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Groups of Automorphisms of a Conus

Grupy automorfizmów stożka

Группы автоморфизмов конуса

Let R^n denote the set of all real n-tuples $x = (x^1, \ldots, x^n)$ and let $\langle , \rangle$ denote the Euclidean scalar product in R^n.

Definition 1. A subset G of R^n will be called an open conus, if it satisfies the following conditions:

$0 \notin G$,

G is an open set in R^n,

if $x \in G$, then $\lambda x \in G$ for all $\lambda > 0$.

Let G be an open conus. We recall, that a function $f: G \to R$ is homogeneous of a positive degree k, if it satisfies the condition

$$f(\lambda x) = \lambda^k f(x) \quad \text{for all } \lambda > 0 \text{ and } x \in G. \tag{1}$$

We know that the condition (1) and the Euler identity

$$kf(x) = x^s f_{|s}(x) \quad \text{for } x \in G \tag{2}$$

are equivalent, f being any function of the class $C^1(G)$; $f_{|s}$ denotes here a partial derivative of f with respect to x^s.

Let $f^1, \ldots, f^n \in C^1(G)$ be homogeneous functions of a positive degree $k^1, \ldots, k^n$ respectively, such that the function $f = (f^1, \ldots, f^n)$ maps G into itself. We denote by G^+ the set of all functions satisfying the above conditions.

Proposition 2.

If $x \in G$ is a fixed point of $f \in G^+$, then there holds

$$\det.\left([f^i_{|s}(x)] - [k^i \delta^i_s]\right) = 0. \tag{3}$$

Proof.

Let $x \in G$ be a fixed point of $f \in G^+$, i.e.

$$(*) \qquad f^i(x) = x^i \quad \text{for } i = 1, \ldots, n.$$

Since f^i are homogeneous functions of the class $C^1(G)$ so they satisfy the Euler identity and we can rewrite $(*)$ in the following form

$$(\ddagger) \qquad (f^i_{|s}(x) - k^i \delta^i_s) x^s = 0 \quad \text{for } i = 1, \ldots, n.$$

This system of equalities has a non-zero solution x, thus (3) is valid.

Q.E.D.

Theorem 3.

A point $x \in G$ is a fixed point of $f \in G^+$ iff

$$(4) \qquad \langle x, F^i(x) \rangle = 0 \quad \text{for } i = 1, \ldots, n,$$

where $F^i(x) = \left(f^i_{|1}(x) - k^i \delta^i_1, \ldots, f^i_{|n}(x) - k^i \delta^i_n\right)$.

Proof.

Let $x \in G$ be a fixed point of $f \in G^+$. Thus we have $(\ddagger)$ or $\langle x, F^i(x) \rangle = 0$.

We assume now that (4) is satisfied. From Euler identity we have

$$k^i f^i(x) - k^i x^i = x^s f^i_s(x) - k^i x^i = (f^i_{|s}(x) - k^i \delta^i_s) x^s = 0,$$

since $k^i \neq 0$, so we obtain

$$f^i(x) = x^i \quad \text{for } i = 1, \ldots, n.$$

Q.E.D.

Let G_h denote a subset of G^+ which consists of all diffeomorphisms $f = (f^1, \ldots, f^n)$ of G such that the functions $f^1, \ldots, f^n$ have the same positive degree. Obviously, the identity id_G, is an element of G_h and if $f, g \in G_h$ then $f \circ g \in G_h$. If $f \in G_h$ is a homogeneous function of degree $k > 0$, then there is

$$f^{-1}(\lambda f(x)) = (f^{-1} \circ f)(\lambda^{1/k} x) = \lambda^{1/k} x = \lambda^{1/k} f^{-1}(f(x))$$

hence $f^{-1} \in G_h$. Thus we have

Theorem 4.

The G_h with a composition $\circ$ of functions consitutes a group.

Definition 5.

Each G_h is called a group of automorphisms of a conus G. We obtain from the theorem 3.

Theorem 6.

An isotropy group of a fixed point $x \in G$ is characterized by the conditions

$$(5) \qquad \langle x, \tilde{F}^i(x) \rangle = 0 \quad \textit{for } i = 1, \ldots, n$$

where $\tilde{F}^i(x) = \left(f^i_{|1}(x) - k\delta^i_1, \ldots, f^i_{|n}(x) - k\delta^i_n\right)$.

We give an example of a Lie subgroup of automorphisms of a conus $G \subset R^n$.

Let us fix an arbitrary non-negative even integer p. We will consider functions of the following form

$$(6) \qquad {}^{ab}f : x \mapsto a|x|^b x \quad \text{for } x \neq 0$$

where

$$|x| := (x^1)^p + \ldots + (x^n)^p$$

and a is a positive real number and b satisfies the inequality $1+pb > 0$.

Obviously, the functions ${}^{ab}f^i(x) = a|x|^b x^i$, $i = 1, \ldots, n$ are homogeneous of the same degree $1+pb > 0$. Since

$$\det.[{}^{ab}f^i_{|j}(x)] = a^n(1+pb)|x|^{bn} \neq 0 \quad \text{for } x \neq 0$$

and ${}^{ab}f$ is an injection, so ${}^{ab}f$ is a diffeomorphism. It is easy to see, that the set

$$F(p) = \{{}^{ab}f \mid a > 0,\ 1+pb > 0\}$$

with the composition of functions constitute a Lie group for any non--negative even integer p.

We know that a Lie group is locally isomorphic with its parameter group, so their Lie algebras are isomorphic. We have to find Lie algebra of a parameter group F_p of the group $F(p)$.

The composition in F_p is given by the rule

$$(7) \qquad (a, b) * (c, d) = (ac^{1+pb}, b+d+pbd).$$

We give the chart

$$\mu(x^1, x^2) = (x^1 - 1, x^2)$$

in a neighbourhood of the unity (1,0).

Since

$$(8) \qquad f(x^1, x^2, y^1, y^2) = \mu\big(\mu^{-1}(x^1, x^2) * \mu^{-1}(y^1, y^2)\big)$$

$$= (x^1 + y^1 + x^1y^1 + px^2y^1 + \text{higher degree terms},\ x^2 + y^2 + px^2y^2)$$

so we have

$$[(x^1, x^2),\ (y^1, y^2)]_p = \left(p\begin{vmatrix} y^1 & y^2 \\ x^1 & x^2 \end{vmatrix},\ 0\right).$$

Thus we obtain

Theorem 7.

The Lie algebra of $F(p)$ is isomorphic with the Lie algebra $(R^2, [\,,\,]_p)$.

Remark 8.

Since each straight line passing by 0 is invariant with respect to the group $F(p)$, so $F(p)$ maps a given conus onto itself.

Theorem 9.

The cross-ratio is an invariant of the group $F(p)$.

Proof.

Let $f \in F(p)$ and let

$$\text{(i)} \qquad \begin{aligned} z &= ax + \beta y \\ t &= \gamma x + \delta y . \end{aligned}$$

From the remark 8 it follows the existence of $\lambda, \mu, \varphi, \psi \in R$ such that

$$\text{(ii)} \qquad \begin{aligned} f(z) &= \lambda f(x) + \mu f(y) \\ f(t) &= \varphi f(x) + \psi f(y) . \end{aligned}$$

Substituting f in (ii) by (6) we obtain

$$\text{(iv)} \qquad \begin{aligned} azZ &= \lambda axX + \mu ayY \\ atT &= \varphi axX + \psi ayY , \end{aligned}$$

where $V = |v|^b$.

By comparing (i) and (iv) we can find

$$a = \frac{\lambda X}{Z} \qquad \beta = \frac{\mu Y}{Z}$$

$$\gamma = \frac{\varphi X}{T} \qquad \delta = \frac{\psi Y}{T} .$$

Hence

$$\frac{\beta}{a} : \frac{\delta}{\gamma} = \frac{\mu}{\lambda} : \frac{\psi}{\varphi} .$$

Q.E.D.

Proposition 10.

If $f \in F(2)$ and if g is an orthogonal mapping then there holds

$$g \circ f = f \circ g \quad \text{in } R^n \setminus \{0\} .$$

Proof.

Let g be an orthogonal mapping in R^n and let f be a function (6). We have

$$(g \circ f)(x) = a|x|^b g(x)$$

$$(f \circ g)(x) = a|g(x)|^b g(x) \quad \text{for } x \neq 0 .$$

Since $p = 2$, so $|g(x)| = |x|$.

Q.E.D.

Theorem 11.

The F_p is a solvable group.

Proof.

We put

$$\overline{ab} := a * b * a^{-1} * b^{-1}$$

for $a, b \in F_p$.

It follows from (7) that

$$\overline{ab} = (k, 0)$$

where k is a some positive number. Thus we conclude that $\{\overline{ab} \mid a, b \in F_p\}$ institutes an abelian group. By consequence, the F_p is a solvable group.

Q.E.D.

Theorem 12.

Let $t \to (g^1(t), g^2(t))$ be a 1-parameter group in the F_p such that $\dot{g}^1(0) = \alpha$, $\dot{g}^2(0) = \beta$. This group has the following form

$$p = 0: t \to (e^{\alpha t}, t)$$

$$p \neq 0: t \to (e^{\alpha t}, 0) \quad \text{for } \beta = 0$$

$$t \to (e^{\alpha/\beta u(t)}, u(t)) \quad \text{for } \beta \neq 0$$

where

$$u(t) = p^{-1}(e^{\beta pt} - 1).$$

Proof.

We put

$$x^1(t) = g^1(t) - 1$$

(10)

$$x^2(t) = g^2(t),$$

then we have

$$x^1(0) = x^2(0) = 0 \text{ and } \dot{x}^1(0) = \alpha, \ \dot{x}^2(0) = \beta.$$

We obtain a 1-parameter subgroup in the F_p as a solution of the following system of differential equations

$$\dot{x}^i(t) = \frac{\partial}{\partial y^j} f^i(x^1(t), x^2(t), 0, 0)\dot{x}^j(0) \qquad i = 1, 2$$

where f is given by (8), i.e.

$$\dot{x}^1(t) = \alpha[1 + x^1(t)][1 + px^2(t)]$$

$$\dot{x}^2(t) = \beta[1 + px^2(t)].$$

In view of (10) we obtain (9).

Q.E.D.

We remark that the group F_p may be considered by an arbitrary non-negative even integer p.

Let us take into considerations a group L of affine transformations of R,

$$x \mapsto ax + b.$$

If we compute a Lie bracket of this group then we have

$$[(A_1, B_1), (A_2, B_2)] = (0, A_1 B_2 - A_2 B_1).$$

We see that by $p \neq 0$ the groups L and F_p have isomorphic Lie algebras. Thus the groups F_p may be viewed as generalisations of an automorphism group of the affine line.

STRESZCZENIE

W pierwszej części pracy udowodniono kilka własności odwzorowań typu $R^n \to R^n$ o składowych jednorodnych, zachowujących stożki w R^n. Druga część pracy jest poświęcona zbadaniu konkretnej rodziny grup Lie'go.

РЕЗЮМЕ

В первой части работы доказано несколько свойств отображений из R^n в R^n с однородными компонентами, при которых конусы остаются инвариантами в R^n. Вторая часть работы посвящена изучению конкретного семейства групп Ли.