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# On Some Properties of the Coefficients of Regular Functions with Positive Real Part

O pewnych własnościach współczynników funkcji holomorficznych o dodatniej części rzeczywistej

О некоторых свойствах коэффициентов регулярных функций, вещественная часть которых является положительной

**1.** Recently the studies have been undertaken concerning the problem of finding the radius of the greatest disc in which a given property of a regular function is preserved after a rather general change of its coefficients has been done, [3], [7].

For the given function of the form

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \tag{1}$$

regular in the disc $K = \{z: |z| < 1\}$ we construct a new function $\tilde{f}(z)$ in the following way, [7]. Let $\{n_k\}$ be a finite or infinite subsequence of the sequence of natural numbers and $\{\varepsilon_{n_k}\}$ – a corresponding sequence of complex numbers with elements not all equal zero. Let us put

$$\tilde{a}_m = \begin{cases} a_m & \text{if } m \neq n_k,\ k = 1, 2, \ldots \\ a_{n_k} + \varepsilon_{n_k} a_{n_k} & \text{if } m = n_k,\ k = 1, 2, \ldots \end{cases} \tag{2}$$

and

$$\tilde{f}(z) = \sum_{m=0}^{\infty} \tilde{a}_m z^m \tag{3}$$

where in the case when the sequence $\{n_k\}$ is infinite we assume that the series $\sum_k |\varepsilon_{n_k} a_{n_k}| r^{n_k}$ converges in the interval $\langle 0, 1)$.

**Definition.** Let $T_1, T_2$ be fixed classes of regular functions of the form (1).

a/ For fixed $\{n_k\}$, $\{\varepsilon_{n_k}\}$, $f \epsilon T_1$ we denote by

$$R_1 = R_1(f, \{n_k\}, \{\varepsilon_{n_k}\}, T_1, T_2)$$

the greatest number $r$, $0 < r \leqslant 1$, such that either $\tilde{f}(rz)$ or $r^{-1}\tilde{f}(rz)$ belongs to $T_2$. If the number with this property does not exist, then we put $R_1 = 0$.

b/ For fixed $\{n_k\}$, $\varrho > 0$, $f \epsilon T_1$ we put

$$R_2 = R_2(f, \{n_k\}, \varrho, T_1, T_2) = \inf_{|\varepsilon_{n_k}| \leqslant \varrho} R_1(f, \{n_k\}, \{\varepsilon_{n_k}\}, T_1, T_2)$$

c/ For fixed $\{n_k\}$, $\{\varepsilon_{n_k}\}$ let

$$R_3 = R_3(\{n_k\}, \{\varepsilon_{n_k}\}, T_1, T_2) = \inf_{f \epsilon T_1} R_1(f, \{n_k\}, \{\varepsilon_{n_k}\}, T_1, T_2).$$

d/ For fixed $\{n_k\}$, $\varrho > 0$, we put

$$R_4 = R_4(\{n_k\}, \varrho, T_1, T_2) = \inf_{f \epsilon T} [\inf_{|\varepsilon_{n_k}| \leqslant \varrho} R_1(f, \{n_k\}, \{\varepsilon_{n_k}\}, T_1, T_2)].$$

It can be easily noticed that if $|\varepsilon_{n_k}| \leqslant \varrho$ and $f \epsilon T_1$, then $R_4 \leqslant R_2 \leqslant R_1$ and $R_4 \leqslant R_3 \leqslant R_1$.

The present work gives estimations of the numbers $R_i$ $(i = 1, 2, 3, 4)$ for some families of functions connected with the class of functions regular in the disc $K$ and having the positive real part there. The obtained results are the generalization of those by J. Stankiewicz, [7].

**2.** Let us denote by $P_m^a$ $(0 \leqslant a < 1, -1 < m \leqslant 1)$, (cf. [5]), a family of functions with the following integral representation

$$p(z) = \int_0^{2\pi} \left[ a + (1-a) \frac{1 + e^{it}z}{1 - me^{it}z} \right] d\mu(t), \quad z \epsilon K \tag{4}$$

where $\mu(t)$ is a function of the variable $t$, $0 \leqslant t \leqslant 2\pi$, nondecreasing in this interval and satisfying

$$\int_0^{2\pi} d\mu(t) = 1. \tag{5}$$

It is easy to observe that $\mathscr{P}_1^a = \mathscr{P}^a$, where $\mathscr{P}^a$, [6], is a family of all functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{6}$$

regular in the disc $K$ and satisfying

$$\operatorname{Re}\{p(z)\} > a \quad \text{for } z \epsilon K.$$

Obviously $\mathscr{P}_1^0 = \mathscr{P}^0 = \mathscr{P}$, where $\mathscr{P}$, [1], denotes the known class of all functions of the form (6) with their real parts positive in the disc $K$.

Now we shall determine the estimations of the numbers $R_i$ $(i = 1, 2, 3, 4)$ in the case when $T_1 = \mathscr{P}^a_m$ and $T_2 = \mathscr{P}^\beta$.

**Theorem 1.** *If either the condition*

$$\text{(i)} \qquad \lim_{r\to 1^-} \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k} = +\infty$$

*or the conditions*

$$\text{(ii)} \qquad \lim_{r\to 1^-} \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k} = A < +\infty$$

*and*

$$\text{(iii)} \qquad A + \beta - a > 0$$

*hold, then the number* $R_1 = R_1(p, \{n_k\}, \{\varepsilon_{n_k}\}, \mathscr{P}^a_m, \mathscr{P}^\beta)$ *is greater than or equal to the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(7) \qquad (1+mr)\Big[\beta + \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k}\Big] + [1-(1+m)a]r - 1 = 0.$$

*If both the conditions* (ii) *and*

$$\text{(iv)} \qquad A + \beta - a \leqslant 0$$

*hold, then* $R_1 = 1$.

*There exists a function belonging to the class* $\mathscr{P}^a_m$ *and a sequence* $\{\varepsilon_{n_k}\}$ *such that the result is the best.*

**Proof.** Let the function $p(z)$ of the form (6) belong to the class $\mathscr{P}^a_m$. From (2) and (3) we get

$$(8) \qquad \tilde{p}(z) = p(z) + \sum_k \varepsilon_{n_k} p_{n_k} z^{n_k}, \qquad z \in K.$$

From the results of Molęda, [5], the following sharp estimation immediately follows:

$$(9) \qquad \mathrm{Re}\{p(z)\} \geqslant \frac{1-[1-(1+m)a]r}{1+mr}, \qquad |z| = r,\ 0 < r < 1.$$

On the circle $|z| = r$, $0 < r < 1$, by (8) and (9), the inequality

$$(10) \qquad \mathrm{Re}\{\tilde{p}(z)\} \geqslant \frac{1-[1-(1+m)a]r}{1+mr} - \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k}$$

holds. From above and from the maximum principle it follows that

$$\mathrm{Re}\{\tilde{p}(z)\} > \beta \text{ in the disc } |z| < r$$

when

$$\Phi(r) \equiv \frac{1-[1-(1+m)a]r}{1+mr} - \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k} > \beta. \tag{11}$$

Thus the problem of finding the estimation of the number $R_1$ is connected with the discussion of either the equation (7) or the equation

$$L(r) \equiv \sum_k |\varepsilon_{n_k} p_{n_k}| r^{n_k} + P(r) = 0 \tag{12}$$

where

$$P(r) = \frac{[1+m\beta-(1+m)a]r+\beta-1}{1+mr}. \tag{13}$$

Consider the following cases:

a/ Suppose that the condition (i) is fulfilled. By (12) we have $L(0) = \beta-1 < 0$. The function $P(r)$ defined by (13) increases in the interval $(0,1)$. Thus the equation (12) and consequently the equation (7) has a unique root in this interval what, by the definition of $R_1$, completes the proof of the theorem in the case in question.

b/ Suppose now that the conditions (ii) and (iii) are fulfilled. Then $\lim\limits_{r\to 1^-} L(r) = A+\beta-a > 0$. Since the function $L(r)$ increases in $(0,1)$ and $L(0) < 0$, therefore the equation (7) has a unique root in $(0,1)$. This proves the assertion in the considered case.

c/ If, in turn, the conditions (ii) and (iv) are fulfilled, then $L(r)$ takes negative values in $(0,1)$ and therefore the function $\Phi(r)$, given by (11), satisfies the inequality

$$\Phi(r) > \beta \text{ for every } r,\ 0 < r < 1.$$

Hence $R_1 = 1$.

Let us consider a function $p^*(z)$ of the form

$$p^*(z) = a+(1-a)\frac{1+z}{1-mz}, \quad z \in K \tag{14}$$

and a sequence $\varepsilon^*_{n_k} = -(-1)^{n_k}|\varepsilon^*_{n_k}|$. Then for $z = -r$, $0<r<1$, by (8), we obtain

$$\bar{p}^*(-r) = \frac{1-[1-(1+m)a]r}{1+mr} - \sum_k |p^*_{n_k}\varepsilon^*_{n_k}| r^{n_k}.$$

Thus it follows that

$$R_1(p^*, \{n_k\}, \{\varepsilon^*_{n_k}\}, \mathcal{P}^a_m, \mathcal{P}^\beta) = r_0$$

where $r_0$ is a root, lying in $(0, 1)$, of the equation (7) corresponding to the function $p^*(z)$ and to the sequence $\varepsilon^*_{n_k}$. That means that the given estimation of $R_1$ is the best. The proof of Theorem 1 has been completed.

**Remark 1.** In the case when $m = 0$ Theorem 1 is obviously reduced to the following formulas:

$$R_1(p, \{n_k\}, \{\varepsilon_{n_k}\}, \mathscr{P}^a_0, \mathscr{P}^\beta) \geqslant \frac{1-\beta}{1-a+|\varepsilon_1 p_1|} \quad \text{if } |\varepsilon_1 p_1| + \beta - a > 0,$$

$$R_1(p, \{n_k\}, \{\varepsilon_{n_k}\}, \mathscr{P}^a_0, \mathscr{P}^\beta) = 1 \quad \text{if } |\varepsilon_1 p_1| + \beta - a \leqslant 0.$$

**Theorem 2.** *If either the condition*

$$\text{(i)} \qquad \lim_{r\to 1^-} \sum_k |p_{n_k}| r^{n_k} = +\infty$$

*or the conditions*

$$\text{(ii)} \qquad \lim_{r\to 1^-} \sum_k |p_{n_k}| r^{n_k} = A < +\infty$$

*and*

$$\text{(iii)} \qquad \varrho A + \beta - a > 0$$

*hold, then the number* $R_2 = R_2(p, \{n_k\}, \varrho, \mathscr{P}^a_m, \mathscr{P}^\beta)$ *is greater than or equal to the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(15) \qquad (1+mr)\Big[\beta + \varrho \sum_k |p_{n_k}| r^{n_k}\Big] + r[1-(1+m)a] - 1 = 0.$$

*If both the conditions* (ii) *and*

$$\text{(iv)} \qquad \varrho A + \beta - a \leqslant 0$$

*hold, then* $R_2 = 1$.

*There exists a function belonging to the class* $\mathscr{P}^a_m$ *and a sequence* $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}| \leqslant \varrho$, *such that the result is the best.*

**Proof.** Consider the lower bound of the function $\Phi(r)$ defined in (11) with respect to the sequences $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}| \leqslant \varrho$. Then the relation (10) implies the following inequality:

$$\operatorname{Re}\{\tilde{p}(z)\} \geqslant \frac{1-[1-(1+m)a]r}{1+mr} - \varrho \sum_k |p_{n_k} r^{n_k}|, \qquad |z| = r,\ 0 < r < 1.$$

Hence, analogously to the proof of Theorem 1 we obtain the estimation of $R_2$.

Notice finally that for the sequence $\{\varepsilon^*_{n_k}\} = \{-(-1)^{n_k}\varrho\}$ and for $p^*(z)$ defined in (14) we have

$$\operatorname{Re}\{\tilde{p}^*(z)\}\Big|_{z=-r} = \frac{1-[1-(1+m)a]r}{1+mr} - \varrho\sum_k |p^*_{n_k}|r^{n_k}.$$

Hence it follows that the estimation of $R_2$ is the best.

**Remark 2.** If $m = 0$, then the following conditions hold:

$$R_2(p, \{n_k\}, \varrho, \mathscr{P}^a_0, \mathscr{P}^\beta) \geqslant \frac{1-\beta}{1-a+\varrho|p_1|} \quad \text{if } \varrho|p_1|+\beta-a > 0$$

and

$$R_2(p, \{n_k\}, \varrho, \mathscr{P}^a_0, \mathscr{P}^\beta) = 1 \quad \text{if} \quad \varrho|p_1|+\beta-a \leqslant 0.$$

**Theorem 3.** 1° *Let* $m \neq 0(-1 < m \leqslant 1)$. *If either the condition*

$$\text{(i)} \qquad \lim_{r\to 1^-}\sum_k |\varepsilon_{n_k}||m|^{n_k-1}r^{n_k} = +\infty$$

*or the conditions*

$$\text{(ii)} \qquad \lim_{r\to 1^-}\sum_k |\varepsilon_{n_k}||m|^{n_k-1}r^{n_k} = A < +\infty$$

*and*

$$\text{(iii)} \qquad (1-a)(1+m)A+\beta-a > 0$$

*hold, then the number* $R_3 = R_3(\{n_k\}, \{\varepsilon_{n_k}\}, \mathscr{P}^a_m, \mathscr{P}^\beta)$ *is greater than or equal to the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(16) \quad (1+mr)\Big[\beta+(1-a)(1+m)r\sum_k |\varepsilon_{n_k}||mr|^{n_k-1}\Big]+[1-(1+m)a]r-1 = 0.$$

*If both the conditions* (ii) *and*

$$\text{(iv)} \qquad (1-a)(1+m)A+\beta-a \leqslant 0$$

*hold, then* $R_3 = 1$.

2° *If* $m = 0$, *then*

$$(17) \qquad R_3 \geqslant \frac{1-\beta}{(1-a)(1+|\varepsilon_1|)} \quad \textit{when } (1-a)|\varepsilon_1|+\beta-a > 0,$$

$$R_3 = 1 \quad \textit{when } (1-a)|\varepsilon_1|+\beta-a \leqslant 0.$$

*There exists a sequence* $\{\varepsilon_{n_k}\}$ *such that the result is sharp.*

**Proof.** Let $m \neq 0$. It is known that the sharp estimation of coefficients $p_n$ of the function having the form (6) and belonging to the class $\mathscr{P}^a_m$ is the following, [5],

$$(18) \qquad |p_n| \leqslant (1-a)(1+m)|m|^{n-1}, \quad n = 1, 2, \ldots$$

Hence, taking into accout the condition (10), for $|z| = r$, $0 < r < 1$, we get the inequality

$$\mathrm{Re}\{\bar{p}(z)\} \geqslant \frac{1-[1-(1+m)a]r}{1+mr} - (1-a)(1+m)\sum_k |\varepsilon_{n_k}|\,|m|^{n_k-1} r^{n_k}.$$

From the above relation and from the definition of $R_3$, the way of reasoning being analogous to that used in the proof of Theorem 1, we get the first part of the Theorem 3.

Let $m = 0$. From the formulas (4), (5) and rules (2) and (3) we obtain the condition

$$\bar{p}(z) = p(z) + \varepsilon_1 p_1 z, \quad z \in K \tag{19}$$

where

$$p(z) = 1 + (1-a)z\int_0^{2\pi} e^{it}\,d\mu(t), \quad z \in K. \tag{20}$$

The conditions (17) are now easy to obtain.

Finally, is can be easily noticed that for the sequence $\varepsilon^*_{n_k} = -(-1)^{n_k} \times |\varepsilon^*_{n_k}|$ the function $p^*(z)$ defined by (14) is the extremal function. Thus the proof is completed.

**Theorem 4.** 1° *Let* $m \neq 0$ $(-1 < m \leqslant 1)$. *If either the condition*

$$\lim_{r\to 1^-} \sum_k |mr|^{n_k-1} = +\infty \tag{i}$$

*or the conditions*

$$\lim_{r\to 1^-} \sum_k |mr|^{n_k-1} = A < +\infty \tag{ii}$$

*and*

$$\varrho(1-a)(1+m)A + \beta - a > 0 \tag{iii}$$

*hold, then the number* $R_4 = R_4(\{n_k\}, \varrho, \mathscr{P}^a_m, \mathscr{P}^\beta)$ *is equal to the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(1+mr)\Big[\beta + \varrho(1-a)(1+m)r\sum_k |mr|^{n_k-1}\Big] + [1-(1+m)a]r - 1 = 0. \tag{21}$$

*If both the conditions* (ii) *and*

$$\varrho(1-a)(1+m)A + \beta - a \leqslant 0 \tag{iv}$$

*hold, then* $R_4 = 1$.

2° *If* $m = 0$, *then*

$$R_4 = \begin{cases} \dfrac{1-\beta}{(1-a)(1+\varrho)} & \text{when} \quad \varrho(1-a)+\beta-a > 0, \\ 1 & \text{when} \quad \varrho(1-a)+\beta-a \leqslant 0. \end{cases} \tag{22}$$

*The estimations are sharp.*

**Proof.** Let $m \neq 0$. The definition of the number $R_4$ implies that we should take the lower bound of $\Phi(r)$ defined in (11) with respect to all functions $p \in \mathscr{P}_m^a$ and all sequences $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}| \leqslant \varrho$. Then the inequality

$$\operatorname{Re}\{\tilde{p}(z)\} \geqslant \frac{1-[1-(1+m)a]r}{1+mr} - \varrho(1-a)(1+m)\sum_k |m|^{n_k-1} r^{n_k}$$

for $|z| = r$, $0 < r < 1$, results from (10). Therefore, we get the equation (21). After the discussion, similar to the one in previous theorems, of the equation we get the first part of Theorem 4.

If $m = 0$, then the assertion of Theorem 4 follows immediately from the definition of $R_4$ and from (19) and (20).

The sharpness of the estimations is realized by the sequence $\varepsilon^*_{n_k} = -(-1)^{n_k}\varrho$ and by the function $p^*(z)$ defined in (14).

The foregoing theorem seems to be of the most considerable interest for us. It states that every function $\tilde{p}(z)$ constructed from an arbitrary function $p \in \mathscr{P}_m^a$ according to the rules (2) and (3), where $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}| \leqslant \varrho$, is an arbitrary sequence of complex numbers corresponding to a fixed sequence $\{n_k\}$ and to a fixed positive number $\varrho$, satisfies in the disc with the radius equal $R_4$ the condition

$$\operatorname{Re}\{\tilde{p}(z)\} > \beta$$

and, moreover, that $R_4$ cannot be made greater.

**Remark 3.** Denote by $\mathscr{P}_{c,M}$, [2], a family of functions of the form (6) regular in the disc $K$ and satisfying

$$|p(z)-c| < M \quad \text{for } z \in K$$

where $c$, $M$ are arbitrary fixed numbers satisfying

$$|1-c| < M \leqslant \operatorname{Re}\{c\}.$$

The definition of the class $\mathscr{P}_m^a$, $-1 < m < 1$, and of the family $\mathscr{P}_{c,M}$ implies the inclusion

$$\mathscr{P}_m^a \subset \mathscr{P}_{c,M}$$

where

$$c = \frac{1-am}{1-m}, \quad M = \frac{1-a}{1-m}.$$

The sharp estimations of the coefficients, [2], and the real part, [4], in the class $\mathscr{P}_{c,M}$ are already known. It is worth mentioning here that the applied method does not allow to determine the sharp estimations of $R_3$ and $R_4$ in the case of $T_1 = \mathscr{P}_{c,M}$, $T_2 = \mathscr{P}^\beta$.

**Remark 4.** In the case of $a = \beta = 0$ and $m = 1$ the theorems 1–4 are identical to the respective results of Stankiewicz, [7].

**Remark 5.** In the case of $m = 1$ the theorems 1 – 4 give the estimations of $R_i$ when $T_1 = \mathscr{P}^a$ and $T_2 = \mathscr{P}^\beta$. For example, Theorem 4 results in the following statement:

*If $\{n_k\}$ is an arbitrarily fixed subsequence of the sequence of natural numbers and the condition $\lim\limits_{r\to 1^-} \sum\limits_k r^{n_k} = +\infty$ is satisfied, then the number $R_4(\{n_k\}, \varrho, \mathscr{P}^a, \mathscr{P}^\beta)$ equals to the unique root, lying in the interval $(0, 1)$, of the equation*

$$(1+r)\Big[\beta + 2(1-a)\varrho \sum_k r^{n_k}\Big] + (1-2a)r - 1 = 0. \tag{23}$$

**Remark 6.** Specifying the sequences $\{n_k\}$ and $\{\varepsilon_{n_k}\}$ we obtain from (23) various estimations of $R_4$. In particular

1° If $m = 1$ and $\{n_k\} = \{2k\}$, $k = 1, 2, \ldots$, i.e. when we change only even coefficients of the function $p \in \mathscr{P}^a$, then the formula

$$R_4(\{2k\}, \varrho, \mathscr{P}^a, \mathscr{P}^\beta) = \frac{1-\beta}{1-a+\sqrt{\dfrac{\Delta}{4}}}$$

is valid, where

$$\frac{\Delta}{4} = (a-\beta)^2 + 2\varrho(1-a)(1-\beta).$$

Assuming, moreover, that $\varepsilon_{n_k} = -1$, we get

$$R_4 = \frac{1-\beta}{1-a+\sqrt{(1-a)^2+(1-\beta)^2}}.$$

2° If $m = 1$ and $\{n_k\} = \{2k-1\}$, $k = 1, 2, \ldots$, then

$$R_4(\{2k-1\}, \varrho, \mathscr{P}^a, \mathscr{P}^\beta) = \frac{1-\beta}{(1-a)(1+\varrho)+\sqrt{\Delta/4}}$$

where

$$\frac{\Delta}{4} = (a-\beta)^2 + (1-a)^2(2\varrho+\varrho^2).$$

If we assume additionally that $\varepsilon_{n_k} = -1$, then the function

$$\tilde{p}(z) = 1 + \sum_k p_{2k} z^{2k}, \quad z \in K$$

belongs to $\mathcal{P}^a$. Obviously for $\beta \leqslant a$, the function $\tilde{p}(z) \in \mathcal{P}^\beta$ for $z \in K$.

**3.** In the following section we shall give the estimations of $R_i$ $(i = 1, 2, 3, 4)$ when $T_1$ and $T_2$ are certain classes of $p$-valent functions.

A function $f(z)$ is said to be $p$-valent in a unit disc $K$ if it is regular in $K$ and the equation

$$f(z) = w_0 \tag{24}$$

has, for some $w_0$, $p$ roots in $K$ and if for an arbitrary complex number $w_0$ the equation (24) has at most $p$ roots in the disc $K$.

Denote by $C_p^a$, $0 \leqslant a < 1$, $p = 1, 2, \ldots$, the family of all functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{+\infty} a_n z^n \tag{25}$$

regular in the disc $K$ and satisfying

$$\mathrm{Re}\left\{\frac{f'(z)}{pz^{p-1}}\right\} > a, \ z \in K. \tag{26}$$

It is known, [8], that the functions belonging to the class $C_p^a$ are $p$-valent in $K$.

Let the function $f(z)$, having the form (25), belong to the class $C_p^a$. Then from the inequality (26) and from the well known, [1], estimations valid in the class $\mathcal{P}$ we obtain the following sharp estimations:

$$|a_n| \leqslant \frac{2p(1-a)}{n}, \ n = p+1, \ p+2, \ldots \tag{27}$$

and

$$\mathrm{Re}\left\{\frac{f'(z)}{pz^{p-1}}\right\} \geqslant \frac{1-r(1-2a)}{1+r}, \ |z| = r, \ 0 < r < 1. \tag{28}$$

Let $\{n_k\}$ be a subsequence of natural numbers greater than $p$. For the given function $f \epsilon C_p^a$ we construct, according to the rules (2) and (3), a new function

$$\tilde{f}(z) = z^p + \sum_{m=p+1}^{+\infty} \tilde{a}_m z^m, \; z \epsilon K. \tag{29}$$

Consider next the definitions of $R_i$ corresponding to the functions defined by (25) and (29) and to the families of functions $T_1 = C_p^a$, $T_2 = C_p^\beta$. Employing the method used in the proofs of the theorems 1 – 4 and of the inequalities (27) and (28) we easily verify that the following theorems are true.

**Theorem 5.** *If either the conditin*

(i) $$\lim_{r\to 1^-} \sum_k n_k |\varepsilon_{n_k} a_{n_k}| r^{n_k} = +\infty$$

*or the conditions*

(ii) $$\lim_{r\to 1^-} \sum_k n_k |\varepsilon_{n_k} a_{n_k}| r^{n_k} = A < +\infty$$

*and*

(iii) $$A + p(\beta - a) > 0$$

*hold, then the number* $R_1(f, \{n_k\}, \{\varepsilon_{n_k}\}, C_p^a, C_p^\beta)$ *is not smaller than the unique root, lying in the interval* $(0, 1)$ *of the equation*

$$(1+r)\left[\beta + \frac{1}{p} \sum_k n_k |\varepsilon_{n_k} a_{n_k}| r^{n_k - p}\right] + (1-2a) r - 1 = 0.$$

*If both the conditions* (ii) *and*

(iv) $$A + p(\beta - a) \leqslant 0$$

*hold, then* $R_1 = 1$.
*For the function*

$$f^*(z) = p \int_0^z \left[a + (1-a)\frac{1-\zeta}{1+\zeta}\right] \zeta^{p-1} d\zeta \tag{30}$$

*there exists a sequence* $\varepsilon^*_{n_k} = -(-1)^{n_k} |\varepsilon_{n_k}|$ *such that the result is the best.*

**Theorem 6.** *If either the condition*

(i) $$\lim_{r\to 1^-} \sum_k n_k |a_{n_k}| r^{n_k} = +\infty$$

*or the conditions*

(ii) $$\lim_{r\to 1^-} \sum_k n_k |a_{n_k}| r^{n_k} = A < +\infty$$

*and*

(iii) $$\varrho A + p(\beta - a) > 0$$

*hold, then number* $R_2(f, \{n_k\}, C_p^a, C_p^\beta)$ *is not smaller than the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(1+r)\left[\beta + \frac{\varrho}{p} \sum_k n_k |a_{n_k}| r^{n_k - p}\right] + (1-2a)r - 1 = 0.$$

*If both the conditions* (ii) *and*

(iv) $$\varrho A + p(\beta - a) \leqslant 0$$

*hold, then* $R_2 = 1$.

*For the function* $f^*(z)$ *of the form* (29) *and for the sequence* $\varepsilon_{n_k}^* = -(-1)^{n_k} \varrho$ *the result is the best.*

**Theorem 7.** *If either the condition*

(i) $$\lim_{r\to 1^-} \sum_k |\varepsilon_{n_k}| r^{n_k} = +\infty$$

*or the conditions*

(ii) $$\lim_{r\to 1^-} \sum_k |\varepsilon_{n_k}| r^{n_k} = A < +\infty$$

*and*

(iii) $$2(1-a)A + \beta - a > 0$$

*hold, then the number* $R_3(\{n_k\}, \{\varepsilon_{n_k}\}, C_p^a, C_p^\beta)$ *is not smaller than the unique root, lying in the interval* $(0, 1)$, *of the equation*

$$(1+r)\left[\beta + 2(1-a) \sum_k |\varepsilon_{n_k}| r^{n_k - p}\right] + (1-2a)r - 1 = 0.$$

*If both the conditions* (ii) *and*

(iv) $$2(1-a)A + \beta - a \leqslant 0$$

*hold, then* $R_3 = 1$.

*For the sequence* $\varepsilon_{n_k}^* = -(-1)^{n_k} |\varepsilon_{n_k}|$ *the function* $f^*(z)$ *defined by* (29) *is the extremal function.*

**Theorem 8.** *If either the condition*

(i) $$\lim_{r\to 1^-} \sum_k r^{n_k} = +\infty$$

*or the conditions*

(ii) $$\lim_{r\to 1^-} \sum_k r^{n_k} = A < +\infty$$

*and*

(iii) $$2\varrho(1-a)A+\beta-a > 0$$

*hold, then number* $R_4(\{n_k\}, \varrho, C_p^a, C_p^\beta)$ *is equal to the unique root, lying in the interval* $(0,1)$, *of the equation*

(31) $$(1+r)\Big[\beta+2\varrho(1-a)\sum_k r^{n_k-p}\Big]+(1-2a)r-1 = 0.$$

*If both the conditions* (ii) *and*

(iv) $$2\varrho(1-a)A+\beta-a \leqslant 0$$

*hold, then* $R_4 = 1$.

*The extremal function has the form* (30).

**Remark 7.** In particular, when $p = 1$ and $a = \beta = 0$, then the theorems 5–8 are identical to the respective results of J. Stankiewicz, [7].

**Remark 8.** If we put

$$n_k = 2k, \; k = p, \, p+1, \ldots$$

or

$$n_k = 2k-1, \; k = p+1, \, p+2, \ldots,$$

then it follows from the equation (31) that the number $R_4$ is a positive root of the equation

$$2\varrho(1-a)r^p+r^2(2a-\beta-1)+2r(1-a)+\beta-1 = 0$$

or

$$2\varrho(1-a)r^{p+1}+r^2(2a-\beta-1)+2r(1-a)+\beta-1 = 0,$$

respectively.

Thus in particular we get the relations

$$R_4(\{2k-1\}, \varrho, C_1^a, C_1^\beta) = R_4(\{2k\}, \varrho, C_2^a, C_2^\beta) = R_4(\{2k\}, \varrho, \mathscr{P}^a, \mathscr{P}^\beta)$$

and

$$R_4(\{2k\}, \varrho, C_1^{\alpha}, C_1^{\beta}) = R_4(\{2k-1\}, \varrho, \mathscr{P}^{\alpha}, \mathscr{P}^{\beta}).$$

## REFERENCES


[1] Carathéodory C., *Über den Variabilitätsbereich der Konstanten von positiven harmonischen Funktionen*, Rendiconti di Palermo, 32 (1911), 193-217.

[2] Jakubowski Z.J., *On the Coefficients of Caratheodory Functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 19 (1971), 805-809.

[3] Касянюк С. А., Об одном свойстве функций с положительной вещественной частю, Теория функций, функциональный анализ и их приложения, 1 (1965), 224—227.

[4] Libera R.J., Livingston, A.E., *Bounded Functions with Positive Real Part*, Czechoslovak Math. J., 22(97) (1972), 195-209.

[5] Molęda A., *O pewnych zagadnieniach ekstremalnych w klasach $P_m$ i $S_m^*$ funkcji holomorficznych w kole $|z| < 1$*, Zesz. Nauk. UŁ. Ser. II, 52 (1973), 57-83.

[6] Robertson M.S., *On the Theory of Univalent Functions*, Ann. of Math., 37 (1936), 374-408.

[7] Stankiewicz J., *The Influence of Coefficients on Some Properties of Regular Functions* (to appear).

[8] Umezawa T., *On the Theory of Univalent Functions*, Tôhoku Mathematical Journal, 7 (1955), 212-228.


## STRESZCZENIE


Niech $T_1$ i $T_2$ będą ustalonymi klasami funkcji holomorficznych w kole jednostkowym. Dla danej funkcji $f \in T_1$ konstruujemy funkcję $\tilde{f}(z)$ według reguł (2) i (3), [7].

W pracy tej wyznaczono ostre oszacowania największej liczby $r$, $0 < r \leqslant 1$, takiej, że funkcja $\tilde{f}(r \cdot z)$ albo $r^{-1}\tilde{f}(r \cdot z)$ należy do $T_2$, gdzie rodziny $T_1$ i $T_2$ są związane z klasą funkcji Caratheodory'ego.


## РЕЗЮМЕ


Пусть $T_1$ и $T_2$ будут фиксированными классами регулярных функций в единичном круге. Для данной функции $f \in T_1$ мы конструируем функцию $\tilde{f}$ по правилам (2) и (3) [7].

В этой работе представлены точные оценки самого большого числа $r$, $0 < r \leqslant 1$, такого что функция $\tilde{f}(rz)$ или $r^{-1}\tilde{f}(rz)$ принадлежит к $T_2$, где семейства $T_1$ и $T_2$ являются связанными с классами функции Каратеодори.