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Janowski Alpha-Convex Functions

Funkcje alfa-wypukłe Janowskiego

Альфа — выпуклые функции Яновского

1. Introduction

In this paper we combine the notions of Janowski starlike functions [1] and alpha-convex functions [2] to obtain a new subclass of starlike functions.

Let $f(z) = z+a_2z^2+\ldots$ be regular in the unit disc D and for $a \geqslant 0$ let

$$J\big(a, f(z)\big) = (1-a)\frac{zf'(z)}{f(z)} + a\left(\frac{zf''(z)}{f'(z)}+1\right) \tag{1}$$

We denote by $\mathcal{M}_a$ the class of functions $f(z)$ for which $\operatorname{Re} J\big(a, f(z)\big) > 0$ for $z \in D$. Note that $\mathcal{M}_0 = S^*$ the class of starlike functions. Functions in the class $\mathcal{M}_a$ are called alpha-convex functions and such functions have been shown to be starlike [2].

In [1] W. Janowski investigated properties of the class $S^*(M)$ of regular functions $f(z) = z+a_2z^2+\ldots$ satisfying

$$\left|\frac{zf'(z)}{f(z)} - M\right| < M,\ (M \geqslant 1) \tag{2}$$

for $z \in D$. It is clear that $S^*(M) \subset S^*$ and $S^*(\infty) = S^*$.

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Definition. Let $a \geqslant 0$ and suppose that $f(z) = z + a_2 z^2 + \ldots$ is regular in D with $\dfrac{f(z)\cdot f'(z)}{z} \neq 0$ in $0 < |z| < 1$. If

(3) $$|J(a, f(z)) - M| < M, \ (M \geqslant 1)$$

for $z \in D$, then $f(z)$ is said to be a Janowski alpha-convex function. We denote the class of such functions by $S^*(a, M)$.

Note that $S^*(M) = S^*(0, M)$, $\mathcal{M}_a = S^*(a, \infty)$ and $S^* = S^*(0, \infty)$. In addition we can prove that a Janowski alpha-convex function is both an alpha-convex function and is in Janowski's class $S^*(M)$.

Theorem 1. $S^*(a, M) \subset \mathcal{M}_a \cap S^*(M)$, $a \geqslant 0$, $M \geqslant 1$.

Proof. Let $f(z) \in S^*(a, M)$. From (3) we can see that $\operatorname{Re} J(a, f(z)) > 0$ and hence $S^*(a, M) \subset \mathcal{M}_a$.

Suppose that $f(z) \notin S^*(M)$. Since at the point $z = 0$ condition (2) is satisfied there exists a point $z_0 = r_0 r^{i\theta_0}$ $(0 < r_0 < 1)$ of D such that

(4) $$\left| \frac{zf'(z)}{f(z)} - M \right| \leqslant \left| \frac{z_0 f'(z_0)}{f(z_0)} - M \right| = M$$

for all $|z| \leqslant r_0$. If we let $p(z) = zf'(z)/f(z)$ then (4) becomes

(5) $$|p(z) - M| \leqslant |p(z_0) - M| = M,$$

and from (1) we obtain

(6) $$|J(a, f(z)) - M| = \left| p(z) + a\frac{zp'(z)}{p(z)} - M \right|.$$

If $p'(z_0) = 0$ then by (5) and (6) we obtain

$$|J(a, f(z_0)) - M| = M$$

If $p'(z_0) \neq 0$ then we must have $\arg z_0 p'(z_0) = \arg(p(z_0) - M) = \varphi$, and by (5) and (6) we obtain

$$|J(a, f(z_0)) - M| = \left| M + \frac{a\,|z_0 p'(z_0)|}{M + Me^{i\varphi}} \right| \geqslant M$$

In both cases we obtain $|J(a, f(z_0)) - M| \geqslant M$, which contradicts (2). Hence we must have $|zf'(z)/f(z) - M| < M$ for all $z \in D$, and $f(z) \in S^*(M)$.

The previous theorem shows that $S^*(a, M) \subset S^*(0, M)$. We can show more than this.

Theorem 2. *If $f(z) \in S^*(a, M)$, then $f(z) \in S^*(\beta, M)$ for all $0 \leqslant \beta \leqslant a$.*

Proof. We need only consider the case $0 < \beta < a$. Suppose $f(z) \notin S^*(\beta, M)$. Then there exists $\zeta \in D$ such that

(7) $$|J(\beta, f(\zeta)) - M| \geqslant M.$$

Since $f(z) \in S^*(a, M)$ we have

$$|J(a, f(\zeta)) - M| < M \tag{8}$$

We will show that (7) and (8) imply that $|\zeta f'(\zeta)/f(\zeta) - M| \geqslant M$, which contradicts Theorem 1. If we let $A = \zeta f'(\zeta)/f(\zeta) - M$ and $B = \zeta f''(\zeta)/f'(\zeta) - \zeta f'(\zeta)/f(\zeta) + 1$, then (7) and (8) become

$$|A + \beta B|^2 \geqslant M^2 \text{ and} \tag{9}$$

$$M^2 > |A + aB|^2. \tag{10}$$

After multiplying (9) by a and (10) by β and adding we obtain

$$(a - \beta)|A|^2 > a\beta(a - \beta)|B|^2 + (a - \beta)M^2$$

Since $a - \beta > 0$ we obtain

$$|A|^2 > a\beta|B|^2 + M^2 \geqslant M^2,$$

that is, $|\zeta f'(\zeta)/f(\zeta) - M| \geqslant M$, which is the desired contradiction.

If $f(z) \in S^*(1, M)$, then $f(z)$ must be a convex function. We see by Theorem 2 that if $f(z) \in S^*(a, M)$, $a \geqslant 1$, then $f(z)$ is a convex function.

2. Integral Representation

Theorem 3. *If $f(z) \in S^*(a, M)$, $a > 0$, and if for $0 < \beta < a$ we choose the branch of $[zf'(z)/f(z)]^\beta$ which is equal to 1 when $z = 0$, then the function $F_\beta(z) = f(z)[zf'(z)/f(z)]^\beta$ is in $S^*(M)$.*

Proof. A simple calculation yields

$$\frac{zF'_\beta(z)}{F_\beta(z)} = J(\beta, f(z))$$

Since $f(z) \in S^*(a, M)$, by Theorem 2 we have

$$\left|\frac{zF'_\beta(z)}{F_\beta(z)} - M\right| = |J(\beta, f(z)) - M| < M,$$

for $0 < \beta \leqslant a$. Hence $F_\beta(z) \in S^*(M)$.

Now consider the converse problem. Given the function $F(z) \in S^*(M)$ and $a > 0$. Is the solution $f(z)$ of the differential equation

$$F(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^a, \tag{11}$$

with boundary condition $f(0) = 0$, a function in $S^*(a, M)$? The answer is yes, and our solution provides us with an integral representation formula for functions in $S^*(a, M)$.

Theorem 4. *If $f(z) \in S^*(M)$ and if $a > 0$, then a solution of* (11) *with boundary condition $f(0) = 0$ is given by*

$$f(z) = \left[\frac{1}{a}\int_0^z \frac{F(\zeta)^{1/a}}{\zeta}\, d\zeta\right]^a, \tag{12}$$

and this function is in $S^*(a, M)$.

The proof of this theorem consists of showing that $f(z)$ is well defined, regular in D and is in $S^*(a, M)$. The technique is similar to the one employed before [3, Theorem 5] and is omitted.

3. Distortion Properties

We will set $m = 1-1/M$ and denote by $h(M, \tau; z)$ the function defined by

$$h(M, \tau; z) = \begin{cases} \dfrac{z}{(1-\tau m z)^{(1+m)/m}} & \text{if } m > 0, \\ z e^{\tau z} & \text{if } m = 0, \end{cases}$$

where $|\tau| = 1$. The function $h(M, \tau; z)$ is in $S^*(M)$ and is the extremal function for many problems in this class [1]. If in (12) we take $F(z)$ to be $h(m, \tau; z)$ then we obtain the Janowski alpha-convex function

$$f(a, M, \tau; z) = \begin{cases} \left[\dfrac{1}{a}\displaystyle\int_0^z \zeta^{1/a-1}(1+\tau m\zeta)^{(1+m)/am}\, d\zeta\right]^a, & \text{if } m > 0 \\ \left[\dfrac{1}{a}\displaystyle\int_0^z \zeta^{1/a-1} e^{\tau\zeta/a}\, d\zeta\right]^a, & \text{if } m = 0. \end{cases}$$

These functions will serve as the extremal functions for the class $S^*(a, M)$.

In what follows, use will be made of the hypergeometric functions

$$G(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)z^k}{\Gamma(c+k)k!} \tag{13}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\, du,$$

where $\operatorname{Re} a > 0$ and $\operatorname{Re}(c-a) > 0$. These functions are regular for $z \in D$ [4, pp. 611]. In addition we define the functions

$$K(a, M; r) = \begin{cases} r[G(1/a, (1+m)/am, 1/a+1; mr]^a, & \text{if } m > 0, \\ r\left[\dfrac{1}{a}\displaystyle\int_0^1 u^{1/a-1} e^{ru/a}\, du\right]^a, & \text{if } m = 0 \end{cases} \tag{14}$$

Theorem 5. *If $f(z) \in S^*(a, M)$, $a > 0$, then for $|z| = r$ $(0 < r < 1)$ we have*

$$-K(a, M; -r) \leqslant |f(z)| \leqslant K(a, M; r). \tag{15}$$

Equality holds in both cases for the function $f(a, M, \tau; z)$.

Proof. By Theorem 4 there exists a function $F(z) \in S^*(M)$ such that

$$f(z) = \left(\frac{1}{a}\int_0^z \frac{[F(\zeta)]^{1/a}}{\zeta}\, d\zeta\right)^a,$$

and if we take $z = r$ and integrate along the positive real axis we obtain

$$f(r) = \left(\frac{1}{a}\int_0^r F(x)^{1/a} x^{-1}\, dx\right)^a.$$

Since $F(z) \in S^*(M)$ we have [1, Theorem 7],

$$\frac{x}{(1+mx)^{(1+m)/m}} \leqslant |F(x)| \leqslant \frac{x}{(1-mx)^{(1+m)/m}}, \quad \text{if } m > 0, \tag{16}$$

$$xe^{-x} \leqslant |F(x)| \leqslant xe^{x}, \quad \text{if } m = 0, \tag{17}$$

and hence

$$|f(r)|^{1/a} \leqslant \begin{cases} \dfrac{1}{a}\displaystyle\int_0^r x^{1/a-1}(1-mx)^{-(1+m)/am}\, dx, & \text{if } m > 0, \\ \dfrac{1}{a}\displaystyle\int_0^r x^{1/a-1} e^{x/a}\, dx, & \text{if } m = 0. \end{cases}$$

Making the change of variables $x = ru$ and raising both sides to the a power and using (13) and (14) we obtain $|f(r)| \leqslant K(a, M; r)$ And applying the above argument to $e^{-i\theta}f(ze^{i\theta})$, which is in $S^*(M)$ if $f(z) \in S^*(M)$, we obtain $|f(z)| \leqslant K(a, M; r)$.

Consider the straight line L joining 0 to $f(z) = Re^{i\varphi}$. Since $f(z)$ is starlike, L is the image of a Jordan arc γ in D connecting 0 and $z = re^{i\theta}$. The image of γ under the mapping $f(z)^{1/a}$ will in general consist of many line segments emanating from the origin, each of length

$$R^{1/a} = |f(z)|^{1/a} = \int_\gamma |df(\zeta)^{1/a}/d\zeta|\,|d\zeta| = S_\gamma |df(\zeta)^{1a}|.$$

Since $f(z) \in S^*(a, M)$, there exists a function $F(z) \in S^*(M)$ such that

$df(\zeta)^{1/a}/d\zeta = F(\zeta)^{1/a}/a\zeta$. Thus if $\varrho = |\zeta|$, we obtain from (16)

$$(18)\qquad R^{1/a} = \frac{1}{a}\int_{\gamma}\left|\frac{F(\zeta)^{1/a}}{\zeta}\right||d\zeta| \geqslant \frac{1}{a}\int_{\gamma} x^{1/a-1}(1+mx)^{-(1+m)/am}|d\zeta|$$

$$\geqslant \frac{1}{a}\int_{0}^{r} x^{1/a-1}(1+mx)^{-(1+m)/am}dx$$

for $m > 0$. By substituting $x = ru$ and using (13) and (14), we obtain $|f(z)| \geqslant -K(a, M; -r)$. The case $m = 0$ makes use of (17) and (18) and is omitted.

Note that functions in $S^*(a, M)$ are bounded for $a \geqslant 0$ and $M \geqslant 1$.

Corollary 5.1. *If* $f(z) \in S^*(a, M)$ *and* $f(z) = z + a_2z^2 + \ldots$, *then* $|a_2| \leqslant (2M-1)/M(1+a)$, *and this inequality is sharp.*

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STRESZCZENIE

W pracy tej autorzy wprowadzają nową rodzinę funkcji gwiaździstych $S^*(a, M)$ określoną warunkiem (3). Dla klasy tej otrzymali m. in. dokładne oszacowania $|f(z)|$ od dołu i od góry oraz dokładne oszacowanie od góry dla $|a_2|$.

РЕЗЮМЕ

В данной работе авторы ввели новый класс $S^*(a, M)$ звёздообразных функций определённых условием (3).

Для этого класса получили точные оценки $|f(z)|$ снизу и сверху и точную оценку сверху для $|a_2|$.