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Instytut Matematyki, Politechnika Łódzka, Łódź

KRYSTYNA DOBROWOLSKA

On some Extremal Problems in the Class of Quasi-Starlike Functions

Pewne problemy ekstremalne w klasie funkcji quasi-gwiaździstych

Некоторые экстремальные проблемы для класса квази-звездообразных функций

I. Dziubiński [2] introduced the class $\mathcal{G}^M$ of the so called quasi-starlike functions $g(z)$ determined by the equation

$$G(g(z)) = \frac{1}{M} G(z) \quad \text{in } |z| < 1, \tag{1}$$

where M fixed, $M > 1$, and $G(z)$ is starlike in the unit disc. He also introduced the subclasses $\mathcal{G}_m^M$ of the class $\mathcal{G}^M$, where $G(z)$ is of the form

$$G(z) = \frac{z}{\prod\limits_{k=1}^{m} (1-\sigma_k z)^{\beta_k}} \quad \text{in } |z| < 1, \tag{2}$$

$$\sigma_k = e^{i\varphi_k}, \tag{3}$$

$$\sum_{k=1}^{m} \beta_k = 2, \tag{4}$$

and $\varphi_k (k = 1, \ldots, m)$ run over all real numbers, $\sigma_l \neq \sigma_n$ for $l \neq n$, $\beta_k (k = 1, \ldots, m)$ run over all positive numbers.

Let

$$g(z) = a_0 + a_1(z-z_0) + \ldots, \tag{5}$$

$$G(z) = A_0 + A_1(z-z_0) + \ldots, \tag{6}$$

where z_0 denotes a fixed point of the unit disc.

Suppose that

$$H = H(X_0, \ldots, X_N, Y_0, \ldots, Y_N) \tag{7}$$

is a real-valued function of $2N+2$ real variables, defined in an open and sufficiently large set V, $H \in C^1$, $\operatorname{grad} H \neq 0$ at every point of V.

Given a function $g(z)$, let

$$H_g = H(\xi_0, \ldots, \xi_N, \eta_0, \ldots, \eta_N), \tag{7'}$$

where

$$\xi_n + i\eta_n = a_n, \quad n = 0, 1, \ldots, N.$$

We shall prove the following theorems for the class $\mathscr{G}^M$:

Theorem 1. *If the functional H_g attains its extremal value for a function $g(z)$ of the class $\mathscr{G}_m^M$, then this function satisfies the following differential equation*

$$\frac{g'(z)}{g(z)} \frac{\mathscr{L}(g(z))}{\mathscr{R}(g(z))} = \frac{1}{z} \frac{\mathscr{L}(z)}{\mathscr{R}(z)}, \tag{8}$$

where

$$\mathscr{L}(\zeta) = \left(\sum_{k=0}^{N+1} \frac{b_k}{(\zeta - a_0)^k} - \frac{B_k}{(\zeta - z_0)^k} + \frac{\bar{b}_k}{\left(\frac{1}{\zeta} - \bar{a}_0\right)^k} + \frac{\bar{B}_k}{\left(\frac{1}{\zeta} - \bar{z}_0\right)^k} \right) + \mu + \bar{\mu}, \tag{9}$$

$$\mathscr{R}(\zeta) = \sum_{k=1}^{N+1} \left(\frac{d_k \zeta}{(\zeta - a_0)^k} - \frac{D_k \zeta}{(\zeta - z_0)^k} - \frac{\bar{d}_k \frac{1}{\zeta}}{\left(\frac{1}{\zeta} - \bar{a}_0\right)^k} + \frac{\bar{D}_k \frac{1}{\zeta}}{\left(\frac{1}{\zeta} - \bar{z}_0\right)^k} \right), \tag{10}$$

$$B_l = \sum_{k=l-1}^{N} H_k\big((k-l+2) a_{k-l+2} z_0^2 + 2(k-l+1) a_{k-l+1} z_0 + (k-l) a_{k-l}\big), \quad l = 0, 1, \ldots, N+1, \tag{11}$$

$$b_l = \sum_{k=l-1}^{N} H_k (a_k^{(l-1)} a_0^2 + 2a_k^{(l)} a_0 + a_k^{(l+1)}), \quad l = 0, 1, \ldots, N+1, \tag{12}$$

$$D_l = \sum_{k=l-1}^{N} H_k \sum_{n=1}^{k-l+2} n a_n c_{k-l-n+2}, \quad l = 0, 1, \ldots, N+1, \tag{13}$$

$$d_l = \sum_{k=l-1}^{N} H_k \sum_{n=1}^{k-l+2} n a_n \sum_{i=l-1}^{k-n+1} a_i^{(l-1)} c_{k-n-i+1}, \quad l = 0, \ldots, N+1, \tag{14}$$

$$\mu = (D_1 + D_0 z_0) \frac{z_0}{c_0} - (d_1 + d_0 z_0) \frac{a_0}{c_0 a_1}, \tag{15}$$

$$S_0 = \log \frac{G(z_0)}{z_0}, \quad S_l = \frac{1}{l} \sum_{i=1}^{m} \frac{\beta_i \sigma^l}{(1 - \sigma_i z_0)^l}, \quad l = 1, 2, \ldots, \tag{16}$$

(17) $$s_0 = \log\frac{G(a_0)}{a_0},\ s_l = \frac{1}{l}\sum_{t=1}^{m}\frac{\beta_t\sigma^l}{(1-\sigma_t a_0)^l},\quad l = 1, 2, \ldots,$$

(18) $$H_l = H'_{X_l}(X_0, \ldots, Y_N) - iH'_{Y_l}(X_0, \ldots, Y_N),\ l = 0, 1, \ldots, N,\ H^{-1} = 0,$$

(19) $$c_l = (1+S_l z_0)^{-l-1}\times$$

$$\times\begin{vmatrix} 1+S_1 z_0 & 0 & \ldots & 0 & z_0 \\ S_1+2S_2 z_0 & 1+S_1 z_0 & \ldots & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (l-1)S_{l-1}+lS_l z_0 & (l-2)S_{l-2}+(l-1)S_{l-1}z_0 & \ldots & 1+S_1 z_0 & 0 \\ lS_l+(l+1)S_{l+1}z_0 & (l-1)S_{l-1}+lS_l z_0 & \ldots & S_1+2S_2 z_0 & 0 \end{vmatrix},\ l = 0, \ldots, N+1$$

Moreover, the numbers $\bar{\sigma}_t$, $t = 1, \ldots, m$ which appear in (2) *and* (3) *are roots of $\mathscr{R}(\zeta)$ and double roots of function of the form*

(20) $$\tilde{\mathscr{R}}(\zeta) = \sum_{k=2}^{N+1}\frac{1}{k-1}\left(\frac{d_k}{(\zeta-a_0)^{k-1}} - \frac{D_k}{(\zeta-z_0)^{k-1}} + \frac{\bar{d}_k}{\left(\frac{1}{\zeta}-\bar{a}_0\right)^{k-1}} + \right.$$

$$\left. - \frac{\bar{D}_k}{\left(\frac{1}{\zeta}-\bar{z}_0\right)^{k-1}}\right) + d_1\log\frac{1-z_0/\zeta}{1-\frac{a_0}{\zeta}} + \bar{d}_1\log\frac{1-\bar{z}_0\zeta}{1-\bar{a}_0\zeta} + \eta + \bar{\eta},$$

where

(21) $$\eta = -\tfrac{1}{2}\sum_{k=1}^{N+1}(d_k s_{k-1} - D_k S_{k-1}).$$

Proof. Since the family $\mathscr{G}^M$ is compact and because of the continuity of the function (7) the existence of extremal function is evident. We suppose that the extremal function belongs to the class $\mathscr{G}^M_m$.

From relations (2), (3) and (6) it follows that the coefficients A_k, $k = 0, 1, \ldots$ are functions of the parametres $\varphi_1, \ldots, \varphi_m, \beta_1, \ldots, \beta_m$; simultaneously from (1) it follows that between the coefficients of the functions (5) and (6) the relations

22) $$\left\{\frac{\partial^k}{\partial z^k}\left(G(g(z)) - \frac{1}{M}G(z)\right)\right\}_{z=z_0} = 0$$

hold. Consequently, the value of the functional H_g for any arbitrary function (5) coincides with the value of the function (7') at the point $P = (a_0, a_1, \ldots, a_N, \varphi_1, \varphi_2, \ldots, \varphi_m, \beta_1, \beta_1, \ldots, \beta_m)$ of an $2N+2m+2$ dimensional space, provided the additional conditions (4) and (22) are satisfied for $k = 0, 1, \ldots, N$.

Applying the method of Lagrange's multipliers for the functions of complex variables [1], we conclude that for the point $P = (a_0, \ldots, a_N, \varphi_1, \ldots, \varphi_m, \beta_1, \ldots, \beta_m)$ connected by (2) and (22)

$$(23)\qquad \sum_{k=0}^{N} \lambda_k \frac{\partial}{\partial a_s}\left(\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big) - \frac{1}{M}G(z)\Big)\right)_{z=z_0} = \frac{\partial H}{\partial a_s} + \overline{\left(\frac{\partial H}{\partial \bar{a}_s}\right)},$$

$$s = 0, 1, \ldots, N,$$

$$(24)\qquad \sum_{k=0}^{N}\left\{\lambda_k \frac{\partial}{\partial \varphi_t}\left(\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big) - \frac{1}{M}G(z)\Big)\right)_{z=z_0} + \overline{\lambda}_k \frac{\partial}{\partial \varphi_t}\overline{\left(\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big) - \frac{1}{M}G(z)\Big)\right)_{z=z_0}}\right\} = 0,$$

$$t = 1, \ldots, m,$$

$$(25)\qquad \sum_{k=0}^{N}\left\{\lambda_k \frac{\partial}{\partial \beta_t}\left(\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big) - \frac{1}{M}G(z)\Big)\right)_{z=z_0} + \overline{\lambda}_k \frac{\partial}{\partial \beta_t}\overline{\left(\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big) - \frac{1}{M}G(z)\Big)\right)_{z=z_0}}\right\} + \lambda = 0,$$

$$t = 1, \ldots, m,$$

where $\lambda, \lambda_0, \lambda_1, \ldots, \lambda_N$ are a nontrivial set of Lagrange's multipliers, are fulfilled.

By suitable transformations the equations (23)-(25) can be written in the form

$$(26)\qquad K\Big(G'\big(g(z)\big)(z-z_0)\Big)^s = H_s, \quad s = 0, 1, \ldots, N,$$

$$(27)\qquad K\Big(G'\big(g(z)\big)f_t^{(1)}(z)\Big) - \hat{K}\big(\hat{G}'(\hat{g}(z))\hat{f}_t^{(1)}(z)\big) = 0, \quad t = 1, \ldots, m,$$

$$(28)\qquad K\Big(G'\big(g(z)\big)f_t^{(2)}(z)\Big) + \hat{K}\Big(\hat{G}'\big(\hat{g}(z)\big)\hat{f}_t^{(2)}(z)\Big) + \lambda = 0, \quad t = 1, \ldots, m,$$

where

$$K\big(h(z)\big) = \sum_{k=0}^{N} \lambda_k \left(\frac{\partial^k h}{\partial z^k}\right)_{z=z_0},$$

$$\hat{K}\big(h(z)\big) = \sum_{k=0}^{N} \overline{\lambda}_k \left(\frac{\partial^k h}{\partial z^k}\right)_{z=\bar{z}_0},$$

$$\hat{h}(z) = \overline{h(\bar{z})},$$

$$f_t^{(1)}(z) = \left(\frac{\sigma_t g(z)}{1-\sigma_t g(z)} - \frac{\sigma_t z}{1-\sigma_t z}\right)\frac{G\big(g(z)\big)}{G'\big(g(z)\big)}, \quad t = 1, \ldots, m,$$

$$f_t^{(2)}(z) = \Big(\big(\log(1-\sigma_t z)\big) - \log\big((1-\sigma_t g(z)\big)\Big)\frac{G\big(g(z)\big)}{G'\big(g(z)\big)}, \quad t = 1, \ldots, m.$$

Next, from (26) and the definition of the operator $K(h)$ it follows that for any function $h_0(z)$ of the form

$$h_0(z) = E_0 + E_1(z-z_0) + \ldots$$

we have

$$(29)\qquad K\big(G'(g(z))h_0(z)\big) = \sum_{k=0}^{N} E_k H_k = \frac{1}{2\pi i}\int_{C(z_0,r)} H^*\left(\frac{1}{z-z_0}\right) h_0(z)\frac{dz}{z-z_0},$$

where

$$H^*\left(\frac{1}{z-z_0}\right) = \sum_{k=0}^{N} \frac{H_k}{(z-z_0)^k}$$

and $C(z_0, r)$ is a sufficiently small circumference with centre at the point z_0.

Using (29) the set of equations (26)-(28) can be written in the form

$$(30)\qquad \frac{1}{2\pi i}\int_{C(z_0,r)} H^*\left(\frac{1}{z-z_0}\right)\left(\frac{\sigma_t g(z)}{1-\sigma_t g(z)} - \frac{\sigma_t z}{1-\sigma_t z}\right)\frac{G(g(z))}{G'(g(z))}\frac{dz}{z-z_0} +$$

$$-\frac{1}{2\pi i}\int_{C(\bar{z}_0,r)} \hat{H}^*\left(\frac{1}{z-\bar{z}_0}\right)\left(\frac{\bar{\sigma}_t \hat{g}(z)}{1-\bar{\sigma}_t \hat{g}(z)} - \frac{\bar{\sigma}_t z}{1-\bar{\sigma}_t z}\right)\frac{\hat{G}(\hat{g}(z))}{\hat{G}'(\hat{g}(z))}\frac{dz}{z-\bar{z}_0} = 0,$$

$$t = 1, \ldots, m,$$

$$(31)\qquad \frac{1}{2\pi i}\int_{C(z_0,r)} H^*\left(\frac{1}{z-z_0}\right)\big(\log(1-\sigma_t z) - \log(1-\sigma_t g(z)\big)\times$$

$$\times\frac{G(g(z))}{G'(g(z))}\frac{dz}{z-z_0} + \frac{1}{2\pi i}\int_{C(\bar{z}_0,r)} \hat{H}^*\left(\frac{1}{z-\bar{z}_0}\right)\big(\log(1-\bar{\sigma}_t z) +$$

$$-\log(1-\bar{\sigma}_t \hat{g}(z))\big)\frac{\hat{G}(\hat{g}(z))}{\hat{G}'(\hat{g}(z))}\frac{dz}{z-\bar{z}_0} + \lambda = 0, \quad t = 1, \ldots, m.$$

Further, let us introduce the functions

$$(32)\qquad \mathscr{R}(\zeta) = \frac{1}{2\pi i}\int_{C(z_0,r)} H^*\left(\frac{1}{z-z_0}\right)\left(\frac{\frac{1}{\zeta}g(z)}{1-\frac{1}{\zeta}g(z)} - \frac{\frac{1}{\zeta}z}{1-\frac{1}{\zeta}z}\right)\frac{G(g(z))}{G'(g(z))}\times$$

$$\times\frac{dz}{z-z_0} + \frac{1}{2\pi i}\int_{C(\bar{z}_0,r)} \hat{H}^*\left(\frac{1}{z-\bar{z}_0}\right)\left(\frac{\zeta\hat{g}(z)}{1-\zeta\hat{g}(z)} - \frac{\zeta z}{1-\zeta z}\right)\frac{\hat{G}(\hat{g}(z))}{\hat{G}'(\hat{g}(z))}\frac{dz}{z-\bar{z}_0}$$

and

$$(33)\qquad \mathscr{L}(\zeta) = \mathscr{R}(\zeta)\frac{G'(\zeta)\zeta}{G(\zeta)}.$$

By applying equation (1) and by the definition of the function $\mathscr{L}(\zeta)$ we get equation (8). Moreover, from (30) and (32) it follows, that the numbers $\bar{\sigma}_t$, $t = 1, 2, \ldots, m$, are roots of the function $\mathscr{R}(\zeta)$, and by (31) double roots of a function

$$(34)\quad \tilde{\mathscr{R}}(\zeta) = \frac{1}{2\pi i}\int\limits_{C(z_0,r)} H^*\left(\frac{1}{z-z_0}\right)\left(\log\left(1-\frac{1}{\zeta}z\right)-\log\left(1-\frac{1}{\zeta}(z)\right)\right)\times$$
$$\times\frac{G(g(z))}{G'(g(z))}\frac{dz}{z-z_0}+\frac{1}{2\pi i}\int\limits_{C(\bar{z}_0,r)}\hat{H}^*\left(\frac{1}{z-\bar{z}_0}\right)\left(\log(1-\zeta z)-\log(1-\zeta\hat{g}(z)\right)\times$$
$$\times\frac{\hat{G}(\hat{g}(z))}{\hat{G}'(\hat{g}(z))}\frac{dz}{z-\bar{z}_0}+\lambda.$$

The functions $\mathscr{R}(\zeta)$ and $\mathscr{L}(\zeta)$ of (32) and (33) are rational functions with the poles only at the points $z_0, 1/\bar{z}_0, g(z_0), 1/g(z_0)$. Further, we prove the formulae (9) and (10) for the functions $\mathscr{R}(\zeta)$ and $\mathscr{L}(\zeta)$ and next formulae (11)-(19) for the coefficients of these functions.

From definitions of the functions $\mathscr{R}(\zeta)$ and $\tilde{\mathscr{R}}(\zeta)$ we can obtain the following formula

$$(35)\qquad \tilde{\mathscr{R}}'(\zeta) = -\frac{1}{\zeta}\mathscr{R}(\zeta)$$

and, further, formula (20).

Theorem 2. *The extremal value of the functional H_g is attained in the class $\mathscr{G}^M$ for function belonging to the class $\mathscr{G}^M_m$, $m \leqslant 2N+1$.*

Proof. Let H^* denote e.g. the maximum of the functional H_g in $\mathscr{G}^M$ and let H^*_k denote an analogous maximum in $\bigcup_{m=1}^{k}\mathscr{G}^M_m$, $k = 1, 2, \ldots$.

1°. First we shall prove that $H^*_k = H^*_{2N+1}$ for $k \geqslant 2N+1$ and consequently

$$(36)\qquad \sup_n H^*_n = H^*_{2N+1}.$$

In fact, because the family $\bigcup_{m=1}^{k}\mathscr{G}^M_m$, $k = 1, 2, \ldots$ is compact, we can find a function of the class $\mathscr{G}^M_l$, $l \leqslant k$, realizing the mentioned maximum. Hence, and from Theorem 1 it follows that there exist such numbers $\bar{\sigma}_t = e^{-i\varphi_t}$, $t = 1, \ldots, l$, $\sigma_k \neq \sigma_j$ for $k \neq j$, which are double roots of the function $\tilde{\mathscr{R}}(\zeta)$. From properties of the function $\tilde{\mathscr{R}}(\zeta)$ it follows that

$$P(\varphi) = \tilde{\mathscr{R}}(e^{i\varphi})$$

is a real-valued function of a real variable and $P(\varphi) \in C^1$. Moreover, it is easy to see that the function $P(\varphi)$ has the period 2π.

Next from (35) we obtain

$$P'(\varphi) = -i\mathscr{R}(e^{i\varphi}).$$

Because the function $\mathscr{R}(\zeta)$ of (32) has $4N+3$ roots we take the inequality $l \leqslant 2N+1$ and, consequently, (36).

2°. In the second part, we prove that

$$H^* \leqslant H^*_{2N+1}. \tag{37}$$

In fact, it is not difficult to observe that we can approximate any quasi-starlike function by the functions of classes $\mathscr{G}^M_m$. Hence, and from (36) it follows that the inequality (37) is true.

Using Theorem 1 and 2 for the functional

$$H_g = g(z_0)\overline{g(z_0)}$$

or

$$H_g = \arg\frac{g(z_0)}{z_0}$$

we obtain sharp estimates of $|g(z_0)|$ and $\arg\frac{g(z_0)}{z_0}$, $|z_0| < 1$, in the class $\mathscr{G}^M$:

Theorem 3. *For every function $g(z)$ of the class $\mathscr{G}^M$ the inequalities*

$$-g^*(-|z|) \leqslant |g(z)| \leqslant g^*(|z|)$$

are satisfied, where $g^*(z)$ is determined by equation

$$\frac{g^*(z)}{(1-g^*(z))^2} = \frac{1}{M}\frac{z}{(1-z)^2}.$$

Theorem 4. *The functional*

$$H_g = \arg\frac{g(z_0)}{z_0}, \quad |z_0| < 1$$

attains its extremal values in the class $\mathscr{G}^M$ for functions $g(z)$ determined by the equation

$$\frac{g(z)}{(1-\sigma g(z))^2} = \frac{1}{M}\frac{z}{(1-\sigma z)^2}$$

in which σ are the suitably chosen roots of the equation

$$\sqrt{1-4\left(1-\frac{1}{M}\right)\frac{\sigma r}{(1+\sigma r)^2}} + \sqrt{1-4\left(1-\frac{1}{M}\right)\frac{\sigma r}{(\sigma+r)^2}}$$
$$= 2\sqrt{1-4\left(1-\frac{1}{M}\right)\frac{\sigma r}{(1+\sigma r)^2}}\sqrt{1-4\left(1-\frac{1}{M}\right)\frac{\sigma r}{(\sigma+r)^2}}.$$

Moreover, from Theorem 3 we have following

Theorem 3'. *Every quasi-starlike function maps the unit disc onto a set containing the disc* $|z| < R_0$, $R_0 = 2M-1-2\sqrt{M(M-1)}$, *only. Moreover, the function* $g(z)$ *determined by the equation*

$$\frac{g(z)}{(1+g(z))^2} = \frac{z}{M(1+z)^2}$$

maps the unit disc onto the disc without the segment $[R_0, 1]$ *of the real axis.*

REFERENCES

[1] **Charzyński, Z.**, *Sur les fonctions univalentes algébriques bornées*, Rozp. Matem. X (1955).

[2] **Dziubiński, I.**, *Quasi-starlike functions*, Ann. Polon. Math. (to appear)

STRESZCZENIE

W pracy tej zajmuję się klasą $\mathcal{G}^M$ funkcji quasi-gwiaździstych wprowadzoną przez I. Dziubińskiego. W szczególności wykazuję, że funkcjonał rzeczywisty H_g o różnym od zera gradiencie osiąga extremum dla pewnych specjalnych funkcji określonych równaniem (8). (Twierdzenie 1 i 2). Ponadto znaleziono oszacowanie wyrażeń $|g(z)|$ oraz $\arg g(z)/z$ w klasie $\mathcal{G}^M$.

РЕЗЮМЕ

В работе автор занимается классом $\mathcal{G}^M$ нормированных квазизвездообразных функций, введенных И. Дзюбиньским. Доказаны следующие теоремы: действительный функционал H_g, для которого $\operatorname{grad} Hg \neq 0$ принимает экстремальное значение для специальных функций определённых уравнением (8). (теорема 1, 2). Найдены также оценки $|g(z)|$, $\arg[g(z)/z]$ для класса $\mathcal{G}^M$.