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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

KAZIMIERZ GOEBEL

On the Structure of Minimal Invariant Sets for Nonexpansive Mappings

O strukturze minimalnych zbiorów niezmienniczych operacji nieoddalających

О структуре минимальных инвариантных множеств слабосжимающих операторов

In 1965 F. E. Browder [2] and D. Göhde [3] proved that each nonempty, closed, bounded and convex subset of an uniformly convex Banach space has fixed point property with respect to nonexpansive mappings. In the same time W. A. Kirk [4] proved the same theorem for convex subsets of a Banach space which are weakly compact and have normal structure in the sense of Brodskii and Milman [1]. The main problem which remains unsolved in this area is whether the same is true for arbitrary weakly compact set. Here we would like to discuss some properties of so-called minimal invariant convex sets for a nonexpansive mapping. The problem mentioned above is equivalent to the question if such sets may consist of more than one point. In spite of a rather negligible progress we hope that our remarks may help to understand the essence of the problem.

Let C be a closed, bounded and convex subset of a Banach space B and let $T: C \to C$ be a nonexpansive mapping i.e. such that:

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for $x, y \in C$. Suppose moreover that C consists of more than one point. A subset $D \subset C$ is said to be T-invariant if $T(D) \subset D$. C is said to be minimal if it does not contain any nonempty proper closed, convex T-invariant subset.

Consider now few constructions of invariant subsets of C.

First notice that $C_1 = \mathrm{Conv}\ T(C)$ is T-invariant so we have the following:

Property I. If C is minimal then $C = C_1$.

Take now any $x_0 \in C$ and $r > 0$. Consider the sequence $x_n = T^n x_0$ and the set

$$C_2 = [z \in C: \limsup \|x_n - z\| \leqslant r].$$

C_2 is convex, closed, T-invariant but may be empty if r is too small, or equal to C if r is too big. However, if there are two points z_1, z_2 such that

$$\limsup \|x_n - z_1\| < \limsup \|x_n - z_2\|$$

then putting $r = \limsup \|x_n - z_1\|$ we find C_2 to be proper and nonempty. Hence

Property II. If C is minimal then $\limsup \|Tx_0^n - z\| = \text{const}$ for $z \in C$.

The same is true for arbitrary sequence y_n such that $\|y_n - Ty_n\|$ tends to zero. In this case the set

$$C_3 = [z \in C\colon \limsup \|y_n - z\| \leqslant r]$$

is T invariant and we have

Property III. If C is minimal then $\limsup \|y_n - z\| = \text{const.}$ for $z \in C$.

Let us now go back to the idea which led to Kirk's result [4]. The point $y \in C$ is said be diametral if

$$\sup [\|y - x\|\colon x \in C] = \operatorname{diam} C$$

If C contains a nondiametral point then C is not minimal. It is so because if there exists $r < d = \operatorname{diam} C$ such that the set

$$C_4 = [z \in C\colon \sup [\|z - x\|\colon x \in C] \leqslant r]$$

is nonempty (obviously it is closed, convex and proper) then either C_1 or C_4 is proper invariant. Indeed if $C_1 = C$ then each point $x \in C$ can be approximated by the points of the form

$$\sum_{i=1}^{n} a_i T u_i$$

where $u_i \in C$ and $a_i \geqslant 0$ for $i = 1, 2, \dots n$ and $\sum_{i=1}^{n} a_i = 1$. But for any $z \in C_4$

$$\Big\|Tz - \sum_{i=1}^{n} a_i T u_i\Big\| \leqslant \sum_{i=1}^{n} a_i \,\|Tz - Tu_i\| \leqslant \sum_{i=1}^{n} a_i \|z - u_i\| \leqslant \sum_{i=1}^{n} a_i r = r$$

implying $Tz \in C_4$, and we have proved.

Property 4. If C is minimal then C consists of only diametral points.

In all we said above there was no need to assume that C is weakly compact. Let us pass now to this specially interesting case (see [2], [4],). The family of all T-invariant nonempty closed and convex subsets of C ordered by inclusion satisfies, in view of weak compactness, the assumptions of Zorn's lemma. It implies

Property 5. If C is weakly compact then C contains a minimal nonempty, closed, convex T-invariant subset.

It may consist of exactly one point — the fixed point of T but the question we pointed at the beginning was whether it must be such.

Suppose now that C is already minimal and weakly compact. Take any sequence $\{x_n\} = \{T^n x_0\}$.

Property 6. If C is minimal and weakly compact then for all $z \in C$

$$\limsup_{i\to\infty} \|x_i - z\| = d = \operatorname{diam} C$$

To prove it notice that if

$$\limsup_{i\to\infty} \|x_i - z\| = \text{const} = r < d$$

then the family of all intersections of C with closed balls centered at C and of radius $\frac{1}{2}(r+d)$ would have the finite intersection property and because of weak closeness of such sets there would exist a point common for all of them. However such point would be nondiametral.

Take now any sequence $\{y_n\}$ such that $\lim\limits_{n\leftarrow\infty} \|y_n - Ty_n\| = 0$. Because each subsequence of $\{y_n\}$ is a sequence of the same type even more can be proved in the same way

Property 7. If C is minimal and weakly compact then for all $z \in C$

$$\lim_{n\to\infty} \|y_n - z\| = d.$$

It implies two next properties

Property 8. If C is minimal and wekly compact then C cannot be covered with a finite number of sets of diameter smaller then d.

Property 9. If C is minimal and weakly compact then C cannot be covered with a finite number of balls centered at C and of radius smaller then d.

Consider now the following construction. Take any $a \in \langle 0, 1)$ and for each $x \in C$ find the solution of the equation

$$y = (1-a)x + aTy$$

Such solution exists because the right hand side of the equation is (with respect to y) a contraction. Denote this solution $F_a x$.

We have

$$F_a x = (1-a)x + aTF_a x.$$

It is easy to verify that each $F_a\colon C \to C$ is nonexpansive and it depends continuously on a. Moreover

$$\|F_a x - TF_a x\| = (1-a)\|x - TF_a x\| \leqslant (1-a)d$$

implying

$$\lim_{a\to 1}\|F_a x - TF_a x\| = 0 .$$

Comparing that with Property 7 we can see that for each $z \in C$, $x \in C$.

$$\lim_{a\to 1}\|F_a x - z\| = d. \qquad (*)$$

Introduce then some new notions. Call the point $y \in C$ almost nondiametral if there exists $\varepsilon > 0$ such that all the path-connected components of the set

$$[x\colon \|x-y\| \geqslant d-\varepsilon] \qquad (*,*)$$

have diameter less then d. Call the set $B \subset C$ almost nondiametral if there exists $\varepsilon > 0$ such that all the path-connected components of the set

$$[x\colon \operatorname{dist}(x, B) \geqslant d-\varepsilon]$$

have the same property (diameter $< d$).

Obviously if x is nondiametral it is also almost nondiametral. However the following example shows that almost nondiametral point may be diametral.

Example. Take C to be the closed convex envelope of all the sequences $e_i = \{0, 0, 0, \ldots 0, 1, 0, 0, \ldots\}$ (1 situated on i-th place) in c_0 space. This set is weakly compact and diametral. It may be also described as

$$C = \left[x = \{x_i\}\colon x_i \geqslant 0, \sum_{i=1}^{\infty} x_i \leqslant 1\right]$$

Obviously $0 = \{0, 0, 0, \ldots\} \in C$ is the diametral point but it is also almost nondiametral because $d = \operatorname{diam} C = 1$ and for $\varepsilon < \frac{1}{2}$ the set $[x\colon x \in C,$ $\|x\| \geqslant 1-\varepsilon]$ consists of infinitely many disjoint path connected components $B_i = [x\colon x \in C,\ x_i \geqslant 1-\varepsilon]$ of diameter equal ε.

Now we can prove

Property 10. If C is minimal and weakly compact then C does not contain any almost nondiametral point.

To prove it notice that if y is almost nondiametral in C then the $F_a x$ must, for a sufficiently close to 1, be contained entirely in one component of $(*, *)$ and it contradicts $(*)$.

The same proof works for

Property 11. If C is minimal and weakly compact then C does not contain any finite or compact almost nondiametral set.

Finally we can prove the following theorem:

Theorem 1: *If C is a nonempty weakly compact convex set such that each closed and convex subset D of C contains almost nondiametral compact subset then C has fixed point property with respect to nonexpansive mappings.*

Proof. Use Properties 5 and 11 to show that the minimal T-invariant subset of C must consist of exactly one point.

This theorem is formally stronger then Kirk's result [4] where it was assumed that C has the normal structure meaning that each convex set $D \subset C$ has a nondiametral point. However we do not know whether this theorem is really stronger. The set C having the property described in our theorem and without normal structure is rather hard to find.

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STRESZCZENIE

W pracy badane są własności minimalnych zbiorów niezmienniczych operacji nieoddalających w przestrzeni Banacha. Udowodnione jest twierdzenie o punkcie stałym, będące formalnym uogólnieniem wyniku W. A. Kirka.

РЕЗЮМЕ

В работе исследуются свойства минимальных инвариантных множеств слабосжимающих операторов в Банаховом пространстве. Доказана была теорема о неподвижной точке, которая является формальным обобщением результата В. А. Кирка.