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The Region of Variability of the Ratio $f(b)/f(c)$ within the Class of Meromorphic and Univalent Functions in the Unit Disc.

Obszar zmienności stosunku $f(b)/f(c)$ w klasie funkcji meromorficznych i jedno-
listnych w kole jednostkowym

Область значений выражения $f(b)/f(c)$ в классе мероморфных и однолистных
функций в единичном круге

1. Introduction.

Let U_p denote the class of functions meromorphic and univalent in the unit disc Δ subject to the conditions

$$(1.1) \quad f(0) = 0, f'(0) = 1, f(p) = \infty$$

where p is fixed, $0 < p < 1$.

Let \mathcal{M}_p be the family of functions meromorphic and univalent in Δ and satisfying the conditions

$$(1.2) \quad \begin{aligned} \text{(i)} \quad & f(0) = 0 \\ \text{(ii)} \quad & f(z_0) = z_0, z_0 \neq 0, z_0 \in \Delta \\ \text{(iii)} \quad & f(p) = \infty, p \neq z_0 \end{aligned}$$

Various problems concerning functions that are holomorphic and univalent in Δ and are normalized by the conditions (i) and (ii) in (1.2) have been considered by many authors. In particular, J. Krzyż [3] found the region of variability of $\varphi(z)$ for fixed $z \in \Delta$, where φ ranges over the whole class of functions that satisfy (i) and (ii) in (1.2).

In the present note we determine the region of variability of $f(z)$, $f \in \mathcal{M}_p$. In the limit case $p = 1$ we obtain the Krzyż's result.

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2. Preliminary remarks

Let $z, z \neq 0, p$, be a fixed point of Δ and let the variability region of $\varphi(z)$ over \mathcal{M}_p be the set $E = \{w: w = \varphi(z), \varphi \in \mathcal{M}_p\}$.

It is clear that E is identical to the set of all possible values of the ratio $z_0 f(z)/zf(z_0)$ for f ranging over the whole class U_p . The set E is closed because the class U_p is compact.

Let ∂E denote a boundary of E and let $\mathcal{C}E$ be the complement of E . A point $P \in \partial E$ is said to be a non-singular boundary point of the set E if there exists a point a , $a \in \mathcal{C}E$, such that $|P-a|$, $P \in E$, attains its minimum with $P = P_0$. It is well-known [5] that the set of non-singular boundary points is everywhere dense in ∂E .

Functions $f \in U_p$ corresponding to the non-singular boundary points of Δ we shall call extremal functions.

In order to determine the set of non-singular boundary points we shall use variational formulas given by the following:

Theorem A. [4]. Suppose that $f \in U_p$, $\zeta, \zeta \neq p$, is a fixed point of Δ , A is an arbitrary fixed complex number, ζ_0 satisfies $|\zeta_0| = 1$ and $a = -p \operatorname{res}_{z=p} f(z)$. Then there exists a positive number λ_0 such that for each $\lambda \in (0, \lambda_0)$ there exist functions belonging to U_p that have the form

(2.1)

$$f^*(z) = f(z) - \lambda \left\{ A \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^2 \frac{2f^2(z)}{f(\zeta) - f(z)} + AP(z, \zeta) + \bar{A}P(z, \bar{\zeta}^{-1}) \right\} + O(\lambda^2),$$

(2.2)

$$f^2(z) = f(z) + \lambda P(z, \zeta_0) + O(\lambda^2)$$

where

$$(2.3) \quad P(z, u) = f(z) - zf'(z) \frac{u+z}{u-z} + af^2(z) \frac{u+p}{u-p}.$$

If w_0 is an interior point of $\mathcal{C}f(\Delta)$, then

$$(2.4) \quad f^{**}(z) = f(z) - \lambda A \frac{f^2(z)}{w_0 - f(z)} + O(\lambda^2)$$

belongs to U_p .

3. A differential equation for the extremal functions

Let $b, c, b \neq c, \neq 0 \neq p$ be fixed points of Δ and let $F(f)$ denote the expression $f(b)/f(c)$, $f \in U_p$. We prove now

Lemma 1. The functions corresponding to the non-singular boundary points of the set E satisfy the differential equation

$$(3.1) \quad e^{-i\theta} \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^2 \frac{(f(b)-f(c)f(\zeta))}{(f(b)-f(\zeta))(f(c)-f(\zeta))} = Q(\zeta)$$

for $\zeta \in A \setminus \{p\}$; here $0 < \theta < 2\pi$ and $Q(\zeta)$ is a rational function such that

$$(3.2) \quad Q(\zeta) \geq 0, |\zeta| = 1$$

holds.

Proof. Suppose that f corresponds to a non-singular boundary point of E . Then for a suitably chosen a , $a \in E$ we have

$$(3.3) \quad |F(f) - a| = \min_{g \in U_p} |F(g) - a|$$

Let f^* be given by (2.1). Then

$$|F(f^*) - a|^2 = |F(f) - a|^2 - \lambda |F(f) - a| \Re \left\{ A \left[\frac{P(b, \zeta) - P(c, \zeta)}{f_b - f_c} e^{-i\theta} + \right. \right. \\ \left. \left. + e^{i\theta} \frac{P(b, \bar{\zeta}^{-1}) - P(c, \bar{\zeta}^{-1})}{f_b - f_c} + e^{-i\theta} \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^2 \frac{(f_b - f_c)f(\zeta)}{(f_b - f(\zeta))(f_c - f(\zeta))} \right] \right\} + O(\lambda^2)$$

where $f_b = f(b)$, $f_c = f(c)$, $f = f(\zeta)$ and $\theta = \arg [F(f) - a]$.

In view of (3.3) the real part of the expression in the braces must be equal to zero for each $\zeta \in A \setminus \{p\}$. Since $\arg A$ can be chosen in an arbitrary manner we obtain the condition

$$(3.4) \quad e^{-i\theta} \left(\frac{\zeta f'}{f} \right)^2 \frac{(f_c - f_b)f}{(f_c - f)(f_b - f)} = e^{-i\theta} \frac{P(b, \zeta) - P(c, \zeta)}{f_b - f_c} + \\ + e^{i\theta} \frac{P(b, \bar{\zeta}^{-1}) - P(c, \bar{\zeta}^{-1})}{f_b - f_c} = Q(\zeta)$$

which holds for $\zeta \in A \setminus \{p\}$.

Now let us apply the formula (2.2) to the extremal function f . Then

$$|F(f^*) - a|^2 = |F(f) - a|^2 + \lambda |F(f) - a| \Re \left\{ e^{-i\theta} \frac{P(b, \zeta_0) - P(c, \zeta_0)}{f_b - f_c} \right\} + O(\lambda^2)$$

from which we obtain

$$\Re \left\{ e^{-i\theta} \frac{P(b, \zeta_0) - P(c, \zeta_0)}{f_b - f_c} \right\} \geq 0.$$

From the last we obtain (3.2).

Now (2.3) and (3.4) yield

$$(3.5) \quad Q(\zeta) = A_1 + A_2 \frac{\zeta + b}{\zeta - b} + A_3 \frac{\zeta + c}{\zeta - c} + A_4 \frac{\zeta + p}{\zeta - p} + \\ + \bar{A}_1 + \bar{A}_2 \frac{1 + \bar{b}\zeta}{1 - \bar{b}\zeta} + \bar{A}_3 \frac{1 + \bar{c}\zeta}{1 - \bar{c}\zeta} + \bar{A}_4 \frac{1 + \bar{p}\zeta}{1 - \bar{p}\zeta}.$$

Hence Lemma 1 has been proved.

If we apply (2.4) to (3.3) we can convince ourselves that the set $\mathcal{C}(\Delta)$, where f is an extremal function, has no interior points so that the set is the whole plane slit along a finite number of arcs.

The condition (3.4) has been established for $\zeta \in \Delta \setminus \{p\}$. However, it is well-known that it holds also on $|\zeta| = 1$. Hence, the extremal functions map Δ onto the whole plane cut along a finite number of analytic arcs.

4. The form of $Q(\zeta)$

We have proved that $Q(\zeta)$ is a rational function. It is easy to see

$$\overline{Q(\bar{\zeta})} = Q(\zeta)$$

which implies that the roots of the equation $Q(\zeta) = 0$ are symmetric w.r.t. the unit circumference. Moreover (3.4) shows that $Q(\zeta) \neq 0$ for $\zeta \neq 0, \infty$, and $|\zeta| \neq 1$. Since the equation $Q(\zeta) = 0$ has at most 6 roots and the roots on $|\zeta| = 1$ have an even order of multiplicity then $Q(\zeta)$ must have the form

$$(4.11) \quad Q(\zeta) = A \frac{\zeta(\zeta - k)^2(\zeta - l)^2}{(b - \zeta)(c - \zeta)(p - \zeta)(1 - p\zeta)(1 - c\zeta)(1 - b\zeta)}$$

where $k = e^{ia}$, $l = e^{ib}$ are the points on $|\zeta| = 1$ which are carried by f onto the endpoints of the arc $f(|\zeta| = 1)$. Of course, $f'(k) = f'(l) = 0$.

Hence f maps Δ onto the whole plane cut along one analytic arc with endpoints $f(k), f(l)$.

Since $f(|\zeta| = 1)$ is an analytic arc, the points k, l divide the unit circumference into two arcs a_1, a_2 with common endpoints which have the same length in the metric $|Q(e^{i\theta})|^{\frac{1}{2}} d\theta$. Hence

$$\int_a^\beta |Q(e^{i\theta})|^{\frac{1}{2}} d\theta = \int_\beta^{a+2\pi} |Q(e^{i\theta})|^{\frac{1}{2}} d\theta$$

from which we obtain

$$(4.2) \quad \int_0^{2\pi} \varphi(\theta) \sin \frac{\theta - \beta}{2} \sin \frac{\theta - a}{2} d\theta = 0,$$

where

$$\varphi(\theta) = |(b - e^{i\theta})(c - e^{i\theta})(p - e^{i\theta})|^{-1}.$$

Now (4.2) can be written in the form

$$(4.3) \quad k + l = \bar{D} + Dk \cdot l$$

where

$$(4.4) \quad D = \int_0^{2\pi} e^{i\theta} \varphi(\theta) d\theta / \int_0^{2\pi} \varphi(\theta) d\theta.$$

On the other hand $Q(e^{i\theta}) \geq 0$, so that it follows that

$$k \cdot l = \bar{A} |A|^{-1} = e^{-i\psi}$$

holds. Finally we find that $Q(\zeta)$ has the form

$$(4.5) \quad Q(\zeta) = |A| \frac{\bar{k}\bar{l}\zeta(\zeta-k)^2(\zeta-l)^2}{(b-\zeta)(c-\zeta)(p-\zeta)(1-p\zeta)(1-\bar{c}\zeta)(1-\bar{b}\zeta)}$$

Now, if we compare the Laurent coefficients of both sides of (2.1) near the point $\zeta = 0$ and if we use (3.5), then we obtain

$$A = e^{-i\theta} \frac{f_b - f_c}{f_b \cdot f_c} pbck^{-2}l^{-2}.$$

Thus we have proved the following:

Theorem 1. If $b, c, b \neq c \neq p \neq 0$ are given points of the unit disc and if $t \in (0, 2\pi)$, then the functions corresponding to the non-singular boundary points of E satisfy the differential equation

(4.6)

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^2 \frac{f_b \cdot f_c \cdot f(\zeta)}{(f_b - f_c)(f_c - f(\zeta))} = \frac{bpc\zeta(1 - (D + \bar{D}e^{it})\zeta + e^{+it}\zeta^2)^2}{(b-\zeta)(c-\zeta)(p-\zeta)(1-p\zeta)(1-\bar{c}\zeta)(1-\bar{b}\zeta)}$$

and map Δ onto the whole plane cut along one analytic arc.

5. The region E

In this section we shall determine the region of variability of the ratio $f(b)|f(c)$ within the class U_p .

Let z_0, z_1, z_2, z_3 denote the points $0, b, c, p$, respectively,

$$(5.1) \quad R(\zeta) = \left[\frac{z_1 z_2 z_3}{\zeta(z_1 - \zeta)(z_2 - \zeta)(z_3 - \zeta)(1 - \bar{z}_1 \zeta)(1 - \bar{z}_2 \zeta)(1 - \bar{z}_3 \zeta)} \right]^{\frac{1}{2}}$$

$$S(\zeta) = 1 - (D + \bar{D}e^{it})\zeta + e^{it}\zeta^2$$

and let F_k be the value of the integral $\int_{\gamma} R(\zeta)(\zeta) d\zeta = I(z_k)$ taken along a closed curve γ situated in Δ that starts from $z = l$ and incloses the single critical point z_k ($k = 0$ to 3). This closed curve can be reduced to a loop formed by the straight line lz_k , the circle of infinitesimal radius about z_k , and the straight line $z_k l$.

Then

$$F_k = 2 \int_l^{z_k} S(\zeta) R(\zeta) d\zeta, \quad k = 0, 1, 2, 3$$

where the integral is taken along the straight line lz_k .

Let

$$(5.2) \quad \Omega_k = F_k - F_0, \quad k = 1, 2, 3.$$

It is well-known [6] that

$$I(\gamma) = I(\zeta) + \sum_{k=1}^3 m_k \Omega_k$$

or

$$I(\gamma) = F_3 - I(\zeta) + \sum_{k=1}^3 m'_k \Omega_k$$

where $I(\gamma)$ denotes the integral taken along a curve γ jointing l to ζ .

In the case under considerations there is the possibility of eliminating one of the constants Ω_k .

Since (4.2) holds, we have $\int_{|\zeta|=1} R(\zeta) S(\zeta) d\zeta = 0$ and hence

$$F_3 - F_2 + F_1 - F_0 = 0 = \int_{|\zeta|=1} R(\zeta) S(\zeta) d\zeta$$

Therefore

$$\Omega_3 - \Omega_2 + \Omega_1 = 0$$

Thus (5.3) takes the form

$$I(\gamma) = I(\zeta) + \sum_1^3 m_k \Omega_k$$

(5.4)

$$I(\zeta) = F_3 - I(\zeta) + \sum_1^3 m_k \Omega_k$$

Let us write (4.1) in the form

$$\frac{v_0 dv}{\sqrt{v(v_0-v)(v_0-qv)}} = S(\zeta) R(\zeta) d\zeta$$

where $v_0 = f(b)$, $q = f(b)/f(c)$, $v = f(\zeta)$.

If we set

$$v = v_0 \left(x + \frac{1+q}{3q} \right)$$

$$g_2 = \frac{4(q^2 - q + 1)}{3q^2}, \quad g_3 = \frac{4(1+q)(q-2)(2q-1)}{27q^3}$$

then we have

$$(5.5) \quad \frac{2dx}{\sqrt{v_0} \sqrt{4x^3 - g_2 x - g_3}} = R(\zeta) S(\zeta) d\zeta$$

and $g_2^3 - 27g_3^2 \neq 0$.

Let $\wp(u; \Omega_1, \Omega_2)$ be the elliptic function of Weierstrass defined by the formula

$$\wp(u; \Omega_1, \Omega_2) = u^{-2} + \sum_{-\infty}^{+\infty}' [(u+n\Omega_1+m\Omega_2)^{-2} - (n\Omega_1+m\Omega_2)^{-2}].$$

If we integrate (5.5), then we obtain

$$(5.6) \quad f(\zeta) = A\wp\left(\int_p^\zeta S(\zeta) R(\zeta) d\zeta\right) + B$$

where A, B are constants and we have made use of the fact that $f(p) = \infty$.

The formulas (5.4) show that we can write (5.6) in the form

$$f(\zeta) = A\wp\left(\int_0^\zeta S(u) R(u) du + \frac{1}{2}(\Omega_1 + \Omega_2)\right) + B$$

and hence the function f is single valued.

Finally, because $f(0) = 0$ we have

$$(5.7) \quad f(\zeta) = A\left[\wp\left(\int_0^\zeta S(x) R(x) dx + \frac{1}{2}(\Omega_1 + \Omega_2)\right) - e_2\right]$$

where $e_2 = \wp\left(\frac{1}{2}(\Omega_1 + \Omega_2)\right)$ and A has to be chosen so that $f'(0) = 1$.

Formula (5.7) gives us the form of the extremal functions.

If we set $\zeta = b, \zeta = c$ then we obtain ($k = 1, 3$)

$$(5.8) \quad \frac{f(b)}{f(c)} = \frac{e_3 - e_2}{e_1 - e_2}, \quad e_k = \wp\left(\frac{\Omega_k}{2}\right)$$

Let $\lambda(\tau)$ be the modular function defined as a conformal mapping of the domain $\{0 < \Re \tau < 1\} \cap \{|\tau - \frac{1}{2}| > \frac{1}{2}\}$ onto the upper half plane such that $\lambda(0) = 1, \lambda(1) = \infty, \lambda(\infty) = 0$.

Then [1]

$$(5.9) \quad \frac{e_2 - e_3}{e_1 - e_3} = \frac{\lambda(\tau)}{\lambda(\tau) - 1} = \lambda(\tau + 1)$$

which combines with (5.8) to yield

$$(5.10) \quad \frac{f(b)}{f(c)} = \lambda\left(1 - \frac{\Omega_1(t)}{\Omega_2(t)}\right) = w(t)$$

As t varies from 0 to 2π the quotient $\Omega_1(t)/\Omega_2(t)$ describes a circle. Hence, the set of all boundary points of the region E lies on the curve (5.10).

Thus we have established the following

Theorem 2. *The region E is a closed set whose boundary is given by the equation*

$$w(t) = \lambda \left(1 + \frac{\Omega_1(t)}{\Omega_2(t)} \right), \quad t \in \langle 0, 2\pi \rangle$$

where λ is the modular function defined by (5.9) and $\Omega_1, \Omega_2, \operatorname{Im} \Omega_2/\Omega_1 > 0$ are given by the formulas

$$\Omega_k = \int_{\gamma_k} S(x) R(x) dx, \quad k = 1, 2$$

γ_k is a loop that incloses the points 0 and z_k ($k = 1, 2$) and that leaves the other critical points outside.

6. The limit case $p = 1$

It follows from (4.3) that $\lim_{p=1} D(p) = 1$. Hence $k = 1$ or $l = 1$ and we obtain

$$\frac{f(b)}{f(c)} = \lambda \left(1 + \frac{\Omega'_1(t)}{\Omega'_2(t)} \right); \quad f \in S$$

where

$$\Omega'_k = \int_{\gamma_k} \frac{(1 - e^{it}x) dx}{\sqrt{x(z_1 - x)(z_2 - x)(1 - \bar{z}_1 x)(1 - \bar{z}_2 x)}}$$

This is a well-known result due to J. G. Krzyż [3].

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STRESZCZENIE

W pracy podano dokładny obszar zmienności wyrażenia $f(b)/f(c)$ w klasie meromorficznych i jednolistnych funkcji.

РЕЗЮМЕ

В работе определена точная область значений выражения $f(b)/f(c)$ в классе мероморфных и однолистных функций.

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VOL. XIX SECTIO A 1965

1. Z. Bogucki: On a Theorem of M. Biernacki Concerning Subordinate Functions
O twierdzeniu M. Biernackiego dotyczącym funkcji podporządkowanych.
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 5. Z. Lewandowski: On some Problems of M. Biernacki Concerning Subordinate Functions and on some Related Topics.
O pewnych zagadnieniach M. Biernackiego dotyczących podporządkowania funkcji i pewnych problemach pokrewnych.
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Pewne uwagi o funkcjach gwiaździstych względem punktów symetrycznych.
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Rodziny elementów płaskich $P(M)$, stanowiących uogólnienie płaszczyzn ściśle stycznych krzywej.
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 12. D. Szynal: Certaines inégalités pour les sommes de variables aléatoires et leur application dans l'étude de la convergence de séries et de suites aléatoires.
Pewne nierówności dla sum zmiennych losowych i ich zastosowanie w badaniu zbieżności szeregów i ciągów losowych.

ANNALES
UNIVERSITATIS MARIAE
VOL. XX SECTIO A

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CZASOPISMA

1968-1970

1. G. Labelle: On the Theorems of Gauss-Lucas
O twierdzeniach Gaussa-Lucasa i Grace
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Podporządkowanie obszarowe a nierówności modułów dla pewnych klas funkcji holomorficznych w kole jednostkowym.
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Obszar zmienności $\log f'(z)$ w pewnych podklasach funkcji prawie wypukłych.
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Promień wypukłości pewnych podklas funkcji prawie wypukłych.
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Pewne problemy ekstremalne w klasach funkcji a -kątowo gwiaździstych.
7. J. Stankiewicz: On some Classes of Close-to-convex Functions.
O pewnych podklasach funkcji prawie wypukłych.
8. A. Wesołowski: Relations entre la subordination et l'inégalité des modules dans le cas des majorantes appartenant à la classe $N(p, 0: q)$.
Zależność między podporządkowaniem i nierównością modułów w przypadku majorant należących do klasy $N(p, 0: q)$.
9. A. Żmurek: Sur les relations entre les plans osculateurs à k -dimensions d'une courbe dans l'espace euclidien à n -dimensions.
Zależności między k -wymiarowymi płaszczyznami ściśle stycznymi krzywej w przestrzeni euklidesowej n -wymiarowej.

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