n\right)$, $g(z) = \left(z - \sum_{n=2}^{\infty} b_n z^n\right)$, $a_n \geqslant 0$, $b_n \geqslant 0$ należą do specjalnych podklas funkcji jednolistnych.

W szczególności rozważany jest problem, do jakiej klasy należy funkcja $h(z)$, jeśli o funkcjach f, g założymy, że należą do klas $T^*(a)$ lub $C(a)$, gdzie $T^*(a)$ jest podklasą funkcji a-gwiaździstych a $C(a)$ podklasą funkcji a-wypukłych.

РЕЗЮМЕ

В работе рассматриваются функции вида

$$h(z) = f(z)*g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

где $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $a_n \geqslant 0$, $b_n \geqslant 0$, принадлежат к специальным подклассам однолистных функций. В особенности рассматриваемая проблема, которому классу принадлежымт функция $h(z)$, если $f(z)$ и $g(z)$ принадлежат соотвемственно $T^*(a)$ и $C(a)$.