ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXII/XXIII/XXIV, 18 SECTIO A 1968/1969/1970

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An Extremal Problem for Univalent Functions Associated with the Darboux Formula

Pewien problem ekstremalny dla funkcji jednolistnych

Некоторая экстремальная проблема для однолистных функций

1. Let $f(z)$ be a regular function in a convex domain D and z_1, z_2 two fixed points in D. Then there is a point ζ on the straight line segment $\overline{z_1 z_2}$ and there is a complex number λ, $|\lambda| \leqslant 1$, such that

(1) $$f(z_1)-f(z_2) = \lambda f'(\zeta)(z_1-z_2).$$

This is the well known Darboux formula.

If the function f is univalent in D, then $\lambda \neq 0$.

Let F be a compact class of regular and univalent functions in D. A natural problem which arises is to find the minimum value of $|\lambda|$ for all $f \in F$.

The author proposed this problem in 1966 at the Conference on Analytic Functions in Łódź [1].

2. The aim of the present paper is to give, by an elementary way, a lower estimation of $|\lambda|$ in the class S of functions $f(z) = z+a_2z^2+\ldots$ regular and univalent in the unit disc $D = \{z\colon |z| < 1\}$.

Let $z_1, z_2 \in D$, $|z_1| \leqslant |z_2|$. From (1) we have

(2) $$|\lambda| = \frac{1}{|z_1-z_2|}\left|\frac{f(z_1)-f(z_2)}{f'(\zeta)}\right|$$

where

$$\zeta = (1-t)z_1+tz_2,\ t \in (0, 1).$$

Let us write

$$\frac{f(z_1)-f(z_2)}{f'(\zeta)} = \frac{f(z_1)-f(z_2)}{f'(z_1)}\,\frac{f'(z_1)}{f'(\zeta)}.$$

If we denote

$$g(u) = f\left(\frac{u+z_1}{1+\bar{z}_1 u}\right), \; u \in D,$$

we have $g(0) = f(z_1), g(-z_1) = 0, g(u_0) = f(z_2)$, where

$$u_0 = \frac{z_2 - z_1}{1-\bar{z}_1 z_2}. \tag{3}$$

Further

$$g'(0) = (1-|z_1|^2)f'(z_1), \; g'(u_1) = (1-|z_1|^2)f'(\zeta),$$

where

$$u_1 = \frac{\zeta - z_1}{1-\bar{z}_1\zeta}. \tag{4}$$

Then

$$\frac{f(z_2)-f(z_1)}{f'(z_1)} = (1-|z_1|^2)h(u_0), \; \frac{f'(z_1)}{f'(\zeta)} = \frac{1}{h'(u_1)}$$

where

$$h(u) = \frac{g(u)-g(0)}{g'(0)}, \; u \in D. \tag{5}$$

It is clear that the function h belongs to S.

From (2) we deduce

$$|\lambda| = \frac{1-|z_1|^2}{|z_1-z_2|}\left|\frac{h(u_0)}{h'(u_1)}\right| \tag{6}$$

where u_0 and u_1 are given by (3) and (4).

Using the well known estimates of the moduli of the function and of its first derivative in the class S, we obtain

$$\left|\frac{h(u_0)}{h'(u_1)}\right| \geqslant \frac{|u_0|}{(1+|u_0|)^2}\,\frac{(1-|u_1|)^3}{1+|u_1|}.$$

Since $|u_1| \leqslant |u_0|$, we have

$$\frac{(1-|u_1|)^3}{1+|u_1|} \geqslant \frac{(1-|u_0|)^3}{1+|u_0|}$$

and from (6) we deduce, finally, the estimation

$$|\lambda| > \frac{1-|z_1|^2}{|1-\bar{z}_1 z_2|}\left(\frac{1-|u_0|}{1+|u_0|}\right)^3 \tag{7}$$

where

$$u_0 = \frac{z_2-z_1}{1-\bar{z}_1 z_2}.$$

3. The estimate (7) is not the best possible. In virtue of (6) we remark that the sharp estimate of $|\lambda|$ in the class S could be find if we know the precise bounds of the ratio $f(z_1)/f'(z_2)$ where z_1, z_2 are fixed points in D and f ranges over S.

Such kind of problem was solved by J. Krzyż for the ratio $f(z_1)/f(z_2)$, [2].

4. Let S_R denote the subclass of S consisting of these functions having real coefficients. For f fixed in S_R the ratio

$$\frac{f(a)}{af'(x)} \tag{8}$$

where a, $0 < a < 1$, is fixed, has an extremal value if x, $0 < x < a$, is such that

$$f''(x) = 0. \tag{9}$$

We put the problem to find the sharp estimation of (8) where x is a solution of (9) and f ranges over S_R.

Let f be the extremal function in S_R, and let x be a solution of (9). Consider a variation f_ε of f given by the formula (see [3])

$$f_\varepsilon(\zeta) = f(\zeta) + \varepsilon V(\zeta, z) + o(\varepsilon)$$

where

$$V(\zeta, z) = f(\zeta)P(\zeta, z),\ |\zeta| < 1,\ |z| < 1,$$

$$P(\zeta, z) = 2\,\mathrm{re}[AQ(\zeta, z)],\ A - \text{arbitrary complex number},$$

$$Q(\zeta, z) = \frac{f(\zeta)}{f(\zeta)-f(z)} - \left[\frac{f(z)}{zf'(z)}\right]^2 \left[\frac{\zeta f'(\zeta)}{f(\zeta)} - \frac{z(\zeta^2-1)}{(\zeta-z)(z\zeta-1)} + 1\right].$$

The equation (9), where f is replaced by f_ε, has a solution $x_\varepsilon = x + \varepsilon h + o(\varepsilon)$, where h is real.

The condition of extremality of f is given by

$$\mathrm{re}\left[\frac{V(a, z)}{f(a)} - \frac{V'(x, z)}{f'(x)} - h\frac{f''(x)}{f'(x)}\right] \geqslant 0.$$

Since $f''(x) = 0$ and

$$\frac{V(a, z)}{f(a)} = P(a, z) = 2\,\mathrm{re}[AQ(a, z)]$$

$$\frac{V'(x, z)}{f'(x)} = P(x, z) + \frac{f(x)}{f'(x)}P'(x, z) = 2\,\mathrm{re}\left\{A\left[Q(x, z) + \frac{f(x)}{f'(x)}Q'(x, z)\right]\right\}$$

we obtain the condition

$$\mathrm{re}\left\{A\left[Q(a, z) - Q(x, z) - \frac{f(x)}{f'(x)}Q'(x, z)\right]\right\} \geqslant 0.$$

Since A is arbitrary, we deduce that the extremal function $w = f(z)$ must verify the differential equation

$$(10)\quad \left(\frac{zw'}{w}\right)^2 \frac{w[c^2+(b-2c)w]}{(b-w)(c-w)^2} = z\left[\frac{(1-a^2)\alpha}{(a-z)(1-az)} + \frac{2x^3+(1-4x^2-x^4)z+2x^3z^2}{(x-z)^2(1-xz)^2}\right]$$

where $b = f(a)$, $c = f(x)$, $\alpha = af'(a)/f(a)$.

The equation (10) is of the form

$$(11)\quad \left(\frac{zw'}{w}\right)^2 \frac{w[c^2+(b-2c)w]}{(b-w)(c-w)^2} = z\frac{a_0+a_1z+a_2z^2+a_1z^3+a_0z^4}{(a-z)(1-az)(x-z)^2(1-xz)^2}$$

where

$$a_0 = x^2(1-a^2)\alpha + 2x^3a$$

$$a_1 = a(1-4x^2-x^4) - 2x^3(1+a^2) - 2x(1+x^2)(1-a^2)\alpha.$$

Letting $z \to 0$, we obtain $x^2a = a_0b$.

The polynomial $a_0+a_1z+a_2z^2+a_1z^3+a_0z^4$ has a double root k, where $k = \pm 1$. Suppose $k = 1$. Then

$$\frac{1+a}{1-a}\alpha = \frac{1+2x-x^2}{(1-x)^2}$$

and the equation (11) becomes

$$\left(\frac{zw'}{w}\right)^2 \frac{w[c^2+(b-2c)w]}{(b-w)(c-w)^2} = \frac{a_0z(1-z)^2(1+2lz+z^2)}{(a-z)(1-az)(x-z)^2(1-xz)^2}$$

where

$$l = \frac{a_1+2a_0}{a_0}$$

Making the substitution $w = cu$, we obtain

$$(12)\quad \left(\frac{zu'}{u}\right)^2 \frac{u[1+(p-2)u]}{(p-u)(1-u)^2} = \frac{a_0z(1-z)^2(1+2lz+z^2)}{(a-z)(1-az)(x-z)^2(1-xz)^2}$$

where

$$u(x) = 1,\ u'(x) = f'(x)/f(x),\ p = u(a) = f(a)/f(x).$$

The equation (12) together with the conditions $u(x) = 1$, $u(a) = p$, $u''(x) = 0$, permits a numerical calculation of

$$\frac{f(a)}{af'(x)} = \frac{u(a)}{au'(x)}.$$

REFERENCES

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STRESZCZENIE

Niech S oznacza klasę wszystkich unormowanych i jednolistnych funkcji określonych w kole jednostkowym, a S_R podklasę klasy S o współczynnikach rzeczywistych.

Dla dowolnych $z_1, z_2, |z_1| \leqslant |z_2| < 1$ i $f \in S$ mamy

$$f(z_1) - f(z_2) = \lambda f'(\zeta)(z_1 - z_2)$$

gdzie ζ leży na odcinku $\overline{z_1 z_2}$ i $|\lambda| < 1$. W pracy tej znaleziono minimum wartości $|\lambda|$ dla wszystkich $f \in S_R$.

РЕЗЮМЕ

Пусть S обозначает класс всех нормированных однолистных функций, определенных в единичном круге, а S_R — подкласс класса S, включающий функции с действительными коэффициентами. Для любых $z_1, z_2, |z_1| \leqslant |z_2| < 1$ и $f \in S$ есть

$$f(z_1) - f(z_2) = \lambda f'(\zeta)(z_1 - z_2)$$

где ζ лежит на отрезке $\overline{z_1 z_2}$ и $|\lambda| < 1$.

В работе представлена проблема нахождения минимума значения $|\lambda|$ для всех $f \in S_R$.