ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXII/XXIII/XXIV, 19 SECTIO A 1968/1969/1970

Instytut Matematyki, Uniwersytet Łódzki, Łódź

JANINA PAŁKA

Sharp Estimates of $|P(w)|$, $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$, $\operatorname{Arg} P'(w)$ in a Class of Univalent Polynomials

Ostre oszacowania $|P(w)|$, $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$, $\operatorname{Arg} P'(w)$ w klasie wielomianów jednolistnych

Точные оценки $|P(w)|$, $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$, $\operatorname{Arg} P'(w)$ в классе однолистных полиномов

1. Introduction and statement of results

The author is concerned with univalent polynomials of the form

(1) $$\tilde{P}(w) = w + C_2 w^2 \quad (C_2 \neq 0)$$

and

(2) $$\tilde{P}(w) = w + C_2 w^2 + C_3 w^3 \quad (C_3 \neq 0),$$

considered in the largest domain D such that $0 \in D$ and $|P(w)| < 1$, for $w \in D$. Given $\tilde{P}$ let P denote its analytic extention to the whole finite plane.

Let $\mathfrak{p}_2$ denote the class of all such polynomials of the form (1) and $\mathfrak{p}_3$ — the class of all such polynomials of the form (1) or (2). The classes in question were introduced by Charzyński [1] and applied, together with analogous classes of higher orders, to various basic problems in the theory of univalent functions (cf. e.g. [4] and [3]). The coefficient problem within these classes has been investigated in [2] and [5].

The following results are obtained by the author.

Theorem 1. *Given an arbitrary* w *let* $r = |w|$ *and let* sup *stand for* $\sup\limits_{P \in \mathfrak{p}_2}$. *Then*

(3) $$\sup |P(w)| = \sup |P(r)| = r + \tfrac{1}{4} r^2,$$

(4) $$\sup \operatorname{Arg}[P(w)/w] = \sup \operatorname{Arg}[P(r)/r] = \arctan[r(16-r^2)^{-\frac{1}{2}}],$$

$$(5)\qquad \sup|P'(w)| = \sup|P'(r)| = 1+\tfrac{1}{2}r,$$

$$(6)\qquad \sup\operatorname{Arg}P'(w) = \sup\operatorname{Arg}P'(r) = \arctan[r(4-r^2)^{-\frac{1}{2}}]+a,$$

where $a = \operatorname{Arg} w$.

All the extremal functions are given by the formulae:

$$(7)\qquad P^*(w) = w+\tfrac{1}{4}e^{-i\vartheta}w^2 \textit{ in the case of } (3) \textit{ and } (5),$$

$$(8)\qquad P^*(w) = w+\tfrac{1}{16}[r-i(16-r^2)]e^{-i\vartheta}w^2 \textit{ in the case of } (4),$$

$$(9)\qquad P^*(w) = w-\tfrac{1}{8}[r-i(4-r^2)^{\frac{1}{2}}]e^{-i\vartheta}w^2 \textit{ in the case of } (6),$$

where ϑ *is real.*

Theorem 2. *Given an arbitrary* w *let* $r=|w|$ *and let* sup *stand for* $\sup\limits_{P\in\mathfrak{P}_3}$. *Then*

$$(10)\qquad \sup|P(w)| = \sup|P(r)| = \tfrac{1}{27}r[(27+r^2)^{\frac{1}{2}}+r]^2,$$

$$(11)\qquad \sup\operatorname{Arg}[P(w)/w] = \sup\operatorname{Arg}[P(r)/r] = \arctan[2r(27-4r^2)^{-\frac{1}{2}}],$$

$$(12)\qquad \sup|P'(w)| = \sup|P'(r)| = \tfrac{1}{9}[9+2r^2+2r(12+r^2)^{\frac{1}{2}}],$$

$$(13)\qquad \sup\operatorname{Arg}P'(w) = \sup\operatorname{Arg}P'(r) = \arctan\{4r[(9-r^2)/(9-4r^2)(27-4r^2)]^{\frac{1}{2}}\}+a.$$

All the extremal functions are given by the formulae:

$$(14)\qquad P^*(w) = w+2(27+r^2)^{-\frac{1}{2}}e^{-i\vartheta}w^2+\tfrac{2}{27}[(27+r^2)^{\frac{1}{2}}+r](27+r^2)^{-\frac{1}{2}}e^{-2i\vartheta}w^3 \textit{ in the case of } (10),$$

$$(15)\qquad P^*(w) = w+2i(27-4r^2)^{\frac{1}{2}}(27-2r^2)^{-1}e^{-i\vartheta}w^2 - 2(27-2r^2)^{-1}e^{-2i\vartheta}w^3 \textit{ in the case of } (11),$$

$$(16)\qquad P^*(w) = w+\tfrac{4}{3}(12+r^2)^{-\frac{1}{2}}e^{-i\vartheta}w^2 + \tfrac{2}{27}[(12+r^2)^{\frac{1}{2}}+r](12+r^2)^{-\frac{1}{2}}e^{-2i\vartheta}w^3 \textit{ in the case of } (12),$$

$$(17)\qquad P^*(w) = w+\tfrac{2}{81}\{r(9-4r^2)-i[(9-4r^2)\times(27-4r^2)(9-r^2)]^{\frac{1}{2}}\}(r^2-3)^{-1}e^{-i\vartheta}w^2+\tfrac{2}{2187}\times\{8r^4-72r^2+243+2ri[(9-4r^2)(27-4r^2)(9-r^2)]^{\frac{1}{2}}\}\times(r^2-3)^{-1}e^{-2i\vartheta}w^3 \textit{ in the case of } (13),$$

where ϑ *is real.*

The proofs are based on a theorem of Charzyński [1], (cf. Section 2) and, in the case of $\mathfrak{p}_3$, on the same basic lemma (cf. Section 3) which is hoped to be of an independet interest (cf. Section 5).

The theorems proved in this paper were presented to the Conference on Analytic Functions in Lublin on August 23, 1970 (cf. [6]).

Here I should like to express my thanks to Prof. Z. Charzyński for suggesting the problem and to Doc. J. Ławrynowicz for helpful suggestions concerning the proofs.

2. The case of $\mathfrak{p}_2$

We proceed to prove Theorem 1. Our proof is based on the following known results (cf. [1], p. 20, [7], p. 122, and [2], p. 28).

Theorem A. *If E is a holomorphic function of the complex variables $z_2, \ldots, z_k$ $(k \leqslant m)$ in a sufficiently large domain then in the class $\mathfrak{p}_m$ there exist extremal polynomials for which the functional* $\operatorname{re} E(C_2, \ldots, C_k)$ *attains its maximum. Moreover, if*

$$P^*(w) = w + C_2^* w^2 + \ldots + C_n^* w^n \quad (C_n^* \neq 0,\ k \leqslant n \leqslant m) \tag{18}$$

is such a polynomial, $w_1, \ldots, w_l$ denote all the distinct zeros of the derived polynomial, and $\gamma_1, \ldots, \gamma_l$ — the multiplicity of these zeros, respectively then $w_1, \ldots, w_l$ lie on the boundary of the domain μ_{P^*}, and there exist numbers $\varrho_\lambda \geqslant 0$ $(\lambda = 1, \ldots, l)$ satisfying the relations

$$\operatorname{res}_{w_\lambda}\{P^*(w)\Phi_k(w)/w^2 P^{*'}(w)\} = \varrho_\lambda \quad (\lambda = 1, \ldots, l),$$

where

$$\Phi_k(w) = \sum_{\varkappa=2}^{k} w^{1-\varkappa} \sum_{\nu=\varkappa}^{k} (\nu - \varkappa + 1) C_{\nu-\varkappa+1}^* E'_{z_\nu}(C_2^*, \ldots, C_k^*) \quad (C_1^* = 1).$$

Theorem B. *Any polynomial, satisfying the necessary condition formulated in Theorem A, belongs to the class $\mathfrak{p}_m$.*

We confine ourselves to that part of Theorem 1 which concerns $|P(w)|$, since the proof of the remaining parts is quite similar.

We adopt the notation of Theorem A. By Theorem A there exists in $\mathfrak{p}_2$ an extremal polynomial

$$P^*(w) = w + C_2^* w^2 \quad (C_2^* \neq 0)$$

for which the functional

$$\operatorname{re} E(C_2) = \operatorname{re} \log|P(w)| = \operatorname{re} \log(r + C_2 r^2)$$

attains its maximal value and, consequently, the estimate $|P(w)| \leqslant |P^*(r)|$ is best possible in $\mathfrak{p}_2$. Moreover, w_1 and, consequently, C_2^* can be obtained by solving the system of algebraic equations

$$w_1 + C_2^* w_1^2 = \sigma_1 \qquad (|\sigma_1| = 1),$$

$$1 + 2C_2^* w_1 = 0,$$

$$\operatorname{res}_{w_1}[P^*(w)\Phi(w)/w^2 P^{*\prime}(w)] = \varrho_1 \qquad (\varrho_1 \geqslant 0),$$

where $\Phi(w) = r/w(1+C_2^* r)$. Direct calculation gives that $w_1 = 2$ or -2, where the first solution can easily be excluded. Consequently $C_2^* = \frac{1}{4}$ and, since $\varrho_1 = r/(4+r) \geqslant 0$ and $\sigma_1 = -1$ i.e. σ_1 lies on the unit circle, then, by Theorem B, we conclude that (3) holds and all the extremal functions are given by (7) with real ϑ.

3. The basic lemma

In this section we formulate and prove the basic lemma announced in Section 1.

Lemma 1. *Let P be a polynomial of the form (2) such that the zeros w_1 and w_2 of its derivative satisfy relations*

(19) $$|P(w_1)| = |P(w_2)| = 1.$$

Then either

(20) $$|w_1| = |w_2|$$

or

(21) $$1/|w_1| + 1/|w_2| = \tfrac{2}{9}.$$

Proof. Relations $P(w_1) = P(w_2) = 0$ yield $C_2 = -(w_1+w_2)/(2w_1w_2)$ and $C_3 = 1/(3w_1w_2)$, whence (19) can be written in the form

(22) $$\tfrac{1}{36}(w_1\overline{w}_1/w_2\overline{w}_2)[w_1\overline{w}_1 - 3(w_1\overline{w}_2 + \overline{w}_1 w_{20}) + 9w_2\overline{w}_2] = 1,$$

(23) $$\tfrac{1}{36}(w_2\overline{w}_2/w_1\overline{w}_1)[9w_1\overline{w}_1 - 3(w_1\overline{w}_2 + \overline{w}_1 w_2) + w_2\overline{w}_2] = 1.$$

Subtracting (23) from (22) we obtain

$$2(|w_1|^2 - |w_2|^2) = 9(|w_1|^2/|w_2|^2 - |w_2|^2/|w_1|^2)$$

whence either (20) or (21) follows.

4. The estimates for $\mathfrak{p}_3$

We proceed to prove Theorem 2. As in the case of Theorem 1 our proof is based on Theorems A and B. We confine ourselves to that part of Theorem 2 which concerns $|P(w)|$ since the proof of the remaining parts is quite similar.

We adopt the notation of Theorem A. By Theorem A there exists in $\mathfrak{p}_3$ an extremal polynomial

$$(24)\qquad P^*(w) = w + C_2^* w^2 + C_3^* w^3 \quad (C_3^* \neq 0)$$

for which the functional

$$(25)\qquad \operatorname{re} E(C_2, C_3) = \operatorname{re}\log|P(w)| = \operatorname{re}\log(r + C_2 r^2 + C_3 r^3)$$

attains its maximal value and, consequently, the estimate $|P(w)| \leqslant |P^*(r)|$ is best possible in $\mathfrak{p}_3$. Moreover, w_1, w_2 and, consequently, C_2^*, C_3^* can b. obtained by solving the system of algebraic equations

$$(26)\qquad w_k + C_2^* w_k^2 + C_3^* w_k^3 = \sigma_k \quad (|\sigma_k| = 1),$$

$$(27)\qquad 1 + 2C_2^* w_k + 3C_3^* w_k^2 = 0,$$

$$(28)\qquad \operatorname{res}_{w_k}[P^*(w)\Phi(w)/[w^2 P^{*\prime}(w)] = \varrho_k \quad (\varrho_k \geqslant 0),$$

where $k = 1, 2$ and

$$\Phi(w) = r(1 + 2C_2^* r + rw^{-1})/[w(1 + C_2^* r + C_3^* r^2)].$$

We eliminate first C_2^* and C_3^*: equations (27) yield $C_2^* = -(w_1 + w_2)/(2w_1 w_2)$ and $C_3^* = 1/(3w_1 w_2)$.

Next we consider, separately, the cases $P^*(w) = (1 - w/w_1)^2$ and $P^*(w) = (1 - w/w_1)(1 - w/w_2)$, where $w_1 \neq w_2$. It can easily be checked that the second possibility is the case. Then equations (28), after eliminating ϱ_k become

$$(29)\qquad (3w_2 - w_1)(w_2 - r)/(w_1 - w_2)[6w_1 w_2 - 3r(w_1 + w_2) + 2r^2]$$
$$= (3\overline{w}_2 - \overline{w}_1)(\overline{w}_2 - r)/\{(\overline{w}_1 - \overline{w}_2)[6\overline{w}_1\overline{w}_2 - 3r(\overline{w}_1 + \overline{w}_2) + 2r^2]\},$$

$$(30)\qquad (3w_1 - w_2)(w_1 - r)/(w_2 - w_1)[6w_1 w_2 - 3r(w_1 + w_2) + 2r^2]$$
$$= (3\overline{w}_1 - \overline{w}_2)(\overline{w}_1 - r)/\{(\overline{w}_2 - \overline{w}_1)[6\overline{w}_1\overline{w}_2 - 3r(\overline{w}_1 + \overline{w}_2) + 2r^2]\}.$$

Now we apply Lemma 1. By this lemma we have again to consider two possibilities: (23) and (22). The first possibility yields

$$|C_2^*| = (\tfrac{2}{27} + \tfrac{2}{3}|w_1|^{-2} - 3|w_2|^{-4})^{\frac{1}{2}} \leqslant \tfrac{1}{3}$$

and

$$|C_3^*| = (\tfrac{2}{81}|w_1|^{-2} - \tfrac{1}{9}|w_1|^{-4})^{\frac{1}{2}} \leqslant \tfrac{1}{27}.$$

Hence $|P^*(w)| \leqslant |\hat{P}(w)|$ for every w, where

$$\hat{P}(w) = w + \tfrac{2}{9}\sqrt{3}w^2 + \tfrac{2}{27}w^3$$

is a polynomial belonging to $\mathfrak{p}_3$ (cf. [2], p. 27). Then the maximum of $\operatorname{re}E(C_2, C_3)$ is attained in the case (22).

Let us introduce the notation

$$|w_1| = |w_2| = \tau, \exp i \operatorname{Arg} w_1 = s_1, \exp i \operatorname{Arg} w_2 = s_2.$$

Then the first of equations (26) and the equations (29), (30) give

$$\tau^2[10-3(s_1^2+s_2^2)/(s_1 s_2)] = 36 \tag{31}$$

$$(s_1-3s_2)(\tau s_2-r)/[6\tau^2 s_1 s_2-3r\tau(s_1+s_2)+2r^2] = -s_1(s_2-3s_1)(\tau-rs_2)/[6\tau^2-3r\tau(s_1+s_2)+2r^2 s_1 s_2], \tag{32}$$

$$(s_2-3s_1)(\tau s_1-r)/[6\tau^2 s_1 s_2-3r\tau(s_1+s_2)+2r^2] = -s_2(s_1-3s_2)(\tau-rs_1)/[6\tau^2-3r\tau(s_1+s_2)+2r^2 s_1 s_2]. \tag{33}$$

Now we add and subtract the both sides of (32) and (33), and introduce the notation

$$s_1+s_2 = s, \quad s_1 s_2 = \delta.$$

Consequently, (32) and (33) give

$$[\tau(8\delta-3s^2)+2rs]/[6\tau^2\delta-3r\tau s+2r^2] = -[\tau(8\delta-3s^2)+2r\delta s]/[6\tau^2-3r\tau s+2r^2\delta], \tag{34}$$

$$(3\tau s-4r)/[6\tau^2\delta-3r\tau s+2r^2] = (3\tau s-4r\delta)/[6\tau^2-3r\tau s+2r^2\delta]. \tag{35}$$

Finally, we rearrange equation (35) and divide the both sides of (35) by the corresponding sides of (34). Thus we obtain

$$3\tau^2 s(1-\delta)-4r\tau^2(1-\delta^2)+r^2\tau s(1-\delta) = 0, \tag{36}$$

$$3\tau s(9-2\tau^2)-r(\delta+1)(27-8\tau^2)-2r^2\tau s = 0. \tag{37}$$

Here we recall that we have also the third equation for s_1, s_2 and τ, namely (31).

In order to solve the system (31), (36), (37) we observe that (36) implies the following possibilities:

(a) $\delta = 1$, (b) $3\tau^2 s-4r\tau^2(1+\delta)+r^2\tau s = 0$.

We begin with (a). In this case (37) and (31) become

$$3\tau s(9-2\tau^2)-2r(27-8\tau^2)-2r^2\tau s = 0 \tag{38}$$

and

$$3\tau^2 s^2-16\tau^2+36 = 0 \tag{39}$$

respectively. Hence, after eliminating s, we get

$$144t^2-12(135+8r^2)t^2+8(729+81r^2+2r^4)t- -9(729+135r^2+4r^4) = 0 \tag{40}$$

where $t = \tau^2$. Equation (40) have three solutions:

$$(41) \qquad t_1 = \tfrac{1}{12}(4r^2+27),\ t_2 = \tfrac{1}{6}[27+r^2-r(27+r^2)^{\frac12}],$$
$$t_3 = \tfrac{1}{6}[27+r^2+r(27+r^2)^{\frac12}].$$

In order to make the proper choice we notice that, by (39) and $\delta = 1$, we have $s = 3\varepsilon[(4t-9)/(3t)]^{\frac12}$, where $\varepsilon = 1$ or -1. Consequently

$$C_2^* = -(w_1+w_2)/(2w_1w_2) = -\tfrac12 st^{-\frac12} = -\varepsilon t^{-1}[(4t-9)/3]^{\frac12},$$
$$C_3^* = 1/(3w_1w_2) = 1/(3t),$$

whence

$$|P^*(w)| = |r+C_2^*r^2+C_3^*r^3| = r|1-\varepsilon t^{-1}[(4t-9)/3]^{\frac12}r+1/(3t)r^2|.$$

The above expression attains its maximum for $t = t_2$ and $\varepsilon = -1$ or $t = t_3$ and $\varepsilon = 1$. If $t = t_3$ and $\varepsilon = 1$, then

$$s = \tfrac19[27+2r^2+2(27+r^2)^{\frac12}] > 2$$

which contradicts the definition of s:

$$s = s_1+s_2 = \exp i\,\mathrm{Arg}w_1+\exp i\,\mathrm{Arg}w_2.$$

Thus $t = t_2$ and $\varepsilon = 1$, and this yields

$$w_k = \tfrac13\{-[(27+r^2)^{\frac12}-r]+i(-1)^k2^{-\frac12}[27-r^2+r(27+r^2)^{\frac12}]^{\frac12}\} \qquad (k = 1, 2),$$
$$C_2^* = 2/(27+r^2)^{\frac12}, \qquad C_3^* = \tfrac{2}{27}[(27+r^2)^{\frac12}+r]/(27+r^2)^{\frac12},$$
$$\varrho_1 = \varrho_2 = (27+r^2)^{-\frac12} > 0$$
$$\sigma_k = -\tfrac19\{(27+r^2)^{\frac12}-r+i(-1)^k2^{\frac12}[27-r^2+r(27+r^2)^{\frac12}]^{\frac12}\} \qquad (k = 1, 2),$$

i.e. σ_1 and σ_2 lie on the unit circle. Then, by Theorem 3, we conclude that (10) holds and all the extremal functions are given by (14) with real ϑ, provided that (b) gives either the same result or does not correspond to the extremum of $|P(w)|$.

Thus there remains to consider (b). Under the same notation as in the preceding case we get $t = r^2$, $s = \delta+1$, $|\delta| = 1$, whence $|P^*(w)| = \frac16 r|3\delta-1| \leqslant \frac23 r$. Consequently (b) does not correspond to the extremum of $|P(w)|$, and this completes the proof.

5. Conclusion

The considerations of Section 4 show that in the case of the four functionals discussed there Lemma 1 enables us to simplify the system of equations arising from Theorem A because (20) is the case, while (21) can easily be excluded. Thus it is natural to consider the two following problems:

(i) Find an analogue of Lemma 1 in the case of polynomials of higher degrees,

(ii) Determine the class of all functionals E appearing in Theorem A such that the zeros $w_1, \ldots, w_l$ of the derivative of an extremal polynomial satisfy the relations:

$$|w_1| = |w_2|, \ldots, |w_{l-1}| = |w_l| \quad (l \text{ even}),$$

$$|w_1| = |w_2|, \ldots, |w_{l-2}| = |w_{l-1}|, \ \operatorname{im} w_l = 0 \quad (l \text{ odd}).$$

The author has tackled the first problem and got the following partial result.

Lemma 2. *Let*

$$P(w) = w + C_2 w^2 + C_3 w^3 + C_4 w^4 \ (C_4 \neq 0)$$

and let the zeros w_1, w_2 and w_3 of the derivative P' satisfy relations

$$(42) \qquad |P(w_1)| = |P(w_2)| = |P(w_3)| = 1,$$

and

$$(43) \qquad |w_3| \neq |w_1|, \ |w_3| \neq |w_2|.$$

Then we have either (20) *or*

$$(44) \quad 33 - 144(1/|w_1^2| + 1/|w_2^2| + 1/|w_3^2|) + [12|w_2^2|(\eta_3|w_3^2| - \eta_1|w_1^2|) - 6\eta_1(\eta_3|w_3^2| - \eta_2|w_2^2|)]/[|w_2^2|(|w_3^2| - |w_1^2|)(|w_3^2| - |w_1^2|)] + [12|w_2^2|(|w_3^2| - |w_1^2|) - 6\eta_3(|w_3^2| - |w_2^2|)][|w_1^2| + |w_2^2| - 32|w_3^2| + 144(1/|w_1^2| + 1/|w_2^2|)|w_3^2| + 6\eta_1\eta_3/|w_2^2| - 2(\eta_1 + \eta_2 + \eta_3)]/[(12|w_2^2| - 6\eta_3)(|w_3^2| - |w_1^2|)(|w_3^2| - |w_2^2|)] = 0$$

where $\eta_1 = w_2\overline{w}_3 + w_3\overline{w}_2$, $\eta_2 = w_1\overline{w}_3 + w_3\overline{w}_1$, $\varphi_3 = w_1\overline{w}_2 + w_2\overline{w}_1$.

Moreover, (20) *holds if and only if*

$$(45) \qquad \operatorname{Arg} w_3 = \tfrac{1}{2}(\operatorname{Arg} w_1 + \operatorname{Arg} w_2).$$

Proof. Relations $P(w_1) = P(w_2) = P(w_3) = 0$ yield

$$C_2 = -(w_1w_2 + w_2w_3 + w_3w_1)/(2w_1w_2w_3), \ C_3 = (w_1 + w_2 + w_3)/(3w_1w_2w_3),$$
$$C_4 = -1/(4w_1w_2w_3),$$

whence (42) can be written in the form

$$(46) \quad \tfrac{1}{144}[w_k\overline{w}_k/(w_l\overline{w}_l w_m\overline{w}_m)](w_k\overline{w}_k)^2 + 36 w_l\overline{w}_l w_m\overline{w}_m + 4w_k\overline{w}_k(w_l\overline{w}_l + w_m\overline{w}_m) - 12[(w_l\overline{w}_l)^2(w_k\overline{w}_m + w_m\overline{w}_k) + (w_m\overline{w}_m)^2(w_k\overline{w}_l + w_l\overline{w}_k)] - 2w_k\overline{w}_k(w_k\overline{w}_l + w_l\overline{w}_k + w_k\overline{w}_m + w_m\overline{w}_k + w_l\overline{w}_m + w_m\overline{w}_l) + 6(w_k\overline{w}_l + w_l\overline{w}_k)(w_k\overline{w}_m + w_m\overline{w}_k) = 1$$

where $l = -\frac{1}{2}(3k^2 - 11k + 4)$, $m = \frac{1}{2}(3k^2 - 13k + 16)$, $(k = 1, 2, 3)$.

Denote the left-hand side of (42) by A_k and consider relations

$$(47)\qquad A_1 - A_2 = 0, \quad \frac{A_3 - A_1}{|w_3^2| - |w_1^2|} - \frac{A_2 - A_3}{|w_2^2| - |w_1^2|} = 0,$$

what is justified in view of (43). Denote again the left-hand side of relations (47) by B_1 and B_2, respectively. The left-hand side of the relation

$$\frac{[2|w_2^2|(|w_3^2| - |w_1^2|) - \eta_3(|w_3^2| - |w_2^2|)]}{[|w_2^2|(|w_3^2| - |w_1^2|)(|w_3^2| - |w_2^2|)]} \cdot \frac{[B_1|w_2^2| - 6(2|w_2^2| - \eta_3)(\eta_1|w_1^2| - \eta|w_2^2|)]}{[2|w_2^2| - \eta_3 + 6(\eta_1|w_1^2| - \eta_2|w_2^2|)]} + B_2 = 0$$

is exactly the product of the expressions (20), (44) and $|w_1^2| + |w_2^2|$.

Finally, by (47), we easily verify that (20) holds if and only if (45) holds.

REFERENCES

[1] Charzyński, Z., *Fonctions univalentes inverses. Polynômes univalentes*, Bull. Soc. Sci. Lettres Łódź 9, 7 (1958), p. 1–21.

[2] Charzyński, Z., and Ławrynowicz, J., *On the coefficients of univalent polynomials*, Colloq. Math. 16 (1967), p. 27–33.

[3] Charzyński, Z., et Śladkowska, J., *Fonctions algébriques et variations analytiques des fonctions univalentes*, Dissertationes Math. 7 (1970), p. 1–80.

[4] Janikowski, J., *Méthodes algébriques et équation de Löwner*, Bull. Soc. Sci. Lettres Łódź 12, 16 (1961), p. 1–9.

[5] Ławrynowicz, J., *On the coefficient problem for univalent polynomials*, Proc. Cambridge Philos. Soc. 64 (1968), p. 87–98.

[6] Pałka, J., *Sharp estimates of* $|P(w)|$, $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$, $\operatorname{Arg}P'(w)$ *in a class of univalent polynomials*, Proc. Conf. Analytic Functions, Lublin 1970, p. 17–18.

[7] Pólya, G., und Szegö, G., *Aufgaben und Lehrsätze aus der Analysis I*, Berlin 1925.

STRESZCZENIE

Praca dotyczy wielomianów jednolistnych postaci (1) i (2) rozważanych w największym obszarze D takim, że $0 \in D$ oraz $|\tilde{P}(w)| < 1$ dla $w \in D$.

Dla danego wielomianu P niech $\tilde{P}$ oznacza jego przedłużenie analityczne na całą płaszczyznę otwartą. Niech dalej $\mathfrak{p}_2$ oznacza klasę wszystkich wielomianów postaci (1) zaś $\mathfrak{p}_3$ klasę wszystkich wielomianów postaci (1) lub (2). Klasy te zostały wprowadzone przez Z. Charzyńskiego w 1958 r. W pracy uzyskano następujące wyniki:

Twierdzenie 1. *Dla dowolnego* w *niech* $r = |w|$. *W przypadku* $\mathfrak{p}_2$ *zachodzi oszacowanie* (3). *Wszystkie funkcje ekstremalne dane są wzorem* (7).

Twierdzenie 2. *Dla dowolnego w niech $r = |w|$. W przypadku* $\mathfrak{p}^3$ *zachodzi oszacowanie* (10). *Wszystkie funkcje ekstremalne dane są wzorem* (14).

Analogiczne wyniki uzyskano dla $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$ oraz $\operatorname{Arg}P'(w)$.

Dowody są oparte na następującym lemacie: Jeśli P jest wielomianem postaci (2), takim że pierwiastki w_1, w_2 jego pochodnej spełniają warunek (19), to zachodzi związek (20) lub (21).

РЕЗЮМЕ

В работе рассмотрены однолистные полиномы вида (1) и (2) в наибольшей области $Д$, такой, что $0 \in Д$ и $|\tilde{P}(w)| < 1$ для $w \in Д$.

Для заданного полинома P пусть $\tilde{P}$ обозначает его аналитическое продолжение на целую открытую плоскость. Пусть дальше $\mathfrak{p}_2$ обозначает класс всех полиномов вида (1), а $\mathfrak{p}_3$ — класс всех полиномов вида (1) или (2). Эти классы были введены З. Хажинским в 1958 г.

Получены следующие результаты.

Теорема 1. *Для произвольного w пусть $r = |w|$. При случае* $\mathfrak{p}_2$ *имеет место оценка* (3). *Все экстремальные функции определены по формуле* (7).

Теорема 2. *Для произвольного w пусть $r = |w|$. При случае* $\mathfrak{p}_3$ *имеет место оценка* (10). *Все экстремальные функции определены по формуле* (14).

Аналогичные результаты получены для $\operatorname{Arg}[P(w)/w]$, $|P'(w)|$, а также $\operatorname{Arg}P'(w)$.

Доказательства основаны на следующей лемме: если P есть полиномом вида (2), таким, что нули w_1, w_2 его производной исполняют условие (19), то имеет место (20) или (21).