ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXII/XXIII/XXIV, 23 SECTIO A 1968/1969/1970

Department of Mathematics, University of Kentucky, Lexington, Kentucky, USA

WIMBERLEY C. ROYSTER and TED J. SUFFRIDGE

Typically Real Polynomials

Wielomiany typowo rzeczywiste

Типично вещественные полиномы

1. Introduction. Let TR denote the class of normalized functions f which are analytic and typically real in the unit disk E. That is, f is of the form $f(z) = z + c_2 z^2 + c_3 z^3 + \ldots$ in E and satisfies in E the condition $\operatorname{Im} f(z) \operatorname{Im} z \geqslant 0$. The class of functions was introduced by Rogosinski and has been studied extensively. In this paper we initiate a study of polynomials $P_n(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n$ which belong to TR, that is $P_n(z)$ is typically real in E. For $n \leqslant 5$ we find the exact bounds for a_k, $k \leqslant n$. We find also the coefficient regions for the cubic $z + a_2 z^2 + a_3 z^3$ and the odd polynomial $z + a_3 z^3 + a_5 z^5$. In what follows the coefficients a_k are real.

2. Main result. Let $R(u)$ be a polynomial such that

$$R(\cos\theta) = \sum_{k=1}^{n} a_k \frac{\sin k\theta}{\sin\theta} = \frac{\operatorname{Im} P_n(e^{i\theta})}{\sin\theta} \tag{1}$$

It follows that $P_n \in TR$ if and only if $R(\cos\theta) \geqslant 0$ for all θ, $-\pi \leqslant \theta \leqslant \pi$. Let $u = \cos\theta$. Then

$$R(u) = \sum_{k=1}^{n} a_k U_{k-1}(u) = 2^{n-1} a_n \sum_{j=0}^{n-1} b_j u^j, \quad -1 \leqslant u \leqslant 1$$

where $U_{k-1}(u)$ is a Tchebychef polynomial of the second kind. For fixed k, we determine the various forms $R(u)$ assumes in order that a_k be extremal.

Lemma 1. *Let b_j be real, $0 \leqslant j \leqslant n-1$ and $b_{n-1} = 1$, suppose $\sum_{j=0}^{n-1} b_j u^j$ is either non-negative or non-positive for all u in $-1 \leqslant u \leqslant 1$. Then there exist unique a_j, $1 \leqslant j \leqslant n$, $a_1 = 1$ such that*

$$\sum_{k=1}^{n} a_k \frac{\sin k\theta}{\sin\theta} = 2^{n-1} a_n \sum_{j=0}^{n-1} b_j u^j$$

and $P_n(z) = \sum_{k=1}^{n} a_k z^k$ belongs to the TR.

Lemma 2. *Let $P_n(z)$ be a polynomial of degree n and let k be fixed, $1 \leqslant k \leqslant n$. Suppose that among all polynomials in the class TR of degree n the k^{th} coefficient a_k assumes its extreme value for $P_n(z)$. Then it suffices to assume that all the zeros of $R(u)$ are real.*

Lemma 3. *Under the hypothesis of Lemma 2, it suffices to take all zeros of $R(u)$ in the closed interval $-1 \leqslant u \leqslant 1$.*

Since all zeros of $R(u)$ lying in the open interval $(-1, 1)$ must be zeros of even multiplicity we have the following result.

Theorem 1. *Let $P_n(z)$ be a polynomial of degree n $(a_n \neq 0)$ and let k be fixed, $1 < k \leqslant n$. If among all polynomials of degree n belonging to the class TR the k^{th} coefficient a_k assumes its extreme value for $P_n(z)$, then $R(u)$ has the form*

$$R(u) = \pm 2^{n-1} a_n (1 \pm u) \prod_{j=1}^{\frac{n-2}{2}} (u - \gamma_j)^2$$

for n even, where $-1 \leqslant \gamma_j \leqslant 1$, $1 \leqslant j \leqslant \dfrac{n-2}{2}$ and

$$R(u) = -2^{n-1} a_n (1 - u^2) \prod_{j=1}^{\frac{n-3}{2}} (u - \gamma_j)^2$$

or

$$R(u) = 2^{n-1} a_n \prod_{j=1}^{\frac{n-1}{2}} (u - \gamma_j)^2$$

for n odd.

3. Coefficient bounds. Using the preceding results we can calculate the extreme values of a_k, $2 \leqslant k \leqslant n$, $2 \leqslant n \leqslant 5$, all bounds are sharp, however, all the coefficients are not extremalized by the same polynomial.

$n = 2 \quad |a_2| \leqslant 1/2$

$n = 3 \quad |a_2| \leqslant 1, \ -1/3 \leqslant a_3 \leqslant 1$

$n = 4 \quad |a_2| \leqslant (1+\sqrt{7})/3, \ -1/3 \leqslant a_3 \leqslant 1, \ a_4 \leqslant 2/3$

$n = 5 \quad |a_2| \leqslant \sqrt{2}, \ -(\sqrt{5}-1)/2 \leqslant a_3 \leqslant (1+\sqrt{5})/2,$

$|a_4| \leqslant 1, \ -1/2 \leqslant a_5 \leqslant 1.$

The calculations of these bounds are lengthy but elementary. Employing the methods in Theorem 1 well yield bounds for coefficients for $n > 5$ but the calculations are very lengthy.

4. Coefficient regions. The equations of the boundary ∂V of the coefficient region V in the a_2, a_3 plane are determined in part by finding the envelope of the family of lines bounding the half-planes $R(u) = 2ua_2 + (4u^2-1)a_3+1 \geqslant 0$. The envelope is the ellipse $a_2^2+4(a_3-1/2)^2 = 1$. It is easily shown that ∂V consists of a portion of the line $2a_2-3a_3 = 1$ between the points $(0, -1/3)$ and $(1/4, 1/5)$, the upper arc of the ellipse between $(1/4, 1/5)$ and $(-1/4, 1/5)$ and the portion of the line $-2a_2-3a_3 = 1$ between the points $(-4/5, 1/5)$ and $(0, -1/3)$.

The boundary of the coefficient region in the a_3, a_5 plane in the case of the odd fifth degree polynomial can be found in a similar manner.

The proofs of these results are to appear in Publicationes Debrecen.

REFERENCE

[1] Rogosinski, W., *Über positive harmonische Entwicklungen and typisch-reelle Potenzreihen*, Math. Zeit. 35 (1932), 93–121.

STRESZCZENIE

Niech $P_n(z) = z+a_2z^n+\ldots+a_nz^n$ będzie unormowanym wielomianem typowo rzeczywistym w kole jednostkowym. Autorzy wykazują, że gdy P_n jest wielomianem, dla którego k-ty współczynnik osiąga maksymalną co do modułu wartość $(1 < k < n)$, to wyrażenie $R(\cos\theta) = \operatorname{Im}\{P_n(e^{i\theta})/\sin\theta\}$ musi mieć jedną z trzech postaci

1. $R(u) = 2^{n-1}a_n(1\pm u)\prod_{j=1}^{n-2/2}(u-\gamma_j)^2, \ -1 \leqslant \gamma_j \leqslant 1, \ 1 \leqslant k \leqslant \frac{n-2}{2},$

2. $R(u) = 2^{n-1}a_n(1-u^2)\prod_{j=1}^{n-3/2}(u-\gamma_j)^2$

3. $R(u) = 2^{n-1}a_n\prod_{j=1}^{n-1/2}(u-\gamma_j)^2, \ -1 \leqslant \gamma_j \leqslant 1.$

Można stąd otrzymać dokładne oszacowanie a_k przy $2 \leqslant k \leqslant n$ dla $2 \leqslant n \leqslant 5$.

РЕЗЮМЕ

Пусть $P_n(z) = z + a_2 z^2 + \ldots + a_n z^n$ будет нормированным типично вещественным полиномом в единичном круге. Доказано, что, если P есть полиномом, k-ый коэффициент которого принимает максимум по модулю $(1 < k < n)$, то выражение $R(\cos\theta) = \operatorname{Im}\{P_n(e^{i\theta})/\sin\theta\}$ должно иметь один из трех видов

1. $R(u) = 2^{n-2} a_n (1 \pm u) \prod_{j=1}^{n-2/2} (u-\gamma_j)^2, \; -1 \leqslant \gamma_j \leqslant 1, \; 1 \leqslant k \leqslant \frac{n-2}{2}$

2. $R(u) = 2^{n-1} a_n (1-u^2) \prod_{j=1}^{n-3/2} (u-\gamma_j)^2$

3. $R(u) = 2^{n-1} a_n \prod_{j=1}^{n-1/2} (u-\gamma_j)^2, \; -1 \leqslant \gamma_j \leqslant 1.$

Отсюда можно вывести точную оценку a_k при $2 \leqslant k \leqslant n$, для $2 \leqslant n \leqslant 5$.