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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

DOMINIK SZYNAL AND WIESŁAW ZIĘBA

On Infinitely Divisible Generalized Distributions in R_k

O uogólnionych rozkładach nieskończenie podzielnych w R_k

Об обобщенных, безгранично делимых функциях распределения в R_k

1. Introduction. The concept of generalized probability distribution in R_1 and suggestions of its applications have been given in [1]. That distribution is generated by real function satisfying some regular conditions.

The purpose of this note is to extend that concept to R_k and to give some facts analogous to those that are well-known in the classical probability theory. The main result of this note gives us a canonical representation of the Lévy-Khinchine's type of infinitely divisible generalized distribution. As particular cases we obtain some results of [2].

2. The generalized probability distribution. Now we are going to introduce a concept generalized probability distribution in R_k as well as its characteristic function.

Definition 1. A generalized distribution function is a function $V(\mathbf{x}) = V(x_1, x_2, \ldots, x_k)$ on R_k with the following properties:

(1) V is continuous to the left in each variable

(2) $V(\mathbf{x}) \to 0$ as any one coordinate of $\mathbf{x}$ goes to $-\infty$, and $V(\mathbf{x}) \to 1$ as all coordinates of $\mathbf{x}$ go to ∞.

(3) $\operatorname{var}_{R_k} V(\mathbf{x}) = \int_{R_k} |dV(\mathbf{x})| < \infty$.

The class of all functions satisfying (1), (2) and (3) will be denoted by $\mathscr{C}$.

The distribution function V generates on $(R_k, \mathscr{B}_k)$ a countable additive set function defined by Lebesgue-Stieltjes integral

$$P_k(A) = \int_A dV(\mathbf{x}),$$

where $A \in \mathscr{B}_k$ and $P_k(R_k) = 1$.

The triplet $(R_k, \mathscr{B}_k, P_k)$ is called the quasiprobability space.

Definition 2. Fourier-Stieltjes transform of V i.e.

$$\varphi(\mathbf{t}) = \int_{R_k} e^{i\langle \mathbf{t},\mathbf{x}\rangle} dV(\mathbf{x}),$$

where $\langle \mathbf{t}, \mathbf{x}\rangle$ is the scalar product of vectors $\mathbf{t}$ and $\mathbf{x}$ is called the characteristic function of V.

One can immediately obtain

(a) $\varphi(\mathbf{O}) = 1$, where $\mathbf{O} = (0, 0, \ldots, 0)$,

(b) $\varphi(-\mathbf{t}) = \overline{\varphi(\mathbf{t})}$,

(c) $|\varphi(\mathbf{t})| \leqslant \operatorname{var}_{R_k} V(\mathbf{x})$.

In what follows we need the following theorems:

Theorem 1. *For every* $V \in \mathscr{C}$, *we have*

$$V(\mathbf{x}) = a_1 F_1(\mathbf{x}) + a_2 F_2(\mathbf{x}),$$

where F_1 *and* F_2 *are distribution functions in the classical sense and* $a_1 + a_2 = 1$.

Theorem 2. *Every* $V \in \mathscr{C}$ *has at most a countable set of discontinuity hiperplanes.*

Theorem 3. *If* $V \in \mathscr{C}$, *then* $\varphi(\mathbf{t})$ *is uniformly continuous on* R_k.

Proof. Let $\operatorname{var}_{R_k} V(\mathbf{x}) = L < \infty$. Then for given any $\varepsilon > 0$ there exists $T > 0$ such that

$$\int_{K(T)} |dV(\mathbf{x})| > L - \frac{\varepsilon}{3},$$

where $K(T) = \{\mathbf{x} : |x_i| < T, i = 1, 2, \ldots, k\}$.

By continuity of $e^{i\langle h,x\rangle}$, there exists $\delta > 0$ such that if $\|\mathbf{h}\| < \delta$ and $\mathbf{x} \in K(T)$, then

$$|e^{i\langle \mathbf{h},\mathbf{x}\rangle} - 1| < \frac{\varepsilon}{3L}$$

holds, where $\mathbf{h} = (h_1, h_2, \ldots, h_k)$ and $\|\mathbf{h}\| = \sqrt{\langle \mathbf{h}, \mathbf{h}\rangle}$. Therefore, we have

$$|\varphi(\mathbf{t}+\mathbf{h}) - \varphi(\mathbf{t})| \leqslant \int_{R_k} |e^{i\langle \mathbf{t}+\mathbf{h},\mathbf{x}\rangle} - e^{i\langle \mathbf{t},\mathbf{x}\rangle}||dV(\mathbf{x})|$$

$$\leqslant \int_{K(T)} |e^{i\langle \mathbf{h}+\mathbf{t},\mathbf{x}\rangle} - e^{i\langle \mathbf{t},\mathbf{x}\rangle}||dV(\mathbf{x})| + \int_{R_k \setminus K(T)} |e^{i\langle \mathbf{t}+\mathbf{h},\mathbf{x}\rangle} - e^{i\langle \mathbf{t},\mathbf{x}\rangle}||dV(\mathbf{x})|$$

$$\leqslant \int_{K(T)} |e^{i\langle \mathbf{h},\mathbf{x}\rangle} - 1||dV(\mathbf{x})| + 2\int_{R_k \setminus K(T)} |dV(\mathbf{x})| < \varepsilon,$$

independently of $\mathbf{t}$, which completes the proof.

Definition 3. A class $\mathscr{T} \subset \mathscr{C}$ of generalized distributions is said to be tight if

1° for any given $\varepsilon > 0$ there exists a positive number T_ε such that for every $V \in \mathscr{T}$

$$\int_{R_k \setminus K(T_\varepsilon)} |dV(\mathbf{x})| < \varepsilon$$

and if

2° there exists a positive constant C such that $\operatorname{var}_{R_k} V(\mathbf{x}) < C$ for every $V \in \mathscr{T}$.

Definition 4. A sequence $\{V_n, n \geqslant 1\}$ of generalized distributions is said to be weakly convergent to V ($V_n \to V$) if $\lim_{n\to\infty} V_n(\mathbf{x}) = V(\mathbf{x})$ at every point of continuity of V.

If $V_n \to V$ as $n \to \infty$ and $V \in \mathscr{C}$, then a sequence $\{V_n, n \geqslant 1\}$ is said to be completly convergent to V ($V_n \underset{n\to}{\rightrightarrows} V$).

By Theorem 1 we have

Theorem 4. *Every $V \in \mathscr{C}$ is uniquely determined by its characteristic function.*

Theorem 5. *If $\mathscr{W} \subset \mathscr{C}$ is such that for every $V \in \mathscr{W}$*

$$\operatorname*{var}_{R_k} V(\mathbf{x}) < C,$$

where C depends on $\mathscr{W}$ only, then $\mathscr{W}$ is weak compact in the sense of the weak convergence.

Theorem 6. *Let $\{V_n, n \geqslant 1\}$ be a sequence of generalized distributions. If there exists C such that $\operatorname{var}_{R_k} V_n(\mathbf{x}) < C$ independently of n and if $V_n \to V$ as $n \to \infty$, then for any given $K(T)$ for which*

$$\int_{\overline{K(T)} \cap \overline{R_k \setminus K(T)}} |dV(\mathbf{x})| = 0$$

and for an arbitrary continuous and bounded function, defined on $K(T)$

$$\lim_{n\to\infty} \int_{K(T)} f(x)\, dV_n(\mathbf{x}) = \int_{K(T)} f(x)\, dV(x)$$

holds.

Proof. A set $K(T)$ is compact and $f(\mathbf{x})$ is continuous on $K(T)$ so $f(\mathbf{x})$ being continuous on the compact set $K(T)$ is uniformly continuous. Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{y}\| < \delta \text{ implies } |f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{3C}.$$

Now, let us divide k-dimensional cube by the means of $(k-1)$ dimensional hiperplanes continuity of V into cubes $K(T_i)$, $i = 1, 2, \ldots, M$ with diametres less than δ. Let $f_\varepsilon(\mathbf{x})$ denote a simple function being constant on every $K(T_i)$ and equal to the value of $f(\mathbf{x})$ at the point being the centre of $K(T_i)$. Then

$$\Big|\int\limits_{K(T)} f(\mathbf{x})\,dV_n(\mathbf{x}) - \int\limits_{K(T)} f(\mathbf{x})\,dV(\mathbf{x})\Big| \leqslant \Big|\int\limits_{K(T)} [f(\mathbf{x})-f_\varepsilon(\mathbf{x})]\,dV_n(\mathbf{x})\Big|$$

$$+\Big|\int\limits_{K(T)} f_\varepsilon(\mathbf{x})\,dV_n(\mathbf{x}) - \int\limits_{K(T)} f_\varepsilon(\mathbf{x})\,dV(\mathbf{x})\Big| + \Big|\int\limits_{K(T)} [f(\mathbf{x})-f_\varepsilon(\mathbf{x})]\,dV(\mathbf{x})\Big|$$

$$\leqslant \frac{\varepsilon}{3C}\,2C + \Big|\int\limits_{K(T)} f_\varepsilon(\mathbf{x})\,d[V_n(\mathbf{x})-V(\mathbf{x})]\Big|.$$

On the basis of the weak convergence of $\{V_n, n \geqslant 1\}$ there exists n_0 such that for $n > n_0$

$$|V_n(x) - V(x)| < \varepsilon/3\cdot 2^k mM$$

at every point $\mathbf{x}$ which is a vertex at least one of cubes $K(T_i)$, where $m = \max\limits_{x\in K(T)} |f(\mathbf{x})|$.

Hence

$$\Big|\int\limits_{K(T_i)} f_\varepsilon(\mathbf{x})\,d[V_n(\mathbf{x})-V(\mathbf{x})]\Big| < \frac{\varepsilon}{3M},$$

and therefore

$$\Big|\int\limits_{K(T)} f(\mathbf{x})\,dV_n(\mathbf{x}) - \int\limits_{K(T)} f(\mathbf{x})\,dV(\mathbf{x})\Big| < \varepsilon,$$

which completes the proof.

Theorem 7. *If $V_n \to V$ as $n\to\infty$ and $\{V_n\}\subset\mathcal{T}$, where $\mathcal{T}$ is tight, then*

$$\lim_{n\to\infty} \varphi_n(\mathbf{t}) = \varphi(\mathbf{t}),$$

where $\varphi_n(\mathbf{t})$ and $\varphi(\mathbf{t})$ are the characteristic functions of V_n and V respectively.

Proof. Let us note

$$|\varphi_n(\mathbf{t}) - \varphi(\mathbf{t})| = \Big|\int\limits_{R_k} e^{i<t,x>}\,dV_n(\mathbf{x}) - \int\limits_{R_k} e^{i<t,x>}\,dV(\mathbf{x})\Big|$$

$$\leqslant \Big|\int\limits_{R_k\setminus K(T)} e^{i<t,x>}\,dV_n(\mathbf{x})\Big| + \Big|\int\limits_{R_k\setminus K(T)} e^{i<t,x>}\,dV(\mathbf{x})\Big|$$

$$+\Big|\int\limits_{K(T)} e^{i<t,x>}\,d[V_n(\mathbf{x})-V(\mathbf{x})]\Big|.$$

By the hypothesis and Theorem 6 for any given $\varepsilon > 0$ there exist T and n_0 such that

$$|\varphi_n(\mathbf{t}) - \varphi(\mathbf{t})| < 3\varepsilon$$

for $n \geqslant n_0$.

The following example shows that the condition $V_n \rightrightarrows V$ is not sufficient for $\varphi_n(\mathbf{t}) \to \varphi(\mathbf{t})$.

Let $k = 1$ and

$$V_n(x) = \begin{cases} 0 \text{ for } x \leqslant -2n-1, \\ 1 \text{ for } -2n-1 < x \leqslant -2n, \\ 0 \text{ for } -2n < x \leqslant 0, \\ 1 \text{ for } 0 < x \leqslant 2n, \\ 0 \text{ for } 2n < x \leqslant 2n+1, \\ 1 \text{ for } 2n+1 < x. \end{cases}$$

It is easily to see that $V_n(x) \rightrightarrows V(x)$ as $n \to \infty$, where

$$V(x) = \begin{cases} 0 \text{ for } x \leqslant 0 \\ 1 \text{ for } x > 0 \end{cases}$$

and

$$\varphi_n\left(\frac{\pi}{2}\right) = 1 - 2(-1)^n, \varphi(t) = 1.$$

Theorem 8. *If a sequence* $\{\varphi_n(\mathbf{t}), n \geqslant 1\}$ *of characteristic functions of* V_n, *where* $\{V_n, n \geqslant 1\} \subset \mathcal{T}$ *and* $\mathcal{T}$ *is tight, converges to* f *at every point* $\mathbf{t} \in R_k$, *then* $V_n \rightrightarrows V$ *as* $n \to \infty$ *and* f *is characteristic function of* V.

Proof. By Theorem 5 there is a subsequence $\{V_{n_k}, k \geqslant 1\}$ of the sequence $\{V_n, n \geqslant 1\}$ such that $V_{n_k} \to V$, where V is a function of bounded variation. By the assumption that $\{V_{n_k}\} \subset \mathcal{T}$ and on the basis of Theorem 7

$$\varphi_{n_k}(\mathbf{t}) \to \varphi(\mathbf{t}) = f(\mathbf{t}) \text{ as } n_k \to \infty$$

where

$$\varphi(\mathbf{t}) = \int_{R_k} e^{i\langle \mathbf{t}, \mathbf{x}\rangle} dV(\mathbf{x}).$$

Since, by the same reason, every subsequence $\{V_{n_k}\}$ contains a subsequence $\{V_{n_{k_r}}\}$ which weakly converges to some V uniquely determined by $\varphi(\mathbf{t})$, then the sequence $\{V_n, n \geqslant 1\}$ completely converges to V as $n \to \infty$.

The convolution of two generalized distributions V_1 and V_2 in R_k is defined by the formula

$$V_3(\mathbf{z}) = (V_1 * V_2)(\mathbf{z}) = \int_{R_k} V_1(\mathbf{z}-\mathbf{x})\, dV_2(\mathbf{x}).$$

Now, let us observe that if $V_1 \in \mathscr{C}$ and $V_2 \in \mathscr{C}$, then $V_3 \in \mathscr{C}$.

Moreover, it is possible to find such generalized distributions V_1 and V_2 that

$$\operatorname*{var}_{R_k}(V_1 * V_2)(\mathbf{z}) > \max\left(\operatorname*{var}_{R_k} V_1(\mathbf{x}_1), \operatorname*{var}_{R_k} V_2(\mathbf{x}_2)\right)$$

2. Infinitely divisible generalized distributions in R_k.

Definition 5. A generalized distribution V is called infinitely divisible if for any integer n there exists a generalized distribution V_n such that

$$V(\mathbf{z}) = V_n^{(n)}(\mathbf{z}),$$

where $V_n^{(n)}$ denotes n-fold convolutions of V_n and $\{V_n, n \geqslant 1\} \subset \mathscr{T}$, where $\mathscr{T}$ is tight.

Theorem 9. *The characteristic function of infinitely divisible distribution has no zeros.*

Theorem 10. *The convolution of infinitely divisible of generalized distributions is infinitely divisible.*

Theorem 11. *If $\{V_n, n \geqslant 1\}$ is a sequence of infinitely divisible of generalized distributions such that $V_{n,p}^{(p)}(\mathbf{x}) = V_n(\mathbf{x})$ $\{V_{n,p}(\mathbf{x})\} \subset \mathscr{T}$ and*

$$\lim_{n\to\infty} V_n(\mathbf{x}) = V(\mathbf{x}),$$

then $V(\mathbf{x}) \in \mathscr{C}$ and it is infinitely divisible.

The above theorems can be proved analogously to the theorems for classical distributions.

Definition 6. A generalized distribution V is said to be infinitely divisible in narrow sense if it is infinitely divisible and it satisfies the following conditions

(i) $$n \int_{R_k} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\, |dv_n| < C,$$

where C is independent of n, and for any given $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that

(ii) $$n \int_{R_k \setminus K \subset T_\varepsilon)} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\, |dv_n| < \varepsilon,$$

where $K(T_\varepsilon) = \{\mathbf{x} : \bigwedge_{x_i} |x_i| < T_\varepsilon\}$ and ν_n is a countable function of set defined on $\mathscr{B}_k$ corresponding to $V_n(\mathbf{x})$.

Theorem 12. *Let* $\mathbf{a}$, $\mathbf{б}$ *and* μ *denote a vector in* R_k, *a matrix of order* k *and a countable additive function of set such that*

$$\int_{R_k} |d\mu| < \infty \text{ and } \int_{[0]} |d\mu| = 0 \text{ respectively.}$$

The function ψ *defined by*

$$\psi(\mathbf{t}) = i\langle \mathbf{a}, \mathbf{t}\rangle - \tfrac{1}{2}\mathbf{t}\mathbf{б}\mathbf{t}^{\mathrm{T}} + \int_{R_k} \left(e^{i\langle t, x\rangle} - 1 - \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}\right) \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\, d\mu$$

determines uniquely $\mathbf{a}$, $\mathbf{б}$ *and* μ.

The proof of this theorem is analogous to that of classical theory.

Theorem 13. *The logarithm of characteristic function of infinitely divisible in the narrow sense generalized distribution is uniquely represented in the form*

$$\log\varphi(\mathbf{t}) = i\langle \mathbf{a}, \mathbf{t}\rangle - \tfrac{1}{2}\mathbf{t}\mathbf{б}\mathbf{t}^{\mathrm{T}} + \int_{R_k} \left(e^{i\langle t, x\rangle} - 1 - \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}\right) \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\, d\mu,$$

where $\mathbf{a}$, $\mathbf{б}$ *and* μ *are the same as above.*

Proof. On the basis of infinitely divisibility of V, we have

$$\log\varphi(\mathbf{t}) = \lim_{n\to\infty} n(\varphi_n(\mathbf{t}) - 1) = \lim_{n\to\infty} n \int_{R_k} (e^{i\langle \mathbf{t}, \mathbf{x}\rangle} - 1)\, d\nu_n.$$

Let us define a countable additive function of set μ_n by

$$\mu_n(\mu) = n \int_{U} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\, d\nu_n.$$

Then

$$n \int_{R_k} (e^{i\langle \mathbf{t}, \mathbf{x}\rangle} - 1)\, d\nu_n = n \int_{R_k} \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}\, d\nu_n - \tfrac{1}{2} n \int_{R_k} \frac{\langle \mathbf{t}, \mathbf{x}\rangle^2}{1+\|\mathbf{x}\|^2}\, d\nu_n$$

$$+ \int_{R_k} \left(e^{i\langle \mathbf{t}, \mathbf{x}\rangle} - 1 - \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2} + \frac{1}{2}\frac{\langle \mathbf{t}, \mathbf{x}\rangle^2}{1+\|\mathbf{x}\|^2}\right) d\mu_n,$$

where the integral function is defined at point $\mathbf{O}$ as the limit with $\|\mathbf{x}\| \to 0$.

Let us choose a subsequence n'such that $\mu_{n'} \to \mu_0$

$$\frac{1}{2}\mathbf{t}\mathfrak{G}_{n'}\mathbf{t}^T = n'\frac{1}{2}\int_{R_k}\frac{\langle \mathbf{t}, \mathbf{x}\rangle^2}{1+\|\mathbf{x}\|^2}d\nu_{n'} \to \frac{1}{2}\mathbf{t}\mathfrak{G}'\mathbf{t}^T$$

as $n' \to \infty$

Then

$$\log\varphi(\mathbf{t}) = \lim_{n'\to\infty} n' \int_{R_k}\frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}d\nu_{n'} - \frac{1}{2}\mathbf{t}\mathfrak{G}'\mathbf{t}^T +$$

$$+\int_{R_k}\left(e^{i\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{i\langle \mathbf{t},\mathbf{x}\rangle^2}{1+\|\mathbf{x}\|^2} + \frac{1}{2}\frac{\langle \mathbf{t}, \mathbf{x}\rangle^2}{1+\|\mathbf{x}\|^2}\right)\frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}d\mu_0$$

and therefore there exists

$$\lim_{n'\to\infty} n' \int_{R_k}\frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}d\nu_{n},$$

which we denote by $i\langle \mathbf{a}, \mathbf{t}\rangle$.

Now putting $\mu(A) = \mu_0(A\setminus[\mathbf{O}])$, we obtain

$$\log\varphi(\mathbf{t}) = i\langle \mathbf{a}, \mathbf{t}\rangle + \tfrac{1}{2}\mathbf{t}\tau\mathbf{t}^T - \tfrac{1}{2}\mathbf{t}\mathfrak{G}'\mathbf{t}^T + \int_{R_k}\left(e^{i\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}\right)\frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}d\mu,$$

where

$$\frac{1}{2}\mathbf{t}\tau\mathbf{t}^T = \frac{1}{2}\int_{R_k}\frac{\langle \mathbf{t}, \mathbf{x}\rangle^2}{\|\mathbf{x}\|^2}d\mu.$$

Denoting

$$\tfrac{1}{2}\mathbf{t}\mathfrak{G}\mathbf{t}^T = \tfrac{1}{2}\mathbf{t}\mathfrak{G}'\mathbf{t}^T - \tfrac{1}{2}\mathbf{t}\tau\mathbf{t}^T,$$

we have

$$\log\varphi(\mathbf{t}) = i\langle \mathbf{a}, \mathbf{t}\rangle - \frac{1}{2}\mathbf{t}\mathfrak{G}\mathbf{t}^T + \int_{R_k}\left(e^{i\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{i\langle \mathbf{t}, \mathbf{x}\rangle}{1+\|\mathbf{x}\|^2}\right)\frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}d\mu,$$

what completes the proof.

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STRESZCZENIE

W pracy określono uogólnione rozkłady prawdopodobieństwa generowane przez funkcje rzeczywiste spełniające pewne warunki regularności. Głównym wynikiem pracy jest wzór typu Lévy-Chinczyna dla uogólnionych rozkładów nieskończenie podzielnych.

РЕЗЮМЕ

В работе определяются обобщенные функции распределения, индуцированные вещественными функциями, удовлетворяющими некоторым условиям регулярности. Главным результатом работы является формула типа П. Леви-А. Хинчина для обобщенных безгранично-делимых распределений.