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Distribution of Sums of the so-called Inflated Distributions

Rozkłady sum tak zwanych rozkładów „nadętych"

Распределения сумм так называемых „раздутых" распределений

1. Introduction. Recently M. P. Singh [5], S.N. Singh [6] and K. N. Panday [3] have discussed the so-called inflated binomial and inflated Poisson distribution. Each inflated distribution is defined as a mixture of a simple distribution and a degenerate distribution. Analogously, we can define some other inflated discrete distributions. One can observe that a large class of inflated discrete distribution is a particular case of inflated generalized power series distribution (IGPSD) which is defined as follows:

A random variable X is said to have the inflated generalized power series distribution, if

$$(1)\quad P[X=x] = p(x;\theta,a) = \begin{cases} 1-a+a\dfrac{a(x)\,\theta^x}{f(\theta)} & \text{for } x = x_0, \\[2ex] a\dfrac{a(x)\,\theta^x}{f(\theta)} & \text{for } x = x_0+1,\ x_0+2,\ \ldots, \end{cases}$$

where $0 < a \leqslant 1$, $a(x) \geqslant 0$, $f(\theta) = \sum_{x\epsilon T} a(x)\,\theta^x$, for $\theta \epsilon \Omega = \{\theta : 0 < \theta < R\}$, the parameter space, and R is the radius of convergence of the power series of $f(\theta)$, $T = \{x_0, x_0+1, \ldots\}$ is a subset of the set non-negative integers.

In this paper, we consider the distribution of sums of random variables in the case of a IGPSD, and also the distribution of sums of truncated sums of random variables having inflated Poisson or inflated negative binomial distributions. Moreover, we find the distribution of the sums of a generalized inflated binomial distribution (a value $x \neq 0$ is inflated) and the distribution of sums of random variables having a truncated generalized inflated binomial distribution. The results of this paper cover also some generalizatins of the results given in [1], [2], [7].

2. Distribution of sums of random variables with inflated generalized power series distribution. G.P. Patil [4] has shown that the random variable $Z = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having the GPSD has a GPSD, too. The case of the independent random variable having the IGSD is different.

Theorem 1. *If $X_1, X_2, \ldots, X_m$ are the independent random variables having the same inflated generalized power series distributions* (1) *and, if $Y = X_1 + X_2 + \ldots + X_m$, then*

$$P[Y = y] = p(y;\theta, a, m) \tag{2}$$

$$= \begin{cases} \sum_{i=0}^{m} \binom{m}{i} a^i (1-a)^{m-i} \dfrac{a_i(ix_0)\,\theta^{ix_0}}{f_i(\theta)} & \text{for } y = mx_0, \\ \sum_{i=1}^{m} \binom{m}{i} a^i (1-a)^{m-i} \dfrac{a_i(y-(m-i)x_0)\,\theta^{y-(m-i)x_0}}{f_i(\theta)} & \text{for } y = mx_0+1, \\ \qquad mx_0+2, \ldots, & \end{cases}$$

where $a_0(0) = 1, f_0(\theta) = 1, f_i(\theta) = [f(\theta)]^i$ and $a_i(x)$ is the coefficient of θ^x in the expansion of $f_i(\theta)$.

Proof. The theorem will be proved by mathematical induction with respect to m. In the case $m = 2$, and for $y = 2x_0$, we have

$$P[Y = 2x_0] = \sum_{i=0}^{2} \binom{2}{i} a^i (1-a)^{2-i} \frac{a_i(ix_0)\,\theta^{ix_0}}{f_i(\theta)},$$

whereas for $y \neq 2x_0$

$$P[Y = y] = \sum_{x=x_0}^{y-x_0} P[X_1 = x]P[X_2 = y-x]$$

$$= 2a(1-a)\frac{a(y-x_0)\,\theta^{y-x_0}}{f(\theta)} + a^2 \sum_{x=x_0}^{y-x_0} \frac{a(x)\,\theta^x}{f(\theta)} \frac{a(y-x)\,\theta^{y-x}}{f(\theta)}$$

$$= \sum_{i=1}^{2} \binom{2}{i} a^i (1-a)^{m-i} \frac{a_i(y-(2-i)x_0)\,a^{y-(2-i)x_0}}{f_i(\theta)}$$

Hence, formula (2) is valid for $m = 2$. Assuming now that (2) is valid for $m \geqslant 2$ we demonstrate that it is also true for $m+1$. For $y = (m+1)x_0$, we have

$$P[Y = (m+1)x_0] = P[X_{m+1} = x_0]P[X_1 + X_2 + \ldots + X_m = mx_0]$$

$$= \left[1 - a + a\frac{a(x_0)\,\theta^{x_0}}{f(\theta)}\right]\sum_{i=0}^{m}\binom{m}{i}a^i(1-a)^{m-i}\frac{a_i(ix_0)\,\theta^{ix_0}}{f_i(\theta)}$$

$$= \sum_{i=0}^{m}\binom{m}{i}a^i(1-a)^{m-i+1}\frac{a_i(ix_0)\theta^{ix_0}}{f_i(\theta)}$$

$$+ \sum_{i=0}^{m}\binom{m}{i}a^{i+1}(1-a)^{m-i}\frac{a_{i+1}\big((i+1)x_0\big)\,\theta^{(i+1)x_0}}{f_{i+1}(\theta)}$$

$$= \binom{m+1}{0}(1-a)^{m+1} + \sum_{i=1}^{m}\left[\binom{m}{i-1} + \binom{m}{i}\right]a^i(1-a)^{m+1-i}\frac{a_i(ix_0)\,\theta^{ix_0}}{f_i(\theta)}$$

$$+ \binom{m+1}{m+1}a^{m+1}\frac{a_{m+1}\big((m+1)x_0\big)\theta^{(m+1)x_0}}{f_{m+1}(\theta)}$$

$$= \sum_{i=0}^{m+1}\binom{m+1}{i}a^i(1-a)^{m+1-i}\frac{a_i(ix_0)\,\theta^{ix_0}}{f_i(\theta)},$$

and for $y = (m+1)x_0 + 1,\ (m+1)x_0 + 2, \ldots$

$$P[Y = y] = \sum_{x=x_0}^{y-mx_0} P[X_{m+1} = x]P[X_1 + X_2 + \ldots + X_m = y - x]$$

$$= \sum_{i=1}^{m}\binom{m}{i}a^i(1-a)^{m+1-i}\frac{a_i\big(y-(m+1-i)x_0\big)\,\theta^{y-(m+1-i)x_0}}{f_i(\theta)}$$

$$+ a(1-a)^m\frac{a(y - x_0 m)\,\theta^{y-x_0 m}}{f(\theta)}$$

$$+ \sum_{i=1}^{m}\binom{m}{i}a^{i+1}(1-a)^{m-i}\frac{\theta^{y-(m-i)x_0}}{f_{i+1}(\theta)}\sum_{x=x_0}^{y-mx_0} a_1(x)\,a_i\big(y-(m-i)x_0-x\big)$$

$$= \sum_{i=1}^{m}\left[\binom{m}{i-1} + \binom{m}{i}\right]a^i(1-a)^{m+1-i}\frac{a_i\big(y-(m+1-i)x_0\big)\,\theta^{y-(m+1-i)x_0}}{f_i(\theta)}$$

$$+ a^{m+1}\frac{a_{m+1}(y)\,\theta^{y}}{f_{m+1}(\theta)}$$

$$= \sum_{i=1}^{m+1}\binom{m+1}{i}a^i(1-a)^{m+1-i}\frac{a_i\big(y-(m+1-i)x_0\big)\,\theta^{y-(m+1-i)x_0}}{f_i(\theta)}$$

In particular case, when T is the set of all non-negative integers, where $x_0 = 0$, by formula (2) we get

$$(2')\qquad P[Y=y] = p(y;\theta,a,m) = \begin{cases} \sum_{i=0}^{m}\binom{m}{i}a^i(1-a)^{m-i}\frac{a_i(0)}{f_i(\theta)} & \text{for } y=0, \\ \sum_{i=1}^{m}\binom{m}{i}a^i(1-a)^{m-i}\frac{a_i(y)\,\theta^y}{f_i(\theta)} & \text{for } y=1,2,\ldots. \end{cases}$$

Now, let us consider some special cases of the distribution (1).

(a) Inflated binomial distribution. Let $f(\theta) = (1+\theta)^N$ and $\theta = \frac{p}{q}$.

Then $a_i(x) = \binom{iN}{x}$, $f_i(\theta) = \frac{1}{q^{iN}}$, $T = \{0, 1, \ldots, N\}$

and

$$P[X=x] = p(x;p,N,a) = \begin{cases} 1-a+aq^N & \text{for } x=0, \\ a\binom{N}{x}p^x q^{N-x} & \text{for } x=1,2,\ldots,N. \end{cases}$$

In this case the formula (2) has the form:

(3)

$$P[Y=y] = p(y;p,N,m,a) = \begin{cases} (1-a+aq^N)^m & \text{for } y=0, \\ \sum_{i=1}^{m}\binom{m}{i}\binom{Ni}{y}a^i(1-a)^{m-i}p^y q^{Ni-y} & \text{for } y=1, \\ 2,\ldots,Nm. \end{cases}$$

(b) Inflated Poisson distribution. Let $f(\theta) = e^\theta$ and $\theta = \lambda$. Then $f_i(\theta) = e^{\lambda i}$, $a_i(x) = \frac{i^x}{x!}$, $T = \{0, 1, 2, \ldots\}$ and formula (1) is of the form

$$P[X=x] = p(x;\lambda,a) = \begin{cases} 1-a+ae^{-\lambda} & \text{for } x=0, \\ ae^{-\lambda}\frac{\lambda^x}{x!} & \text{for } x=1,2,\ldots. \end{cases}$$

From (2) we have

$$(4)\qquad P[Y=y] = p(y;\lambda,m,a) = \begin{cases} (1-a+ae^{-\lambda})^m & \text{for } y=0, \\ \sum_{i=1}^{m}\binom{m}{i}a^i(1-a)^{m-i}\frac{(\lambda i)^y}{y!}e^{-\lambda i} \\ \text{for } y=1,2,\ldots. \end{cases}$$

(c) **Inflated negative binomial distribution.** Putting in (1) $f(\theta) = (1-\theta)^{-N}$, $\theta = p$ and $a(x) = (-1)^x\binom{-N}{x}$, we have the so-called inflated negative binomial distribution

$$P[X = x] = p(x; p, N, a) = \begin{cases} 1 - a + aq^N & \text{for } x = 0, \\ a\binom{-N}{x}(-1)^x p^x q^N & \text{for } x = 1, 2, \ldots. \end{cases}$$

Noticing that $f_i(\theta) = (1-a)^{-Ni}$ and $a_i(x) = (-1)^x\binom{-Ni}{x}$, we get by (2)

$$(5)\quad P[Y = y] = p(y; p, N, m, a) = \begin{cases} (1 - a + aq^N)^m & \text{for } y = 0, \\ \sum_{i=1}^{m}\binom{m}{i} a^i (1-a)^{m-i}(-1)^y\binom{-Ni}{y} p^y q^{Ni} \\ \text{for } y = 1, 2, \ldots. \end{cases}$$

(d) **Inflated truncated binomial distribution.** If we put $f(\theta) = (1+\theta)^N - 1$, $\theta = \frac{p}{q}$, $a(x) = \binom{N}{x}$ and $T = \{1, 2, \ldots, N\}$, then the distribution (1) is the inflated truncated binomial distribution, i.e.

$$P[X = x] = p(x; p, N, a) = \begin{cases} 1 - a + a\dfrac{Npq^{N-1}}{1-q^N} & \text{for } x = 1, \\ a\binom{N}{x} p^x q^{N-x}/(1-q^N) & \text{for } x = 2, 3, \ldots, N. \end{cases}$$

In this case

$$f_i(\theta) = [(1+\theta)^N - 1]^i = (1-q^N)^i q^{-Ni},$$

$$a_i(x) = \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\binom{Nj}{x}.$$

Then by (2) we get

$$(6)\quad P[Y = y] = p(y; p, N, m, a)$$

$$= \begin{cases} \sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{i-j}\binom{m}{i}\binom{i}{j}\binom{Nj}{i} a^i(1-a)^{m-i}p^i q^{Ni-i}(1-q^N)^{-i} & \text{for } y = m, \\ \sum_{i=1}^{m}\sum_{j=0}^{i}(-1)^{i-j}\binom{m}{i}\binom{i}{j}\binom{Nj}{y+i-m} a^i(1-a)^{m-i}p^{y+i-m}q^{Ni+m-y-i} \\ \cdot(1-q^N)^{-i} \quad \text{for } y = m+1, m+2, \ldots, Nm. \end{cases}$$

(e) Inflated truncated Poisson distribution. Putting in (1) $f(\theta) = e^{\theta}-1$, $\theta = \lambda$, $a(x) = \frac{1}{x!}$ and $T = \{1, 2, \ldots\}$, we have

$$P[X = x] = p(x; \lambda, a) = \begin{cases} 1-a+a\lambda e^{-\lambda}/(1-e^{-\lambda}) & \text{for } x = 1, \\ ae^{-\lambda}\dfrac{\lambda^x}{x!}/(1-e^{-\lambda}) & \text{for } x = 2, 3, \ldots. \end{cases}$$

Because of

$$f_i(\theta) = (e^{\theta}-1)^i = (e^{\lambda}-1)^i,$$

$$a_i(x) = \sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\frac{j^x}{x!},$$

we get by (2)

(7) $$P[Y = y] = p(y; \lambda, m, a)$$

$$= \begin{cases} \sum_{i=0}^{m}\sum_{j=0}^{i}\binom{m}{i}\binom{i}{j}(-1)^{i-j}a^i(1-a)^{m-i}\dfrac{(\lambda j)^i}{i!}e^{-\lambda i}/(1-e^{-\lambda})^i & \text{for } y = m, \\ \sum_{i=1}^{m}\sum_{j=0}^{i}\binom{m}{i}\binom{i}{j}(-1)^{i-j}(1-a)^{m-i}a^i\dfrac{(\lambda j)^{y+i-m}}{(y+i-m)!}e^{-\lambda i}/(1-e^{-\lambda})^i \\ \text{for } y = m+1, m+2, \ldots. \end{cases}$$

(f) Inflated truncated negative binomial distribution. If we put $f(\theta) = (1-\theta)^{-N}-1$, $a(x) = (-1)^x\binom{-N}{x}$ and $T = \{1, 2, 3, \ldots.\}$, then by (1)

$$P[X = x] = p(x; p, N, a) = \begin{cases} 1-a+aNpq^N/(1-q^N) & \text{for } x = 1, \\ a(-1)^x\binom{-N}{x}p^x q^N/(1-q^N) & \text{for } x = 2, 3, \ldots. \end{cases}$$

Observe that in this case

$$f_i(\theta) = [(1-\theta)^{-N}-1]^i = (1-q^N)^i q^{-Ni},$$

$$a_i(x) = \sum_{j=0}^{i}(-1)^{i+x-j}\binom{-Nj}{x}\binom{i}{j},$$

so we obtain by (2)

(8) $$P[Y=y]=p(y;N,p,a)$$

$$=\begin{cases}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{m}{i}\binom{i}{j}\binom{-Nj}{i}(-1)^{2i-j}a^i(1-a)^{m-i}p^iq^{Ni}(1-q^N)^{-i} & \text{for } y=m,\\ \sum_{i=1}^{m}\sum_{j=0}^{i}\binom{m}{i}\binom{i}{j}\binom{-Nj}{y+i-m}(-1)^{y+2i-m-j}a^i(1-a)^{m-i}p^{y+i-m}\cdot & \\ \cdot q^{Ni}(1-q^N)^{-i} \text{ for } y=m+1,\ m+2,\ldots & \end{cases}$$

3. Distribution of sum of truncated sums of random variables. Now we are going to consider the distribution of a sum of truncated sums of random variables. Let $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed random variables having a probability function given by

$$P[Z_i=z]=p(z;p,N,m,a)=\sum_{j=1}^{m}\binom{m}{j}\binom{Nj}{z}a^j(1-a)^{m-j}p^zq^{Nj-z}\cdot$$

$$\cdot\{1-[1-a(1-q^N)]^m\}^{-1} \text{ for } z=1,2,\ldots,Nm,$$

$i=1,2,\ldots n$, where N and m are positive integer numbers, $0<a\leqslant 1$, $0<p<1$, $p+q=1$. It has been shown in [7], that if $Y=Z_1+Z_2+\ldots+Z_n$, then the probability function of the random variable Y is given by

$$P[Y=y]=\{1-[1-a(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\binom{n}{r}\binom{mr}{s}\binom{Ns}{y}\cdot$$

$$a^s(1-a)^{mr-s}[1-a(1-q^N)]^{m(n-r)}p^yq^{Ns-y} \text{ for } y=n,n+1,\ldots,Nmn,$$

and the distribution function of Y is obtained as

$$F(y)=1-\{1-[1-a(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\binom{n}{r}\binom{mr}{s}\cdot$$

$$\cdot a^s(1-a)^{mr-s}[1-a(1-q^N)]^{m(n-r)}I_p(y+1,Ns-y)$$

where $I_p(y+1, Ns-y)$ is the incomplete beta function.

Let us consider the distribution of a sum of truncated sums of the independent random variables having the inflated Poisson distribution.

Theorem 2. *Let $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed random variables with distribution function*

(9) $$P[Z_i=z]=p(z;\lambda,m,a)=\sum_{j=1}^{m}\binom{m}{j}a^j(1-a)^{m-j}e^{-\lambda j}\frac{(\lambda j)^z}{z!}\cdot$$

$$\cdot[1-(1-a+ae^{-\lambda})^m]^{-1} \text{ for } z=1,2,\ldots,\ i=1,2,\ldots,n,$$

where m is a positive integer number, $0 < a \leqslant 1$, $\lambda > 0$.

If $Y = Z_1 + Z_2 + \ldots + Z_n$, *then*

$$(10)\quad P[Y = y] = [1-(1-a+ae^{-\lambda})^m]^{-n} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r} \binom{n}{r} \binom{mr}{s} a^s \cdot$$

$$\cdot (1-a)^{mr-s} (1-a+ae^{-\lambda})^{m(n-r)} e^{-\lambda s} \frac{(\lambda s)^y}{y!} \quad \text{for } y = n, n+1, \ldots.$$

Proof. The characteristic function of the random variables Z_i, $i = 1, 2, \ldots, n$ with (9) is given by

$$\varphi_{Z_i}(t) = \sum_{z=1}^{\infty} e^{itz} \sum_{j=1}^{m} \binom{m}{j} a^j (1-a)^{m-j} e^{-\lambda j} \frac{(\lambda j)^z}{z!} / [1-(1-a+ae^{-\lambda})^m]$$

$$= [1-(1-a+ae^{-\lambda})^m]^{-1} \sum_{j=1}^{m} \binom{m}{j} a^j (1-a)^{m-j} e^{-\lambda j} \sum_{z=1}^{\infty} \frac{(\lambda j)^z}{z!} e^{itz}$$

$$= [1-(1-a+ae^{-\lambda})^m]^{-1} \{[1-a+a\exp(\lambda(e^{it}-1))]^m - (1-a+ae^{-\lambda})^m\}.$$

Hence

$$\varphi_Y(t) = [1-(1-a+ae^{-\lambda})^m]^{-n} \{[1-a+a\exp(\lambda(e^{it}-1))]^m - (1-a+ae^{-\lambda})^m\}^n$$

$$= [1-(1-a+ae^{-\lambda})^m]^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (1-a+ae^{-\lambda})^{m(n-r)} \cdot$$

$$\cdot [1-a+a\exp(\lambda(e^{it}-1))]^{mr}.$$

Using the inversion formula for characteristic functions, we obtain

$$P[Y = y] = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} e^{-ity} \varphi_Y(t)\, dt$$

$$= [1-(1-a+ae^{-\lambda})^m]^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (1-a+ae^{-\lambda})^{m(n-r)}$$

$$\lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} e^{-ity} \sum_{s=0}^{mr} \binom{mr}{s} a^s (1-a)^{mr-s} \exp(s\lambda(e^{it}-1))\, dt$$

$$= [1-(1-a+ae^{-\lambda})^m]^{-n} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r} \binom{n}{r} \binom{mr}{s} a^s (1-a)^{mr-s} \cdot$$

$$\cdot (1-a+ae^{-\lambda})^{m(n-r)} e^{-\lambda s} \frac{(\lambda s)^y}{y!}.$$

Hence the distribution function of Y is given by

$$F(y) = 1 - \sum_{x=y+1}^{\infty} \{[1-(1-a+ae^{-\lambda})^m]^{-n}\} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r} \binom{n}{r} \binom{mr}{s} \cdot$$

$$a^s (1-a)^{mr-s} (1-a+ae^{-\lambda})^{m(n-r)} e^{-\lambda s} \frac{(\lambda s)^x}{x!}$$

$$= 1 - [1-(1-a+ae^{-\lambda})^m]^{-n} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r} \binom{n}{r} \binom{mr}{s} a^s (1-a)^{mr-s} \cdot$$

$$\cdot (1-a+ae^{-\lambda})^{m(n-r)} I_\lambda(y+1)$$

where $I_\lambda(y+1) = \sum_{x=y+1}^{\infty} e^{-\lambda} \frac{(\lambda s)^x}{x!}$.

The distribution of a sum of the truncated sums of the independent random variables having the inflated negative binomial distribution is given by the following

Theorem 3. *Let $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed random variables having the probability function*

$$(11) \qquad P[Z_i = z] = p(z; N, p, m, a) = \sum_{j=1}^{m} \binom{m}{j} \binom{-Nj}{z} (-1)^z a^j (1-a)^{m-j} \cdot$$

$$\cdot p^z q^{Nj} / [1-(1-a+aq^N)^m], \text{ for } z = 1, 2, \ldots, \; i = 1, 2, \ldots, n,$$

where $0 < a \leqslant 1$, $0 < p < 1$, $p+q = 1$, N, m are positive integer numbers.

If $Y = Z_1 + Z_2 + \ldots + Z_n$, then the probability function of the random variable Y is given by

(12)

$$P[Y = y] = [1-(1-a+aq^N)^m]^{-n} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r+y} \binom{n}{r} \binom{mr}{s} \binom{-Ns}{y} \cdot$$

$$\cdot a^s (1-a)^{mr-s} (1-a+aq^N)^{m(n-r)} p^y q^{Ns}, \text{ for } y = n, n+1, \ldots, Nmn,$$

and the distribution function of Y has the form

$$F(y) = 1 - [1-(1-a+aq^N)^m]^{-n} \sum_{r=1}^{n} \sum_{s=1}^{mr} (-1)^{n-r} \binom{n}{r} \binom{mr}{s} a^s (1-a)^{mr-s} \cdot$$

$$\cdot (1-a+aq^N)^{m(n-r)} I_p(y+1, Ns).$$

The proof of this Theorem is similar to the proof of the Theorem 2. For $a = 1$, the formulas (10) and (12) reduce to the classical ones.

4. Distribution of a sum of the generalized inflated binomial distributions. Let X be a random variable having probability function

(13)
$$P[X=x]=p(x;p,N,a)=\begin{cases}1-a+a\binom{N}{x}p^x q^{N-x} & \text{for } x=s,\\ a\binom{N}{x}p^x q^{N-x} & \text{for } x=0,1,\ldots,s-1,\\ & s+1,s+2,\ldots,N.\end{cases}$$

where $0<a\leqslant 1$, $0<p<1$, $p+q=1$.

Theorem 4. *If $X_1, X_2, \ldots, X_m$ are independent random variables having the distribution* (13) *and if $Z=X_1+X_2+\ldots+X_m$, then*

$$(14)\qquad P[Z=z]=\sum_{j=0}^{m}\binom{m}{j}\binom{Nj}{z-s(m-j)}^{*}a^j(1-a)^{m-j}p^{z-s(m-j)}q^{Nj-z+s(m-j)}$$

for $z=0,1,\ldots,Nm$, where

$$\binom{Nj}{z-s(m-j)}^{*}=\begin{cases}0 & \text{for } Nj<z-s(m-j) \text{ or } z-s(m-j)<0\\ \binom{Nj}{z-s(m-j)} & \text{otherwise.}\end{cases}$$

Proof. The characteristic function of each random variable X_i, $i=1,2,\ldots,m$ is given by

$$\varphi_{X_i}(t)=(1-a)e^{its}+a(e^{it}+q)^N.$$

Hence, we have

$$\varphi_Z(t)=[(1-a)e^{its}+a(pe^{it}+q)^N]^m$$
$$=\sum_{j=0}^{m}\sum_{r=0}^{Nj}\binom{m}{j}\binom{Nj}{r}a^j(1-a)^{m-j}p^r q^{Nj-r}e^{it[s(m-j)+r]}.$$

Using the inversion formula for characteristic functions, we obtain

$$P[Z=z]=\lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k}\sum_{j=0}^{m}\sum_{r=0}^{Nj}\binom{m}{j}\binom{Nj}{r}a^j(1-a)^{m-j}p^r q^{Nj-r}e^{it[s(m-j)+r-z]}dt$$
$$=\sum_{j=0}^{m}\sum_{r=0}^{Nj}\binom{m}{j}\binom{Nj}{r}a^j(1-a)^{m-j}p^r q^{Nj-r}\lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k}e^{it[s(m-j)+r-z]}dt$$

Taking into account that

$$\lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} e^{it[s(m-j)+r-z]}dt = \begin{cases} 1 \text{ for } s(m-j)+r-z = 0, \\ 0 \text{ for } s(m-j)+r-z \neq 0, \end{cases}$$

we get

$$P[Z=z] = \sum_{j=0}^{m}\binom{m}{j}\binom{Nj}{z-s(m-j)}^{*} a^j(1-a)^{m-j}p^{z-s(m-j)}q^{Nj+s(m-j)-z}.$$

Let us note that, if $s=0$, then for $z=0$

$$P[Z=0] = \sum_{j=0}^{m}\binom{m}{j}a^j(1-a)^{m-j}q^{Nj} = (1-a+aq^N)^m,$$

and for $z\neq 0$

$$P[Z=z] = \sum_{j=0}^{m}\binom{m}{j}\binom{Nj}{z}a^j(1-a)^{m-j}p^z q^{Nj-z}$$

$$= \sum_{j=1}^{m}\binom{m}{j}\binom{Nj}{z}a^j(1-a)^{m-j}p^z q^{Nj-z}.$$

5. A truncated inflated binomial distribution. Finally let us consider the case of a sum of the truncated inflated binomial variables.

The random variable X is said to have a truncated inflated binomial distribution, if

(15) $$P[X=x] = p(x;p,N,a) = \begin{cases} \left[1-a+a\binom{N}{x}p^x q^{N-x}\right]\Big/(1-aq^N) \text{ for } x=s, \\ a\binom{N}{x}p^x q^{N-x}/(1-aq^N) \text{ for } x=1,2,\ldots, \\ s-1, s+1, s+2, \ldots, N. \end{cases}$$

Using the inversion formula for characteristic functions, it is easy to prove the following

Theorem 6. *If $X_1, X_2, \ldots, X_m$ are independent random variables having the same truncated inflated distribution* (15), *and if $Z = X_1+X_2+\ldots+X_m$, then the probability function of the random variable Z is given by*

(16) $$P[Z=z] = \sum_{j=1}^{m}\sum_{r=0}^{m-j}\binom{m}{j}\binom{m-j}{r}\binom{Nj}{z-rs}(-1)^{m-j-r}a^{m-r}(1-a)^r\cdot$$

$$\cdot p^{z-rs}q^{N(m-r)+rs-z}(1-aq^N)^{-m} \quad \text{for } z=m, m+1, \ldots, Nm.$$

In the case $a=1$, (16) gives the formula for the distribution of the sum of truncated binomial distributions.

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STRESZCZENIE

W pracy podano rozkłady sum niezależnych zmiennych losowych o „nadętych" rozkładach typu uogólnionych szeregów potęgowych.

РЕЗЮМЕ

В работе приводятся распределения сумм независимых случайных величин, имеющих „раздутое" распределение типа обобщенных степенных рядов.