ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXVI, 3 SECTIO A 1972

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

JÓZEF MIAZGA

The Radius of Convexity for a Class of Regular Functions

Promień wypukłości pewnej klasy funkcji regularnych

Радиус выпуклости некоторого класса регулярных функций

1. Introduction

Let S denote the class of functions $f(z) = z + a_2 z^2 + \dots$ analytic and univalent in the unit disc Δ and let $\check{S}(a, \beta)$ be its subclass consisting of functions f subject to the condition

$$\mathrm{Re}\frac{ze^{ia}f'(z)}{f(z)} > \beta\cos a \tag{1}$$

where $\beta \in \langle 0, 1)$ and $a \in (-\pi/2, \pi/2)$.

In the case $\beta = 0$ the class $\check{S}(a, \beta)$ becomes the well known class of Špaček [3], in the case $a = \beta = 0$ it is identical with the class of starlike functions which is usually denoted by S^*.

Let $J(a, \beta, \lambda)$ denote the class of functions of the form

$$\varphi(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\lambda dt$$

where $f \in \check{S}(a, \beta)$ and λ is an arbitrary real number.

In this paper we shall determine the exact value of the radius of convexity in the class $J(a, \beta, \lambda)$. In particular, for $\lambda = 1$, $\beta = 0$ this result is identical with the result earlier obtained by Libera and Ziegler [2]. In view of the well-known connection between the classes of starlike and convex functions this radius of convexity is equal to the radius of starlikeness of the class $\check{S}(a, 0)$

2. The main results

Let P be the class of functions of the form $p(z) = 1+p_1z+\ldots$, $z\in\Delta$, and such that $\operatorname{Re} p(z) > 0$ for $z\in\Delta$.

Lemma 1. [1] *If $p\in P$ and $|z| = r < 1$, then*

$$\left|e^{-ia}p(z)-\frac{1+r^2}{1-r^2}e^{-ia}\right|\leqslant\frac{2r}{1-r^2},\ a \text{ being real.}$$

It is known [2] that if $g\in \check{S}(a, 0)$, $h\in S^*$ then

$$g(z) = z\left[\frac{h(z)}{z}\right]^{\cos a e^{-ia}} \tag{2}$$

holds in Δ for each real a.

Lemma 2. *If $g\in\check{S}(a, 0)$ then the function*

$$f(z) = z\left[\frac{g(z)}{z}\right]^{1-\beta},\ z\in\Delta,\ \beta\in\langle 0, 1) \tag{3}$$

belongs to the class $\check{S}(a, \beta)$ and conversely.

Proof. Let f, g satisfy the condition of the lemma. Then taking the logarithmic derivative we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z}+(1-\beta)\left[\frac{g'(z)}{g(z}-\frac{1}{z}\right]$$

and

$$\operatorname{Re}\frac{ze^{ia}f'(z)}{f(z)} = \beta\cos a+(1-\beta)\operatorname{Re}\frac{ze^{ia}g'(z)}{g(z)} \tag{4}$$

Since g is an element of $\check{S}(a, 0)$ we obtain

$$\operatorname{Re}\frac{ze^{ia}f'(z)}{f(z)} > \beta\cos a$$

Hence $f\in\check{S}(a, \beta)$.

On the other hand if $f\in\check{S}(a, \beta)$ then in view of (4) we have

$$\beta\cos a+(1-\beta)\operatorname{Re}\frac{ze^{ia}g'(z)}{g(z)} = \operatorname{Re}\frac{ze^{ia}f'(z)}{f(z)} > \beta\cos a.$$

Thus

$$\operatorname{Re}\frac{ze^{ia}g'(z)}{g(z)} > 0,\ z\in\Delta$$

and $g\in\check{S}(a, 0)$

Lemma 3. *If $h \in S^*$ then the function of the form*

$$f(z) = z\left[\frac{h(z)}{z}\right]^{(1-\beta)\cos\alpha e^{-i\alpha}}$$

where $z \in \Delta$, $\beta \in \langle 0, 1)$, $\alpha \in (-\pi/2, \pi/2)$ belongs to the class $\check{S}(\alpha, \beta)$.

Proof. It follows immediately from the formula (2) and Lemma 2.

Theorem. *The radius of convexity for the class $J(\alpha, \beta, \lambda)$ is given by the formula*

$$r_c = \begin{cases} 1/[\lambda(1-\beta)\cos\alpha + \sqrt{\lambda^2(1-\beta)^2\cos^2\alpha - 2\lambda(1-\beta)\cos^2\alpha + 1}], & \text{for } \lambda \geqslant 0. \\ 1/[-\lambda(1-\beta)\cos\alpha + \sqrt{\lambda^2(1-\beta)^2\cos^2\alpha - 2\lambda(1-\beta)\cos^2\alpha + 1},] & \text{for } \lambda < 0. \end{cases}$$

The extremal function has the form

$$\varphi(z) = \int_0^z (1 - e^{i\theta}t)^{-2\lambda(1-\beta)\cos\alpha e^{-i\alpha}}\, dt.$$

Proof. In view of Lemma 3 we can represent the function $\varphi(z) \in J(\alpha, \beta, \lambda)$ as follows

$$\varphi(z) = \int_0^z \left[\frac{h(t)}{t}\right]^{\lambda(1-\beta)\cos\alpha e^{-i\alpha}} dt$$

Now taking the logarithmic derivative of $\varphi'(z)$ we obtain

$$\frac{z\varphi''(z)}{\varphi'(z)} = \lambda(1-\beta)\cos\alpha e^{-i\alpha}\frac{zh'(z)}{h(z)} - \lambda(1-\beta)\cos\alpha e^{-i\alpha}$$

In what follows we have

$$\operatorname{Re}\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)}\right\} = (1-\beta)\cos\alpha \operatorname{Re}\left\{\lambda e^{-i\alpha}\frac{zh'(z)}{h(z)}\right\} - \lambda(1-\beta)\cos^2\alpha + 1.$$

Since $zh'(z)/h(z) \in P$ then in view of Lemma 1 we have

$$\operatorname{Re}\left\{\lambda e^{-i\alpha}\frac{zh'(z)}{h(z)}\right\} \geqslant \lambda\frac{(1+r^2)\cos\alpha - 2r}{1-r^2}, \ \lambda \geqslant 0;$$

$$\operatorname{Re}\left\{\lambda e^{-i\alpha}\frac{zh'(z)}{h(z)}\right\} \geqslant \lambda\frac{(1+r^2)\cos\alpha + 2r}{1-r^2}, \ \lambda < 0.$$

Hence

$$\operatorname{Re}\left\{1+\frac{z\varphi''(z)}{\varphi'(z)}\right\} \geqslant \lambda(1-\beta)\cos\alpha\,\frac{(1+r^2)\cos\alpha-2r}{1-r^2} - \lambda(1-\beta)\cos^2\alpha+1, \ \lambda \geqslant 0;$$

$$\operatorname{Re}\left\{1+\frac{z\varphi''(z)}{\varphi'(z)}\right\} \geqslant \lambda(1-\beta)\cos\alpha\,\frac{(1+r^2)\cos\alpha+2r}{1-r^2} - \lambda(1-\beta)\cos^2\alpha+1, \ \lambda < 0.$$

The function φ is convex in a disc $|z| < r$ if the conditions

$$1-\lambda(1-\beta)\cos^2\alpha+\lambda(1-\beta)\cos\alpha\,\frac{(1+r^2)\cos\alpha-2r}{1-r^2} > 0, \ \lambda \geqslant 0;$$

$$1-\lambda(1-\beta)\cos^2\alpha+\lambda(1-\beta)\cos\alpha\,\frac{(1+r^2)\cos\alpha+2r}{1-r^2} > 0, \ \lambda < 0$$

hold. Obviously this can be written as follows

$$(5)\qquad \begin{aligned} &[2\lambda(1-\beta)\cos^2\alpha-1]r^2-2\lambda(1-\beta)\cos\alpha r+1 > 0, \ \lambda \geqslant 0;\\ &[2\lambda(1-\beta)\cos^2\alpha-1]r^2+2\lambda(1-\beta)\cos\alpha r+1 > 0, \ \lambda < 0. \end{aligned}$$

The trinomials in (5) have four roots r_1, r_2, r_3, r_4 given by the formulas

$$r_1 = 1/[\lambda(1-\beta)\cos\alpha+\sqrt{\lambda^2(1-\beta)^2\cos^2\alpha-2\lambda(1-\beta)\cos^2\alpha+1}], \ \lambda \geqslant 0$$

$$r_2 = 1/[\lambda(1-\beta)\cos\alpha-\sqrt{\lambda^2(1-\beta)^2\cos^2\alpha-2\lambda(1-\beta)\cos^2\alpha+1}], \ \lambda \geqslant 0$$

$$r_3 = 1/[-\lambda(1-\beta)\cos\alpha+\sqrt{\lambda^2(1-\beta)^2\cos^2\alpha-2\lambda(1-\beta)\cos^2\alpha+1}], \ \lambda < 0$$

$$r_4 = 1/[-\lambda(1-\beta)\cos\alpha-\sqrt{\lambda^2(1-\beta)^2\cos^2\alpha-2\lambda(1-\beta)\cos^2\alpha+1}], \ \lambda < 0.$$

There are following possible cases

1° If $2\lambda(1-\beta)\cos^2\alpha-1 < 0$ and $\lambda \geqslant 0$ then $r_c = r_1$

2° If $2\lambda(1-\beta)\cos^2\alpha-1 = 0$ then $r_c = r_1 = \cos\alpha$

3° If $2\lambda(1-\beta)\cos^2\alpha-1 > 0$ then $r_c = r_1$

4° If $\lambda < 0$ then $r_c = r_3$

The greatest lower bound of r_c with respect to λ is attained for $+\infty$ or $-\infty$ and it is equal 0.

For $\beta = 0$ we obtain the radius of convexity of the Biernacki's integral within the class of functions of Špaček.

REFERENCES

[1] Ашневиц, И. Я. Улина, Г. В., *Об одластях значений аналитических функций представимых интегралом Стильтеса*, Вестник Ленинградского Чниверситета. 11 (1955), 31-42.

[2] Libera, R. J., Ziegler, M. R., *Regular Functions f(z) for which zf′(z) is α-spiral* (to appear).

[3] Špaček, L., *Přispěvek k teorii funkci prostlčh*, Časopis Pěst. Mat a Fys. 62 (1933), 12-19.

STRESZCZENIE

Niech S będzie klasą funkcji $f(z) = z + a_2 z^2 + \ldots$ regularnych i jednolistnych w kole jednostkowym Δ i niech $\check{S}(\alpha, \beta)$ będzie podklasą klasy S funkcji spełniających warunek (1).

W nocie tej wyznaczono dokładną wartość promienia wypukłości w klasie funkcji $J(\alpha, \beta, \lambda)$ postaci

$$\varphi(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\lambda dt$$

gdzie λ jest dowolną ustaloną liczbą rzeczywistą, $f(z) \in \check{S}(\alpha, \beta)$.

РЕЗЮМЕ

Пусть S будет классом функций $f(z) = z + a_2 z^2 + \ldots$ регулярных и однолистных в единичном круге Δ и пусть $\check{S}(\alpha, \beta)$ будет подклассом класса S функций, отвечающих условию (1).

В работе вычисляется точное значение радиуса выпуклости в классе функций $J(\alpha, \beta, \lambda)$ вида

$$\varphi(z) = \int_0^z \left(\frac{(f(t)}{t}\right) \lambda,$$

где λ является произвольным фуксированным и вещественным числом, $f(z) \in \check{S}(\alpha, \beta)$.