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Z Zakładu Matematycznych Metod Fizyki Zespołowej Katedry Matematyki
Wydziału Mat.-Fiz.-Chem. UMCS
Kierownik Zakładu: doc. dr Zdzisław Lewandowski

LUCJAN IZDEBSKI

A Contribution to the Theory of Subordination

Pewien przyczynek do teorii podporządkowania

К некоторому вопросу подчинений

1. Introduction

Suppose the function $f(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 \geqslant 0$, is regular in $K_\delta = \{z: |z| < \delta\}$ and $F(z) = A_1 z + A_2 z^2 + \ldots$, $A_1 > 0$, regular and univalent in K_δ.

The function $F(z)$ is said to be a domain majorant of $f(z)$ in K_ϱ, $\varrho \leqslant \delta$, resp. $f(z)$ is said to be subordinate to $F(z)$ if the map of K_ϱ under f, i.e. $f(K_\varrho)$, is contained in $F(K_\varrho)$. In this case we write (f, F, ϱ).

On the other hand, $F(z)$ is said to be a modular majorant of $f(z)$ in K_ϱ, if $|f(z)| \leqslant |F(z)|$ for any $z \epsilon K_\varrho$. We write $|f, F, \varrho|$ in the latter case.

Let S be the class of functions regular and univalent in K_1 with usual normalization $f(0) = 0, f'(0) = 1$.

As shown by Z. Lewandowski [1], $|f, F, 1|$ implies $|f', F', 2-\sqrt{3}|$ for any $F \epsilon S$ and any f regular in K_1 and satisfying $f(0) = 0$, $f'(0) \geqslant 0$ and the constant $2-\sqrt{3}$ is the best possible. Even under supplementary assumption of starshapedness of both functions the constant $2-\sqrt{3}$ cannot be improved.

In this paper we obtain a result analogous to the just mentioned. We prove that $|f, F, 1|$ implies (zf', zF', R_0) for any $F \epsilon S$ and any f univalent in K_1 and satisfying $f(0) > 0$, $f'(0) > 0$, where $R_0 = 0{,}143\ldots$ is the positive root of the transcendental equation (4) below. The problem whether R_0 is best possible, remains still open.

2. Preliminary remarks

Suppose P^δ_β $(\beta > 0,\ \delta > 0)$ is the class of all functions $\varphi(z) = \beta + \beta_1 z + \beta_2 z^2 + \ldots$ regular in K_δ and satisfying $\operatorname{re}\varphi(z) > 0$ in K_δ. For any $\varphi \in P^\delta_\beta$ we have

$$(1) \qquad |\arg\varphi(z)| \leqslant \arcsin\big(2r\delta/(r^2+\delta^2)\big) = 2\arctan(r/\delta)$$

where $r = |z| < \delta$. This can be easily deduced from the fact that the domain of variability of $\varphi(z)$ for fixed $z \in K_1$ and varying $\varphi \in P^1_\beta$ is the closed disc with diameter

$$[\beta(1-|z|)/(1+|z|),\ \beta(1+|z|)/(1-|z|)].$$

Note that $\varphi(z) \in P^\delta_\beta$ is equivalent to $\varphi(\delta z) \in P^1_\beta$.

Suppose that S_a, $a > 0$, is the class of functions $f(z) = az + a_2 z^2 + \ldots$ regular and univalent in K_1. It is well known (cf. e.g. [3], p. 42) that for any $f \in S_a$

$$(2) \qquad \left|\big(1 + zf''(z)/f'(z)\big) - (1+|z|^2)/(1-|z|^2)\right| \leqslant 4|z|/(1-|z|^2).$$

Suppose that C_δ is the class of functions

$$\Phi(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$$

of $z \in K_\delta$ and $t \in [t_1, t_2]$ with $a_1(t) > 0$ which satisfy the following conditions:

(A) for any fixed $t \in [t_1, t_2]$ the function $\Phi(z,t)$ is regular and univalent as a function of $z \in K_\delta$.

(B) for any fixed $z \in K_\delta$ the function $\Phi(z,t)$ is continuous in t and has a continuous derivative $\Phi'_t(z,t)$ in $[t_1, t_2]$.

A function $\Phi(z,t) \in C_\delta$ is said to be areally increasing in K_ϱ, $\varrho \leqslant \delta$, if $\big(\Phi(z,t'), \Phi(z,t''), \varrho\big)$ for any $t_1 \leqslant t' < t'' \leqslant t_2$. We write $\Phi \uparrow^\varrho$ in this case.

A function $\Phi(z,t) \in C_\delta$ is said to be absolutely increasing in K_ϱ, $\varrho \leqslant \delta$, if $|\Phi(z,t'), \Phi(z,t''), \varrho|$ for any $t_1 \leqslant t' < t'' \leqslant t_2$. We write $|\Phi| \uparrow^\varrho$ in this case.

A. Bielecki and Z. Lewandowski [4] gave necessary and sufficient conditions of areal and absolute monotoneity. In the sequel we shall use the following

Lemma. If $\Phi(z,t) \in C_\delta$ and

$$(3) \qquad \text{either } |\arg[\Phi'_t(z,t)/z\Phi'_z(z,t,)]| < \frac{\pi}{2}, \text{ or } \Phi'_t(z,t) = 0.$$

for any $t \in [t_1, t_2]$, $z \in K_\varrho$, then $\Phi \uparrow^\varrho$.

3. The main result

Theorem. *If* $f \in S_a$, $F \in S$ *and* $|f, F, 1|$, *then* (zf', zF', R_0), *where* R_0 *is the unique root of the equation*

(4) $$\Omega(r) = \arcsin\big(4r/(1+r^2)\big) + 2 \arctan\big(r/(2-\sqrt{3})\big) = \pi/2$$

contained in the interval $(0, 2-\sqrt{3})$.

Proof. Put

(5) $$\Phi(z, t) = z[f'(z)]^{1-t}[F'(z)]^t,$$

where $0 \leqslant t \leqslant 1$ and the single valued branches of u^{1-t}, u^t become 1 for $u = 1$. We have $\Phi(0, t) = 0$, $\Phi'_z(0, t) = a^{1-t} > 0$ and $\mathrm{re}[z\Phi'_z(z, t)/\Phi(z, t)] > 0$ for $|z| < 2-\sqrt{3}$. Hence $\Phi(z, t) \in C_{2-\sqrt{3}}$. We have, moreover, $\Phi(z, 0) = zf'(z)$, $\Phi(z, 1) = zF'(z)$. Now, $|f, F, 1|$ implies

(6) $$\mathrm{re}[\Phi'_t/\Phi] = \mathrm{re}\log[F'(z)/f'(z)] > 0 \text{ for } z \in K_{2-\sqrt{3}}$$

since $|F'(z)| > |f'(z)|$ in $K_{2-\sqrt{3}}$, [4]. From (1) it follows that

(7) $$\arg[\Phi'_t/\Phi] \leqslant 2 \arctan[r/(2-\sqrt{3})]$$

where $r = |z| < 2-\sqrt{3}$. Now, it follows from the convexity of the domain described by $1 + zf''(z)/f'(z)$ which is a circular disc of the right halfplane in case $|z| < 2-\sqrt{3}$, cf. (2), that also

$$|\arg(z\Phi'_z/\Phi)| = 1 + (1-t)zf''(z)/f'(z) + tzF''(z)/F'(z)$$

belongs to this disc. This implies

(8) $$\big|\arg\big(z\Phi'_z(z)/\Phi(z)\big)\big| \leqslant \arcsin\big(4r/(1+r^2)\big)$$

in case $r = |z| < 2-\sqrt{3}$. It follows from (7) and (8) that

(9) $$|\arg(\Phi'_t/z\Phi'_z)| + |\arg(z\Phi'_z/\Phi)| + |\arg(\Phi'_t/\Phi)| \leqslant \Phi(r)$$

for $|z| = r < 2-\sqrt{3}$ and, consequently, in view of monotoneity of $\Phi(r)$, also $|\arg(\Phi'_t/z\Phi'_z)| < \pi/2$ for $|z| < R_0$ which means $\Phi \uparrow^{R_0}$ in view of Lemma. In particular (zf', zF', R_0) and this proves our theorem. It can be easily verified that $R_0 = 0.143\ldots$

The problem whether R_0 is best possible remains still open.

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Streszczenie

W pracy tej dowodzę następującego twierdzenia:

Jeżeli $f(z)$ i $F(z)$ są funkcjami holomorficznymi i jednolistnymi w kole $|z| < 1$, $f(0) = F(0) = 0$, $f'(0) > 0$, $F'(0) > 0$ i $|f(z)| \leqslant |F(z)|$ dla $|z| < 1$, to $zf'(z) \prec zF'(z)$ w kole $|z| < r_0$, gdzie r_0 nie zależy od szczególnego doboru funkcji f i F. Liczba r_0 jest pierwiastkiem dodatnim równania (4) i wynosi w przybliżeniu 0,143.

Nie wiadomo czy r_0 nie da się zastąpić liczbą większą.

Резюме

В работе доказана следующая теорема. Пусть $f(z)$, $F(z)$ — голоморфные и однолистные функции в единичном круге $|z| < 1$, $f(0) = F(0) = 0$, $f'(0) > 0$, $F'(0) > 0$ и $|f(z)| \leqslant |F(z)|$. Тогда $zf'(z) \prec zF'(z)$ в круге $|z| < r_0$, где $r_0 = 0.143$ — положительный корень уравнения (4).