ANNALES
UNIVERSITATIS MARIAE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XX, 1 SECTIO A 1966

Z Katedry Funkcji Analitycznych Wydz. Mat. Fiz. Chem. UMCS
Kierownik: prof. dr Jan Krzyż

GILBERT LABELLE

On the Theorems of Gauss-Lucas and Grace

O twierdzeniach Gaussa-Lucasa i Grace

О теоремах Гаусса-Льукаса и Греса

Introduction

The theorems of Gauss-Lucas and of Grace have proven themselves to be of fundamental importance in the theory of the zeros of polynomials. We study here some extensions of these results together with a variety of their consequences.

For simplicity of reading we have gathered in section O relevant definitions and notations which will be used in the text. Sections 1 and 2 contain results of the Gauss-Lucas type and Grace's type respectively. Section 3 deals with applications. Section 4 consists in the proofs of all the theorems.

0. Definitions and notations

Let P denote the set of all complex polynomials and for $n \geqslant 0$, P_n denote the set of all complex polynomials of degree n. If $p \in P_n$ and $p(z) = a_n(z-z_1)\dots(z-z_n)$ we denote by $z(p)$ the set $\{z_1, \dots, z_n\}$ of its zeros. If $a_n = 1$, p is said to be monic. For $p, q \in P_n$ where

$$p(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu \quad \text{and} \quad q(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$$

we call the expression

$$\{p, q\} = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} a_\nu b_{n-\nu}$$

the apolarity expression for p and q. In the situation $\{p, q\} = 0$ we say that p and q are apolar while if $|\{p, q\}| = |a_n||b_n|\,\delta^n$ we shall say that p and q are δ-apolar. For $\zeta \in \mathbb{C}$ (the complex plane) and $p \in P_n$, $\mathscr{L}_\zeta p$ defined by $(\mathscr{L}_\zeta p)(z) = np(z) - (z-\zeta)p'(z)$ is the polar derivative of p with respect to the pole ζ. A point $\lambda \in \mathbb{C}$ will be called a $(w_0, w_1, \ldots, w_k)$ —point of a regular function f if $f(\lambda) = w_0, f'(\lambda) = w_1, \ldots, f^{(k)}(\lambda) = w_k$. For $g: A \to \mathbb{C}$ (where $A \subseteq \mathbb{C}$ and $w_0, w_1, \ldots, w_k \in A$ given, we call the quantity $[w_0, w_1, \ldots, w_k]_g$, which is defined recursively by $[w_0]_g = g(w_0)$, $\ldots$, $[w_k]_g = g(w_k)$, $\ldots$, $([w_0, \ldots, w_{k-1}]_g - [w_1, \ldots, w_k]_g)/(w_0 - w_k) = [w_0, \ldots, w_k]_g$ the (usual) difference quotient of order k of g with respect to the points $w_0, w_1, \ldots, w_k$. If g is regular, the difference quotients always have a meaning even when two of the w_i's are equal (using limiting processes).

For a set $S \subseteq \mathbb{C}$, the convex hull of S is denoted as usual by $\operatorname{Conv}(S)$ and if θ is an angle, we write S_θ for the set $S + L_\theta$ where L_θ is the half-line $\{re^{i\theta} | r \geqslant 0\}$, that is, S_θ is that part of $\mathbb{C}$ swept by S when the latter is carried to ∞ along a direction making an angle θ with respect to the positive real axis. A set $C \subseteq \mathbb{C}$ is called a circular region, if it consists of a disk, the exterior of a disk or a half-plane (open or closed). Two sets $S_1, S_2 \subseteq \mathbb{C}$ are said to be separated by two circular regions C_1, C_2 if $S_i \subseteq C_i$, $i = 1, 2$ and $C_1 \cap C_2 = \varphi$. The "distance" $d(S_1, S_2)$ between the two sets S_1 and S_2 is defined, as usual, by $d(S_1, S_2) = \inf\limits_{z_i \in S_i} |z_1 - z_2|$. Note that this is not a distance in the mathematical sense of the term.

1. On the theorem of Gauss-Lucas

This well known theorem [5] states that

$$p \in P \Rightarrow z(p') \subseteq \operatorname{Conv}[z(p)]. \tag{1.1}$$

The result locates the zeros of p' in terms of the convex hull of the set of the zeros of p. Using the $(w_0, w_1, \ldots, w_k)$ — points of p, instead of the zeros of p' (i.e., $(w_0, 0, w_2, \ldots, w_k)$ — points of p) we state the following generalization of (1.1).

Theorem 1.1 *Let $p \in P_n$, $K = \operatorname{Conv}[z(p)]$ and take $w_0, w_1, \ldots, w_k \in \mathbb{C}$, where $0 < k \leqslant n$. Then the set Ω of all the $(w_0, w_1, \ldots, w_k)$ — points of p satisfies*

$$\Omega \subseteq \bigcap_{\nu=0}^{k-1} K_{\theta_\nu}, \ \theta_\nu = (\arg w_\nu - \arg w_{\nu+1}) (\operatorname{mod} 2\pi) \tag{1.2}$$

in the case $w_0 w_1 \ldots w_k \neq 0$ and $\Omega \subseteq K$ in the case $w_0 w_1 \ldots w_k = 0$.

The theorem of Laguerre [5] states that for any polynomial $p \in P_n$, we have

$$[\zeta \notin C, z(p) \subseteq C] \Rightarrow z(\mathscr{L}_\zeta p) \subseteq C \tag{1.3}$$

where C is a circular region. Now, if we note that as $\zeta \to \infty$, the set $z(\mathscr{L}_\zeta p)$ tends to the set $z(p')$ (this is easily seen by looking at the zeros of

$$\left(\frac{1}{\zeta}\,\mathscr{L}_\zeta p\right)(z) = \frac{n}{\zeta}\,p(z) - \left(\frac{z}{\zeta} - 1\right)p'(z)$$

which tends uniformly on every compact to $p'(z)$) we have the qualitative result that as ζ becomes large, the zeros of $\mathscr{L}_\zeta p$ come near to $\mathrm{Conv}[z(p)]$. In this connection we state the quantitative

Theorem 1.2. *Let* $p \in P_n$ *with* $z(p) = \{z_1, \ldots, z_n\}$ *and* $M = \max_\nu |z_\nu|$ *then for every* $\lambda \in z(\mathscr{L}_\zeta p)$ *we have*

$$|\zeta| > M \cdot \left(1 + \frac{2M}{\delta}\right) \Rightarrow d\big(\lambda, \mathrm{Conv}[z(p)]\big) < \delta. \tag{1.4}$$

Note that when $\delta \downarrow 0$ we come back to (1.1). We locate now the set of zeros of linear combinations of the derivatives of p.

Theorem 1.3. *Let* $p \in P_n$ *and* $p^*(z) = \sum_{\nu=0}^{n} a_\nu p^{(\nu)}(z)$ *then*

$$z(p^*) \subseteq \bigcap \left\{ \big(z(p) + C\big) \,\big|\, C \supseteq z(\pi) \right\} \tag{1.5}$$

where $\pi(z) = \sum_{\nu=0}^{n} n^{(n-\nu)} a_{n-\nu} z^\nu$ *and C ranges over the circular regions containing* $z(\pi)$. *The symbol* $n^{(k)}$ *denotes, as usual, the product* $n(n-1)\ldots(n-k+1)$.

In the case $a_0 = 0$, $a_1 = 1$, $a_2 = \ldots = a_n = 0$ we come back to 1.1 as easily seen since $z(\pi) = \{0, \infty\}$ in this case because $\pi(z) = n!z^{n-1} + 0z^n$.

We close this section by giving an extension of (1.1) for Weierstrass' canonical products.

Theorem 1.4. *Let*

$$P(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu}\right) \exp\left(\frac{z}{z_\nu} + \frac{1}{2}\left(\frac{z}{z_\nu}\right)^2 + \ldots + \frac{1}{p}\left(\frac{z}{z_\nu}\right)^p\right) \tag{1.6}$$

be a canonical product of genus p. Then the zeros ζ of P' satisfy

i) $\zeta = 0$

or

ii) $\zeta \cdot \sum_{\nu=1}^{\infty} a_\nu z_\nu^p = \sum_{\nu=1}^{\infty} a_\nu z_\nu^{p+1}$ *for a non-trivial sequence of non-negative numbers* a_ν.

2. On the theorem of Grace

This result [5] reads as follows: given $p, q \in P_n$ then

$$\{p, q\} = 0 \Rightarrow [z(p) \subseteq C_1, z(q) \subseteq C_2 \Rightarrow C_1 \cap C_2 \neq \varphi] \tag{2.1}$$

where C_1 and C_2 are circular regions. That is to say, the sets $z(p)$ and $z(q)$ cannot be separated by two circular regions when p and q are apolar.

We give first a representation theorem for the apolarity condition which will be useful for this section.

Theorem 2.1. *The following representations for $\{p, q\}$ are valid*

a) If $p(z) = \sum_{\nu=1}^{M} a_\nu (z-\sigma_\nu)^n$ *and* $q(z) = \sum_{\mu=1}^{N} \beta_\mu (z-\eta_\mu)^n$ *then*

$$\{p, q\} = \sum_{\nu,\mu} a_\nu \beta_\mu (\eta_\mu - \sigma_\nu)^n \tag{2.2}$$

b) Let $p(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $q(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$ *then*

$$\{p, q\} = b_n L_1 \dots L_{\zeta_n} [p(z)] \tag{2.3}$$

where $L_{\zeta_\nu} = \frac{1}{\nu} \mathscr{L}_{\zeta_\nu}$ *and the* ζ_ν *are the zeros of* q.

c) Let $z_1, \dots, z_n$ *and* $\zeta_1, \dots, \zeta_n$ *be the zeros of* p *and* q *respectively and* C_1, C_2 *be two disjoint circular regions such that* $\{\zeta_1, \dots, \zeta_n\} \subseteq C_1$, $\{z_1, \dots, z_n\} \subseteq C_2$. *Moreover suppose that* C_1 *is a disc with center* ω. *Then* $\exists\ \lambda_1, \dots, \lambda_n \in C_2$ *such that*

$$\{p, q\} = a_n b_n \left(\frac{\omega - z_1}{\omega - \lambda_1}\right) \dots \left(\frac{\omega - z_n}{\omega - \lambda_n}\right) (\zeta_1 - \lambda_1) \dots (\zeta_n - \lambda_n) \tag{2.4}$$

d) If $p(z) = a_n (z-z_1) \dots (z-z_n)$, $q(z) = b_n (z-\zeta_1) \dots (z-\zeta_n)$ *then*

$$\{p, q\} = \frac{a_n b_n}{n!} \sum_{\varphi \in S_n} \prod_{\nu=1}^{n} (\zeta_\nu - z_{\varphi(\nu)}) \tag{2.5}$$

where S_n *is the symmetric group of order* n.

Observe that representation (2.3) immediately imply (2.1) if one takes (1.3) into account.

We will use parts c) and d) of theorem 2.1 to prove

Theorem 2.2 *If* p *and* q *are* δ*-apolar then*

i) The sets of their zeros are not too far from each other. More precisely, $z(p)$ *and* $z(q)$ *cannot be separated by two circular regions* C_1 *and* C_2 *such that*

$$d(C_1, C_2) > \delta. \tag{2.6}$$

The result is best possible.

ii) Their respective zeros are not uniformly too near. More precisely, $\exists z_\nu \in z(p)$, $\exists \zeta_\mu \in z(q)$ such that $d(z_\nu, \zeta_\mu) \geqslant \delta$. (2.7)
The result is best possible.

This result thus extends (2.1) in two directions in the context of δ-apolar polynomials.

Grace, Heawood and Szegö [6] have proved the following generalization of Rolle's theorem.

Theorem *Let $p \in P_n$ be such that $p(-1) = p(1)$ then p' possesses a zero in every circle passing through $\pm i \cot \pi/n$ and in the two half-planes $\operatorname{Re} z \geqslant 0$ and $\operatorname{Re} z \leqslant 0$.*

Using the notation of difference-quotients, the hypothesis $p(-1) = p(1)$ can be written $[-1, 1]_p = 0$ and this theorem gives a conclusion on $z(p')$. Using general difference-quotients of order k we draw now conclusions on $z(p^{(k)})$ without even assuming that the difference-quotients vanish.

Theorem 2.3. *Let $w_0, w_1, \dots, w_k \in \mathbb{C}$ and $p \in P_n$ be monic. Then*

$$d\big(z(p^{(k)}), C\big) \leqslant \left| \frac{[\omega_0, \dots, \omega_k]_p}{\binom{n}{k}} \right|^{\frac{1}{n-k}} \tag{2.8}$$

for every circular region C containing all the zeros of the polynomial (of degree $n-k$)

$$\sum_{\nu=0}^{k} \frac{(\omega_\nu - z)^n}{(\omega_\nu - \omega_0)(\omega_\nu - \omega_1)\dots(\omega_\nu - \omega_{\nu-1})(\omega_\nu - \omega_{\nu+1})\dots(\omega_\nu - \omega_k)} \tag{2.9}$$

This concludes section 2.

3. Applications

1- We start with a few simple consequences of theorems 1.1 and 1.4.

Theorem 3.1 *If $p \in P_n$ where $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is such that two of its successive coefficients a_{ν_0}, a_{ν_0+1} satisfy*

$$|\arg a_{\nu_0} - \arg a_{\nu_0+1}| \leqslant \pi/2 \tag{3.1}$$

then the open right half-plane cannot contain the whole set $z(p)$.

Condition (3.1) can be thought as a condition stating that the arguments of the coefficients of p do not oscillate too much.

Theorem 3.2 *If, in theorem* 1.1, *we have* $\theta_{\nu_0}+\pi=\theta_{\nu_1}$ *for two indices* ν_0 *and* ν_1 *then* $\Omega\subseteq\mathrm{Conv}[z(p)]$.

This theorem says that for a much wider class of points than $z(p')$, a conclusion of type (1.1) still holds.

Theorem 3.3 *If, in theorem* 1.4, *we have* $p=0$ *then conclusion ii) can be written as*

$$ii)^{*}\qquad \zeta\in\overline{\mathrm{Conv}\{z_1, z_2, \ldots\}}.$$

This is the known extension of (1.1) given in [5].

Theorem 3.4 *If, in theorem* 1.4, *all the* z_ν'*s are situated on p half-rays issuing from the origin and separated by equal angles then ii) can be written as ii)* as above.*

Combined utilizations of theorems 1.1, 1.2, 1.3 and 1.4 give rise to a wide class of related results of the type just mentioned.

2- When the polynomials p and q are written in the form

$$p(z)=\sum_{\nu=0}^{n}a_\nu\varphi_\nu(z)\quad\text{and}\quad q(z)=\sum_{\nu=0}^{n}\beta_\nu\varphi_\nu(z)$$

where the φ_ν's form a basis for the vector space P_n, the apolarity expression takes (by bilinearity) the form

$$\{p,q\}=\sum_{\nu=0}^{n}\sum_{\mu=0}^{n}\omega_{\nu\mu}a_\nu\beta_\mu \tag{3.2}$$

where the $\omega_{\nu\mu}'$s depend only on $\{\varphi_\nu\}_{\nu=0}^{n}$. Thus theorem 2.2 permits us to draw conclusions about $z(p)$ in terms of the a_ν's when q and $z(q)$ are known.

We give now a theorem which generalizes well-known theorems of S. Bernstein [1], P.D. Lax [4] and G. Szegö [7] about estimations for $|p'(z)|$. Let U and $U^*\subseteq\mathbb{C}$ be open sets then

Theorem 3.5 *If* $p\in P_n$ *maps* U *into* U^* $\forall z\in U$,

$$|p'(z)|\leqslant nR/\varrho(z) \tag{3.3}$$

where R is the supremum of the radii of all circles contained in U^* *and* $\varrho(z)$ *is the supremum of the radii of all the circles contained in U and containing z. The result is best possible when U is the open unit disk D.*

The cases $U=D$, $U^*=D$, $U=D$, $U^*=D\setminus\{0\}$ and $U=D$, $U^*=\{z\mid |\mathrm{Re}\,z|<1\}$ give Bernstein's, Lax's and Szegö's results respectively as it is immediately seen.

Other applications of this result are, for example,

Theorem 3.6 *If U contains arbitrarily large disks while U^* does not and $p \in P$ maps U into U^* then p must be a constant.*

For example, this simple theorem says that p cannot map an infinite sector into an infinite strip without being constant. This is in any case rather obvious.

Theorem 3.7 *If $p \in P_n$ maps U into U then $\exists z_0 \in U$ such that*

$$|p'(z_0)| \leqslant n. \tag{3.4}$$

3- Let $p \in P_n$ be mapping the open unit disk D into a given set U. The following theorem permits us to find from it a n-parameters family of polynomials $p_{\zeta_1}, \ldots, \zeta_n$ doing the same.

Theorem 3.8 *Let $p \in P_n$ be mapping D into U then for any choice of $\zeta_1, \ldots, \zeta_n \in D$, the polynomial $p_{\zeta_1}, \ldots, \zeta_n$ given by*

$$p_{\zeta_1,\ldots,\zeta_n}(z) = a_0 + a_1 \frac{(\zeta_1 + \ldots + \zeta_n)z}{\binom{n}{1}} + a_2 \frac{(\zeta_1\zeta_2 + \ldots + \zeta_{n-1}\zeta_n)z^2}{\binom{n}{2}} + \\ + \ldots + a_n \frac{\zeta_1 \ldots \zeta_n z^n}{\binom{n}{n}} \tag{3.5}$$

also maps D into U.

One immediate consequence of this result is that $\forall \lambda \in D$, $a_0 + a_n \lambda \in U$ which generalizes the fact that, when $U = D$, we have $|a_0| + |a_n| \leqslant 1$.

Let now $\mathscr{U}_n$ denote the class of normalized univalent polynomials of degree n, [3]. It is well-known (this is Dieudonné's criterion [3]) that for a normalized polynomial $p(z) = z + a_2 z^2 + \ldots + a_n z_n$ we have $p \in \mathscr{U}_n \Leftrightarrow \forall \varphi \in [0, \pi/2]$, $(D_\varphi p)(z) \neq 0$ in D where $D_\varphi p$ denotes the Dieudonné's derivative of p with respect to φ defined by

$$(D_\varphi p)(z) = \begin{cases} p'(z) \text{ if } \varphi = 0, \\ \displaystyle\sum_{\nu=1}^{n} a_\nu \frac{\sin_{\nu\varphi}}{\sin_\varphi} z^{\nu-1} \text{ if } \varphi \neq 0. \end{cases} \tag{3.6}$$

For each set S containing the origin, we define the class $\mathscr{U}_n(S) \subseteq \mathscr{U}_n$ of normalized univalent polynomials of type S by $p \in \mathscr{U}_n(S) \Leftrightarrow p$ is normalized and $\forall \varphi \in \left[0, \frac{\pi}{2}\right]$, $D_\varphi p\colon D \to C \setminus S$. (3.7)

The previous theorem gives the following variational formula for $\mathscr{U}_n(S)$.

Theorem 3.9 *If* $p \in U_n(S)$ *and* $\zeta_1, \ldots, \zeta_{n-1} \in D$ *then*

$$p(\zeta_1, \ldots, \zeta_{n-1}; z) = z + a_2 \frac{(\zeta_1 + \ldots + \zeta_{n-1})}{\binom{n-1}{1}} z^2 + \ldots + a_n \frac{\zeta_1 \ldots \zeta_{n-1}}{\binom{n-1}{n-1}} z^n \quad (3.8)$$

also belongs to $\mathscr{U}_n(S)$ *where* $p(z) = z + a_2 z^2 + \ldots + a_n z^n$.

We immediately infer from this that if $z + \ldots + a_n z^n \in \mathscr{U}_n(S)$ then

$$|a_n| \leqslant \frac{1}{n} \inf_{z \in S} |z - 1| \quad (3.9)$$

and the estimate is best possible.

4. Proofs

Proof of Theorem 1.1 The case where $w_0 \ldots w_k = 0$ being an immediate consequence of (1.1) we need only to look at the case $w_0 \ldots w_k \neq 0$. Let $\zeta \in \Omega$ then ζ being a $(w_0, \ldots, w_k)$ - point of p it is, in particular, a (w_0, w_1) - -point of p and the relation $p'(\zeta) = p(\zeta) \sum_{\nu=1}^{n} \frac{1}{\zeta - z_\nu}$ which can be written as

$$\sum_{\nu=1}^{n} (\zeta - z_\nu)/|\zeta - z_\nu|^2 = \overline{(w_1/w_0)}$$

implies that

$$\zeta = \left(\overline{(w_1/w_0)} + \sum_{\nu=1}^{n} a_\nu z_\nu\right) \Big/ \sum_{\nu=1}^{n} a_\nu \quad (4.1)$$

where $a_\nu = 1/|\zeta - z_\nu|^2$, $\nu = 1, \ldots, n$. That is

$$\zeta \in L_{\theta_0} + K = K_{\theta_0}. \quad (4.2)$$

The point ζ being also a $(w_\nu, w_{\nu+1})$ - point of $p^{(\nu)}$ for $1 \leqslant \nu \leqslant k-1$ we similarly deduce that

$$\zeta \in L_{\theta_\nu} + K^{(\nu)}, \ \nu = 1, \ldots, k-1 \quad (4.3)$$

where $K^{(\nu)} = \mathrm{Conv}[p^{(\nu)}] \subseteq K$ by (1.1).

We thus have

$$\zeta \in L_{\theta_\nu} + K = K_{\theta_\nu}, \ \nu = 1, \ldots, k-1 \quad (4.4)$$

and (4.2) together with (4.4) gives the desired conclusion

$$\zeta \in \bigcap_{\nu=0}^{k-1} K_{\theta_\nu}.$$

Proof of Theorem 1.2 We have $(L_\zeta p)(z) = p(z) \sum_{\nu=1}^{n} (\zeta - z_\nu)/(z - z_\nu)$. Let $z_0 \in z(\mathscr{L}_\zeta p)$. If $z_0 = z_\nu$ for some ν, there is nothing to prove, so we suppose that $z_0 \neq z_\nu$ for every ν and we can write $\sum(\zeta - z_\nu)/(z_0 - z_\nu) = 0$ and obtain that

$$\zeta \cdot \sum (\bar{z}_0 - \bar{z}_\nu)/|z_0 - z_\nu|^2 = \sum (\bar{z}_0 - \bar{z}_\nu) z_\nu / |z_0 - z_\nu|^2. \quad (4.5)$$

Putting $a_\nu = (1/|z_0 - z_\nu|^2)/\sum_\mu (1/|z_0 - z_\mu|^2)$, $\nu = 1, \ldots, n$ we get from (4.5) that

$$\bar{\zeta} = \left(z_0 \sum \beta_\nu \bar{z}_\nu - \sum \beta_\nu |z_\nu|^2\right) \Big/ \left(z_0 - \sum \beta_\nu z_\nu\right), \quad (4.6)$$

where $\beta_\nu = a_\nu / \sum_\mu a_\mu$, $\nu = 1, \ldots, n$.

Suppose now that $d(z_0, \mathrm{Conv}[z(p)]) \geqslant \delta$. Since we have $\sum \beta_\nu z_\nu \in \mathrm{Conv}[z(p)]$ we obtain from (4.6)

$$|\zeta| \leqslant \left(|z_0| \left|\sum \beta_\nu \bar{z}_\nu\right| + \sum \beta_\nu |z_\nu|^2\right) \Big/ \left|z_0 - \sum \beta_\nu z_\nu\right| \leqslant |z_0| M \Big/ \left|z_0 - \sum \beta_\nu z_\nu\right| + M^2/\delta. \quad (4.7)$$

Let us now show that

$$|z_0| \Big/ \left|z_0 - \sum \beta_\nu z_\nu\right| \leqslant (M + \delta)/\delta. \quad (4.8)$$

If $|z_0| \leqslant M + \delta$ this is immediate and if $|z_0| > M + \delta$ we have $\left|z_0 - \sum \beta_\nu z_\nu\right| \geqslant |z_0| - \left|\sum \beta_\nu z_\nu\right| = |z_0| - a$, say, where $0 \leqslant a \leqslant M$. Consequently,

$$|z_0| \Big/ \left|z_0 - \sum \beta_\nu z_\nu\right| \leqslant |z_0|/(|z_0| - a) \leqslant (M + \delta)/(M + \delta - a) \leqslant (M + \delta)/\delta$$

because the maximum of the functions $t/(t - a)$ for $t \in [M + \delta, \infty)$ occurs at $t = M + \delta$. Relation (4.8) thus holds and combined use of (4.7) and (4.8) then gives

$$|\zeta| \leqslant M \cdot (1 + 2M/\delta) \quad (4.9)$$

which contradicts the hypothesis of the theorem.

Proof of Theorem 1.3 To prove this result, we refer to the Grace's apolarity theorem (2.1). For each $\omega \in \mathbb{C}$, define $p_\omega \in P_n$ by

$$p_\omega(z) = p(z + \omega) = \sum_{\nu=0}^{n} \frac{p^{(\nu)}(\omega)}{\nu!} z^\nu$$

and put $q(z) = \sum_{\nu=0}^{n} (-1)^{n-\nu} n^{(n-\nu)} a_{n-\nu} z^\nu$.

By a simple calculation we find that $\{p_\omega, q\} = p^*(\omega)$. Now $\omega \in z(p^*) \Leftrightarrow \{p_\omega, q\} = 0$ and if C is a circular region containing $z(q)$ then, by (2.1), we have that $\exists \nu_0$ such that $z_{\nu_0} - \omega \in C$, that is $\exists \nu_0$ such that $\omega \in z_{\nu_0} - C$ which implies that $\omega \in z(p) - C$. We can thus write $z(p^*) \subseteq z(p) - C$ and putting $\pi(z) = (-1)^n q(-z)$ we finally find that

$$z(\pi) \subseteq C \Leftrightarrow z(p^*) \subseteq z(p) + C$$

which implies (1.5).

Proof of Theorem 1.4 Suppose that the trivial case i) does not occur. If $\zeta = z_{\nu_0}$ for a certain ν_0 we have only to take the sequence $\{a_\nu\}$ where $a_{\nu_0} = 1$ and $a_\nu = 0$ for $\nu \neq \nu_0$ and conclusion ii) holds. If for every ν $\zeta \neq z_\nu$ we have

$$0 = P'(\zeta)/P(\zeta) = \sum_{\nu=1}^{\infty} (\zeta/z_\nu)^p/(\zeta - z_\nu) = \sum_{\nu=1}^{\infty} \frac{(\bar{\zeta} z_\nu)^p}{|z_\nu|^{2p}} \cdot \frac{(\bar{\zeta} - \bar{z}_\nu)}{|\zeta - z_\nu|^2}$$

which implies that

$$\sum_{\nu=1}^{\infty} \frac{z_\nu^p}{|z_\nu|^{2p}} \cdot \frac{(\zeta - z_\nu)}{|\zeta - z_\nu|^2} = 0 \tag{4.10}$$

Equality (4.10) immediately leads to

$$\zeta \cdot \sum_{\nu=1}^{\infty} z_\nu^p/|z_\nu|^{2p} |\zeta - z_\nu|^2 = \sum_{\nu=1}^{\infty} z_\nu^{p+1}/|z_\nu|^{2p} |\zeta - z_\nu|^2$$

and conclusion ii) follows by putting $a_\nu = 1/|z_\nu|^{2p} |\zeta - z_\nu|^2$.

Proof of Theorem 2.1 a) Let $\pi_1(z) = (z-\sigma)^n$ and $\pi_2(z) = (z-\eta)^n$ then $\{\pi_1, \pi_2\} = (\eta - \sigma)^n$ as is easily verified. Equality (2.2) then follows from the bilinearity of $\{p, q\}$.

b) We have $p(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - z_\nu)$ and $q(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu = b_n \prod_{\nu=1}^{n} (z - \zeta_\nu)$.

Write now (as we can always do) the polynomial p in the form

$$p(z) = \sum_{\mu=1}^{N} \omega_\mu (z - \lambda_\mu)^n \tag{4.12}$$

where ω_μ, λ_n and N are suitable constants. We immediately have:

$$\sum_{\mu=1}^{N} \omega_\mu \lambda_\mu^\nu = (-1)^\nu a_{n-\nu}, \quad \nu = 0, 1, \ldots, n. \tag{4.13}$$

Moreover,

$$L_{\zeta_{n-1}}[p(z)] = \sum_{\mu=1}^{N} \omega_\mu (\zeta_n - \lambda_\mu)(z - \lambda_\mu)^{n-1},$$

$$L_{\zeta_{n-1}} L_{\zeta_n}[p(z)] = \sum_{\mu=1}^{N} \omega_\mu (\zeta_n - \lambda_\mu)(\zeta_{n-1} - \lambda_\mu)(z - \lambda_\mu)^{n-2},$$

$$\vdots \qquad\qquad \vdots$$

$$L_{\zeta_1} \ldots L_{\zeta_n}[p(z)] = \sum_{\mu=1}^{N} \omega_\mu (\zeta_n - \lambda_\mu) \ldots (\zeta_1 - \lambda_\mu).$$

That is

$$L_{\zeta_1}\dots L_{\zeta_n}[p(z)] = \frac{(-1)^n}{b_n}\sum_{\mu=1}^{N}\omega_\mu q(\lambda_\mu) = \frac{(-1)^n}{b_n}\sum_{\mu=1}^{N}\omega_\mu\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu\lambda_\mu^\nu$$

$$= \frac{(-1)^n}{b_n}\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu\sum_{\mu=1}^{N}\omega_\mu\lambda_\mu^\nu$$

and this is equal by (4.13) to

$$\frac{(-1)^n}{b_n}\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu(-1)^\nu a_{n-\nu} = \frac{1}{b_n}\{p, q\}$$

which gives (2.3).

c) By (1.3) we have

$$\left.\frac{L_{\zeta_n}[p(z)]}{p(z)}\right|_{z=\omega} = 1-\frac{(\zeta_n-\omega)}{n}\sum_{\nu=1}^{n}\frac{1}{z_\nu-\omega} = 1-\left(\frac{\zeta_n-\omega}{\lambda_n-\omega}\right),$$

$$\left.\frac{L_{\zeta_{n-1}}L_{\zeta_n}[p(z)]}{L_{\zeta_n}[p(z)]}\right|_{z=\omega} = 1-\frac{(\zeta_{n-1}-\omega)}{n-1}\sum_{\nu=1}^{n-1}\frac{1}{z'_\nu-\omega} = 1-\left(\frac{\zeta_{n-1}-\omega}{\lambda_{n-1}-\omega}\right),$$

$$\vdots$$

$$\left.\frac{L_{\zeta_k}\dots L_{\zeta_n}[p(z)]}{L_{\zeta_{k+1}}\dots L_{\zeta_n}[p(z)]}\right|_{z=\omega} = 1-\frac{(\zeta_k-\omega)}{k}\sum_{\nu=1}^{k}\frac{1}{z_\nu^{(n-k)}-\omega} = 1-\left(\frac{\zeta_k-\omega}{\lambda_k-\omega}\right),$$

$$\vdots$$

$$\left.\frac{L_{\zeta_1}\dots L_{\zeta_n}[p(z)]}{L_{\zeta_2}\dots L_{\zeta_n}[p(z)]}\right|_{z=\omega} = 1-\frac{(\zeta_1-\omega)}{1}\sum_{\nu=1}^{1}\frac{1}{z_\nu^{(n-1)}-\omega} = 1-\left(\frac{\zeta_1-\omega}{\lambda_1-\omega}\right)$$

where $z_\nu^{(\mu)}$'s are the zeros of $L_{\zeta_{n-\mu+1}}\dots L_{\zeta_n}[p(z)]$ (which are in C_2) and the λ'_νs are points in C_2. Multiplying now these equalities we find

$$\left.\frac{L_{\zeta_1}\dots L_{\zeta_n}[p(z)]}{p(z)}\right|_{z=\omega} = \left(\frac{\zeta_1-\lambda_1}{\omega-\lambda_1}\right)\dots\left(\frac{\zeta_n-\lambda_n}{\omega-\lambda_n}\right) \tag{4.14}$$

and conclusion (2.4) follows by representation a).

d) Let $S_\nu = \sum z_{\mu_1}\dots z_{\mu_\nu}$ and $T_\nu = \sum\zeta_{\mu_1}\dots\zeta_{\mu_\nu}$ denote the elementary symmetric functions of degree ν of the zeros of p and q respectively. The identities

$$p(z)/a_n = \sum_{\nu=0}^{n}(-1)^{n-\nu}S_{n-\nu}z^\nu \text{ and } q(z)/b_n = \sum_{\nu=0}^{n}(-1)^{n-\nu}T_{n-\nu}z^\nu$$

immediately give

$$\{p, q\}/a_n b_n = \sum_{\nu=0}^{n} \left((-1)^{n-\nu} S_{n-\nu} T_\nu \Big/ \binom{n}{\nu}\right).$$

We thus have to show that

$$\sum_{\nu=0}^{n} \left((-1)^{n-\nu} S_{n-\nu} T_\nu \Big/ \binom{n}{\nu}\right) = \frac{1}{n!} \sum_{\varphi \in S_n} \prod_{\nu=1}^{n} (\zeta_\nu - z_{\varphi(\nu)}) \tag{4.15}$$

Since each of these two expressions is symmetric in $z_1, \dots, z_n$ and in $\zeta_1, \dots, \zeta_n$ it is sufficient to show that the coefficients A_ν and B_ν of

$$z_1 z_2 \dots z_\nu \zeta_1 \zeta_2 \dots \zeta_{n-\nu}$$

in the two sides of (4.15) are equal. In fact, we have: $A_\nu = B_\nu = (-1)^\nu \Big/ \binom{n}{\nu}$ as it is easily verified.

Proof of Theorem 2.2 i) Let $\delta > 0$ and suppose that the conclusion is false; that is the sets $z(p)$ and $z(q)$ can be separated by C_1 and C_2 satisfying (2.6). By symmetry, we can always suppose that the zeros $\zeta_1, \dots, \zeta_n$ of q are all in C_1 (which we will first take to be a disk centered at ω say) and that the zeros $z_1, \dots, z_n$ of p are in C_2. If we put r_1 = radius of C_1 and $r_2 = d(\omega, C_2)$ we must have $\delta < r_2 - r_1$.

By representation c) in theorem 2.1, we deduce that:

$$\begin{aligned} |a_n||b_n|\delta^n = |\{p, q\}| &= |a_n||b_n| \left|\frac{z_1 - \omega}{\lambda_1 - \omega}\right| \dots \left|\frac{z_n - \omega}{\lambda_n - \omega}\right| \cdot |\zeta_1 - \lambda_1| \dots |\zeta_n - \lambda_n| \\ &\geqslant |a_n||b_n||z_1 - \omega| \dots |z_n - \omega| \left(\frac{r_2 - r_1}{r_2}\right)^n \\ &> |a_n||b_n||z_1 - \omega| \dots |z_n - \omega| \frac{\delta^n}{r_2^n}. \end{aligned}$$

We thus have

$$1 > |z_1 - \omega| \dots |z_n - \omega| / r_2^n$$

from which we get the existence of a ν_0, $(1 \leqslant \nu_0 \leqslant n)$ such that $|z_{\nu_0} - \omega| < r_2$ which is a contradiction. In the case where C_1 is a half-plane, an approximation of C_1 by discs is required.

To show that the result is best possible it is sufficient to look at the polynomials

$$p(z) = a_n (z - \sigma_1)^n \text{ and } q(z) = b_n (z - \sigma_2)^n.$$

ii) We use here representation d) of theorem 2.1 which gives

$$|\{p, q\}| = |a_n||b_n|\delta^n \leqslant |a_n||b_n| \left(\frac{1}{n!} \sum_{\varphi} \prod_{\nu=1}^{n} |\zeta_\nu - z_{\varphi(\nu)}|\right)$$

from which we can assert the existence of a $\varphi_0 \in S_n$ such that

$$\prod_{\nu=1}^{n} |\zeta_\nu - z_{\varphi_0(\nu)}| \geq \delta^n$$

and so the existence of ν_0 such that

$$|\zeta_{\nu_0} - z_{\varphi_0(\nu_0)}| \geq \delta.$$

The same example as above shows that the result is best possible.

Proof of Theorem 2.3 We need first the following reformulation of the condition of δ-apolarity (see Szegö [6] for the case $\delta = 0$). We omit the proof which is easy.

Lemma. *Let $l_0, l_1, \dots, l_n \in \mathbb{C}$, $l_n \neq 0$ be given and let Λ be a linear operator defined on P_n which carries $a(z) = a_0 + a_1 z + \dots + a_n z^n$ into the number*

$$\Lambda(a) = l_0 a_n + l_1 a_{n-1} + \dots + l_n a_0. \tag{4.17}$$

Then the polynomials $a(z)$ and $l(z) = \sum_{\nu=0}^{n} (-1)^\nu l_\nu \binom{n}{\nu} z^\nu$ are δ-apolar if and only if

$$|\Lambda(a)| = |a_n| |l_n| \delta^n.$$

Moreover, the polynomial $l(z)$ can be written in the form

$$l(z) = \Lambda\big((x-z)^n\big)$$

where $(x-z)^n = \beta(x)$ is considered as a polynomial in x.

Now define Λ (for polynomials $q \in P_{n-k}$) by

$$\Lambda(q) = \int_0^1 \int_0^{t_1} \dots \int_0^{t_{k-1}} q\big(\omega_1 + (\omega_2 - \omega_1)t_1 + \dots + (\omega_k - \omega_{k-1})t_{k-1} + (\omega_0 - \omega_k)t_k\big)\, dt_k \dots dt_1.$$

This expression is trivially linear in the coefficients of q and a formula of Newton [2] gives, for $p \in P_n$,

$$\Lambda(p^{(k)}) = [\omega_0, \omega_1, \dots, \omega_k]_p. \tag{4.18}$$

Taking, as in the lemma, the polynomial $l(z)$ (of degree $n-k$) associated with Λ and taking (4.18) into account we get

$$l(z) = \Lambda\big(\beta(x)\big) = \Lambda\big((x-z)^{n-k}\big) = [\omega_0, \omega_1, \dots, \omega_k]_r \tag{4.19}$$

where $r(x) = \dfrac{(x-z)^n}{n^{(k)}}$, $n^{(k)} = n(n-1)\dots(n-k+1)$. Now from Lagrange's interpolation formula [2] we have

$$\begin{aligned} l(z) &= \frac{1}{n^{(k)}} \sum_{\nu=0}^{k} \frac{(\omega_\nu - z)^n}{(\omega_\nu - \omega_0)\dots(\omega_\nu - \omega_{\nu-1})(\omega_\nu - \omega_{\nu+1})\dots(\omega_\nu - \omega_k)} \\ &= (-1)^{n-k} \frac{z^{n-k}}{k!} + \dots . \end{aligned} \tag{4.20}$$

Since we also have that

$$p^{(k)}(z) = n^{(k)} z^{n-k} + \dots \tag{4.21}$$

we conclude by the lemma (with n replaced by $n-k$), (4.18), (4.19), (4.20) and (4.21) that $p^{(k)}(z)$ and $l(z)$ are δ-apolar with

$$\delta = \left| \frac{[\omega_0, \dots, \omega_k]_p}{\binom{n}{k}} \right|^{1/n-k}$$

and the result follows by theorem 2.2.

Note that part ii) of theorem 2.2 also gives a conclusion in theorem 2.3.

Proof of Theorem 3.1. This is an immediate consequence of theorem 1.1 if we note that the point $\zeta = 0$ is a $(\omega_0, \dots, \omega_n)$-point of p where $\omega_\nu = \nu! a_\nu$, $0 \leqslant \nu \leqslant k = n$.

Proof of Theorem 3.2. Under the hypothesis $\theta_{\nu_0} + \pi = \theta_{\nu_1}$ for two indices ν_0 and ν_1, conclusion

$$\Omega \subseteq \bigcap_{\nu=0}^{k-1} K_{\theta_\nu}$$

gives $\Omega \subseteq K_{\theta_{\nu_0}} \cap K_{\theta_{\nu_1}} = K$ since K is convex.

Proof of Theorem 3.3. Conclusion ii) of theorem 1.3 with $p = 0$ is merely $\zeta \sum a_\nu = \sum a_\nu z_\nu$ which trivially implies ii)*.

Proof of Theorem 3.4. Considering $P(wz)$ where $|w| = 1$ we can suppose that the p half-lines issuing from the origin are determined by the p^{th} roots of unity. The zeros z_ν of P thus have the form

$$z_\nu = \varrho_\nu \exp(^{2k\pi i/p}), \quad (0 < \varrho_\nu, 0 \leqslant k < p)$$

and ii) of theorem 1.3 gives

$$\zeta \cdot \sum_{\nu=1}^{\infty} a_\nu \varrho_\nu^p = \sum_{\nu=1}^{\infty} a_\nu \varrho_\nu^p z_\nu .$$

Putting now

$$\beta_\nu = a_\nu \varrho_\nu^p \Big/ \sum_{\nu=1}^{\infty} a_\nu \varrho_\nu^p$$

we get

$$\zeta = \sum_{\nu=1}^{\infty} \beta_\nu z_\nu , \sum_{\nu=1}^{\infty} \beta_\nu = 1, \beta_\nu \geqslant 0, \nu = 1, 2, \dots$$

which proves the theorem.

Proof of Theorem 3.5. Let $a \in U^*$ then $z[p(z)-a] \cap U = \varphi$.

Let now $\zeta \in U$ and draw a circle centered at ζ which is completely contained in U. Call this circle $D_\varrho(\zeta)$ where ϱ is its radius. From (1.3) we conclude that the zeros of the polynomial

$$\frac{1}{n}\mathscr{L}_\zeta[p(z)-a] = p(z)-a-\frac{(z-\zeta)}{n}p'(z)$$

are outside $D_\varrho(\zeta)$. That is, for any $z \in D_\varrho(\zeta)$ we have

$$\omega(z) = p(z)-\frac{(z-\zeta)}{n}p'(z) \in U^* \tag{4.22}$$

Let now ζ_1 be an arbitrary point in $D_\varrho(\zeta)$. Another application of (1.3) permits us to write

$$p(z)-\frac{(z-\zeta_1)}{n}p'(z) = \omega(z)+\frac{(\zeta_1-\zeta)}{n}p'(z) \in U^*. \tag{4.23}$$

Since $\omega(z) \in U^*$ we must have

$$\frac{1}{n}|\zeta_1-\zeta|\,|p'(z)| \leqslant R \text{ for exvery } z \in D_\varrho(\zeta).$$

This last relation being true for any $\zeta_1 \in D_\varrho(\zeta)$ we deduce that

$$\varrho|p'(z)|/n \leqslant R \text{ for } z \in D_\varrho(\zeta) \subseteq U,$$

that is

$$|p'(z)| \leqslant nR/\varrho.$$

To show that in the case where $U = D$ (the open unit disc) the inequality is best possible it suffices to check that equality occurs for a polynomial of the form $a_0 + Rz^n$ for a suitable a_0.

Proof of Theorem 3.6. Let $p \in P_n$ and $[D_{\varrho_i}(w_i)]_{i=1}^{\infty}$ be a sequence of disks such that for each i, $D_{\varrho_i}(\omega_i) \subseteq U$ is a disk of radius ϱ_i centered at ω_i. We can always assume that $\varrho_i \uparrow \infty$ and $|\omega_i| \uparrow \infty$. Theorem 3.5 then gives $|p'(\omega_i)| \leqslant nR/\varrho_i \downarrow 0$ which means that $p'(z) \equiv 0$, that is p is a constant.

Proof of Theorem 3.7. This is trivial since there exists a $z_0 \in U$ such that $\varrho(z_0) = R$.

Proof of Theorem 3.8. Let $w \in D$ and consider the polynomial q_ω defined by

$$\begin{aligned} q_\omega(z) &= \prod_{\nu=1}^{n}(z-\zeta_\nu\omega) \\ &= z^n-(\zeta_1+\ldots+\zeta_n)\omega z^{n-1}+\ldots+(-1)^n\zeta_1\ldots\zeta_n\omega^n. \end{aligned} \tag{4.24}$$

We have $z(q_\omega) \subseteq D$. Take $a \notin U$ then $z[p(z)-a] \cap D = \varphi$. From (2.1) we must have

$$\{p(z)-a, q_\omega(z)\} \neq 0$$

that is

$$p_{\zeta_1,\ldots,\zeta_n}(\omega) \neq a. \tag{4.25}$$

Since this is true for every $\omega \in D$ we must conclude that $p_{\zeta_1,\ldots,\zeta_n}(D)$ excludes every point a excluded by U, which means that $p_{\zeta_1,\ldots,\zeta_n}: D \to U$.

Proof of theorem 3.9 Let $p \in U_n(S)$ then $D_\varphi p \in P_{n-1}$ maps D into C/S which implies by (3.5) that

$$(D_\varphi p)_{\zeta_1,\ldots,\zeta_{n-1}}: D \to C/S.$$

But $(D_\varphi p)_{\zeta_1,\ldots,\zeta_{n-1}}(z) = D_\varphi p(\zeta_1, \ldots, \zeta_{n-1}; z)$ which means that

$$p(\zeta_1, \ldots, \zeta_{n-1}; z) \in U_n(S).$$

Conclusion (3.9) is a consequence of (3.7) and (3.8) by taking suitable ζ_ν's.

This work have been supported by the N.R.C. of Canada.
I am very much indebted to Professor Q.I. Rahman for his various valuable suggestions.

REFERENCES

[1] Bernstein, S., *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, *Mémoite de l'Académie Royale de Belgique* (2), vol. 4 (1912), 1–103.

[2] Davis, P. J., *Interpolation and Approximation*, Blaisdell Publishing Company, Toronto, (1963).

[3] Dieudonné, J., *Recherche sur quelques problemes relatifs aux polynômes et aux fonctions bornées d'une variable complexe*, Ann. Ecole Norm. Sup. (3), 48 (1931), 78–81.

[4] Lax, P. D., *Proof of a Conjecture of P. Erdös on the Derivative of a Polynomial*, Bull. Amer. Math. Soc., vol. 50 (1944).

[5] Marden, M., *Geometry of Polynomials*, Amer. Math. Soc., Math. Surveys. No 3, Providence, Rhode Island, (1966).

[6] Szegö, G., *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z., vol. 13 (1922), 28–55.

[7] Szegö, G., *Über einen Satz, Herrn Serge Bernstein*, Schriften der Königsberger Gelehrten Gessellchaft, vol. 22 (1928), 59–70.

Streszczenie

W pracy tej autor otrzymuje kilka twierdzeń określających położenie zer wielomianów otrzymanych przez pewne operacje z wielomianu danego, względnie z dwu danych wielomianów. Wyniki te stanowią uogólnienie klasycznych rezultatów Gaussa-Lucasa, Laguerre'a, Grace i Heawooda.

Резюме

В этой работе автор получает несколько теорем, определяющих распределение нулей полиномов, которые получены из данного либо из двух данных полиномов с помощью некоторых операций. Эти результаты становят обобщение классических результатов Гаусса-Льукаса, Льагэрра, Греса и Хэавода.