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A Class of Domains Determined by an Invariant Property of the Bergman Function

Klasa obszarów określona przez niezmienniczą własność funkcji Bergmana

Класс областей, определенный инвариантным свойством функции Бергмана

In the present paper we shall consider a class of domains in the space $\mathbf{C}^n$ of n complex variables. For a domain $D \subset \mathbf{C}^n$ we denote by $L^2H(D)$ the Hilbert space of functions which are holomorphic and square integrable with respect to Lebesque measure in D. The relevant scalar product is given by $(f, g) = \int\limits_D f\bar{g}d\omega$. It is known that for every point $p \in D$ there exists a unique element $\chi(p) \in L^2H(D)$ such that for all $f \in L^2H(D)$, $f(p) = (f, \chi(p))$ holds. The function $K_D(p, \bar{q}) = (\chi(q), \chi(p))$, $p, q \in D$ is the Bergman function of D. This function has been a source of numerous biholomorphic invariants. If D is equivalent to a bounded domain the distance in D given by

$$\varrho_D(p, q) = \left[1 - \left(\frac{K_D(p, \bar{q}) K_D(q, \bar{p})}{K_D(p, \bar{p}) K_D(q, \bar{q})}\right)^{1/2}\right]^{1/2}$$

is invariant under biholomorphic transformations. The distance between p and q is never greater than one, and is equal one if and only if $K_D(p, \bar{q}) = 0$.

The question whether exists a bounded domain D for which the Bergman function attains zero value was raised first by Lu Qi-Keng in [2] p. 293. In the following we shall call a domain Lu Qi-Keng domain if its Bergman function does not attain zero value. In view of the transformation rule for the Bergman function the class of Lu Qi-Keng domains is invariant under biholomorphic transformations. Furthermore two followings remarks are true.

Remark 1. The Cartesian product of two Lu Qi-Keng domains is a Lu Qi-Keng domain.

Remark 2. If $D_m \subset D$, $m = 1, 2, \ldots$ is a sequence of Lu Qi-Keng domains and $\lim_m D_m = D$ is bounded then D is a Lu Qi-Keng domain. The latter is a conclusion from the recent result by Ramadanov [3] who proved that the sequence $K_{D_m}(p, \bar{q})$ converges locally uniformly to $K_D(p, \bar{q})$ on $D \times D$.

It would not be meaningful, however, to distinguish the class of Lu Qi-Keng domains without giving an example of bounded domain which does not belong to this class. To this aim we prove

Theorem. *Consider in* $\mathbf{C}^1$ *an annulus* $R = \{z: 0 < r < |z| < 1\}$. *For* $r < e^{-2}$, *R is not a Lu Qi-Keng domain.*

Proof. We begin with a series which expresses the Bergman function for R

$$K_R(z, \bar{t}) = \frac{-1}{2\pi z\bar{t} \ln r} + \frac{1}{\pi z\bar{t}} \sum_{n=-\infty}^{\infty}{}' \frac{n z^n \bar{t}^n}{1 - r^{2n}}$$

where in the sum the index $n = 0$ is omitted. It can also be written in the form

$$K_R(z, \bar{t}) = \frac{1}{\pi w}\left[-\frac{1}{\ln \varrho} + \sum_{m=0}^{\infty}\left(\frac{w\varrho^m}{(1-w\varrho^m)^2} + \frac{\frac{\varrho}{w}\varrho^m}{\left(1-\frac{\varrho}{w}\varrho^m\right)^2}\right)\right]$$

Here $w = z\bar{t}$, $\varrho = r^2$, $0 < \varrho < |w| < 1$.

In the above formula we note that the expression in parentheses is real for real w and for w on the distinguished line $|w| = r$ since then $\varrho/w = \bar{w}$. Furthermore it is a continuous function of w positive for positive w, and for $\varrho < e^{-4}$ negative in a neighbourhood of $w = -1$. To see the latter denote the expression in parentheses by $\varphi(w)$. We have

$$\varphi(-1) = \frac{-1}{\ln \varrho} - \sum_{m=0}^{\infty} \frac{\varrho^m}{(1+\varrho^m)^2} - \sum_{m=0}^{\infty} \frac{\varrho^{m+1}}{(1+\varrho^{m+1})^2} < -\frac{1}{\ln \varrho} - \frac{1}{4}$$

The right side is negative for $-\ln \varrho > 4$ or $\varrho < e^{-4}$. We conclude that $\varphi(w)$ must vanish in some interior point of the w-annulus q.e.d.

More detailed investigations of the above example were carried out by P. L. Rosenthal who proved that all doubly connected Lu Qi-Keng domains in $\mathbf{C}^1$ are biholomorphically equivalent to the unit disc with deleted center. In a case of higher connectivity, or $n > 1$ the situation is more complicated. The following problems seem to be open:

1° Is every Lu Qi-Keng domain of finity connectivity in $\mathbf{C}^1$ equivalent to the unit disc with a finite number of points deleted?

2° Is every simply connected, bounded domain in $\mathbf{C}^n$, $n > 1$ a Lu Qi-Keng domain?

3° Is every bounded convex domain in $\mathbf{C}^n$ a Lu Qi-Keng domain?

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STRESZCZENIE

Praca zawiera przykład ograniczonego obszaru w $\mathbf{C}^1$, dla którego funkcja Bergmana przyjmuje wartość zero. Pozwala to na rozważanie klasy obszarów, dla których funkcja Bergmana nie zeruje się. Sformułowane są pewne własności obszarów, należących do tej klasy, oraz kilka otwartych problemów.

РЕЗЮМЕ

В работе дан пример ограниченной области в $\mathbf{C}^1$, для которой функция Бергмана принимает нулевое значение. Это позволяет рассматривать класс областей, для которых функция Бергмана не принимает нулевого значения. Сформулированы некоторые свойства областей, принадлежащих к этому классу, а также несколько неразрешенных проблем.