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Estimates of Parameters of Mixed Model $I \times J$ with Interaction in the Case of Non-Orthogonal Data

Oceny parametrów mieszanego modelu $I \times J$ z interakcją w przypadku danych nieortogonalnych

Оценки параметров смешанной модели $I \times J$ со взаимодействием в случае неортогональных данных

1. Testing of hypothesis and estimating of parameters

The problem of estimating the parameters of the mixed model with interaction for non-orthogonal data has not yet been discussed in my papers [2] and [3]. We are interested in it here under almost the same assumptions and notation as before; now we assume that restrictions concerning the parameters a_i are unweighted [cf. (1)] and that the interaction is significant, so the method of weighted squares of means can be applied.

At first, let us use an example of mixed model 3×3 [cf. [3]) to show that the problem of estimating is closely connected with that of testing the hypothesis of H_A that all the fixed effects a_1, a_2 and a_3 of the classification A are equal to zero, i. e.

(1) $$a_1 = a_2 = a_3 = 0$$

In fact, because of different expectations of mean squares for A and for AB, denoted by MS_A and MS_{AB} respectively, it is not possible to use the test function

(2) $$F = MS_A / MS_{AB}$$

for testing the hypothesis H_A.

However, we can use Satterthwaite's approximate test procedure [4], which requires to find some estimates of parameters. Then one uses approximate test function from [3]:

(3) $$F = MS_A/G$$

where

(4) $$G = K_1 MS_e + K_2 MS_A + K_3 MS_{AB}$$

is a linear combination of mean squares MS_e, MS_A and MS_{AB}. The symbols K_1, K_2 and K_3 denote such numbers that $E(MS_A) = E(G)$, where E is the mathematical expectation.

The degrees of freedom given by the Satterthwaite's method are: $\nu_1 = I-1 = 2$ and

(5) $$\nu_2 = G^2/[(K_1 MS_e)^2/\nu_e + (K_2 MS_A)^2/\nu_A + (K_3 MS_{AB})^2/\nu_{AB}]$$

where $\nu_1 = \nu_A = I-1 = 2$, $\nu_{AB} = (I-1)(J-1) = 4$, and $\nu_e = n-IJ = n-9$. The symbol ν is reserved for degrees of freedom.

In order to find the coefficients K_1, K_2, K_3 it is necessary to compare $E(MS_A)$ with $E(G)$.

Let us note that when the hypothesis H_A is true we obtain (cf. [3], table 1)

(6) $$E(MS_A) = \sigma_e^2 + \sum_{i=1}^{3} d_i \operatorname{Var}(c_i(v)) - \sum_{\substack{i<i' \\ i,i'=1,2,3}} K_{ii'} \operatorname{Cov}(c_i(v), c_{i'}(v))$$

where

(7) $$d_i = \frac{1}{6}\Big(W_i - W_i^2 \Big/ \sum_{i=1}^{3} W_i\Big), \quad K_{ii'} = W_i W_{i'} \Big/ 3 \sum_i W_i.$$

Thus, as it is evident from the form (6) the problem of finding the values of K_1, K_2 and K_3 is equivalent to that of estimating the parameters $\operatorname{Var}(c_i(v))$ and $\operatorname{Cov}(c_i(v), c_{i'}(v))$, $i, i' = 1, 2, 3$. This fact indicates the necessity of estimating the parameters of the model if one wants to test the hypothesis H_A.

2. Estimates of σ_e^2, σ_{ii}' and σ_{ii}

Now we shall deal with the mixed model $I \times J$, any I and any J. At first, let us note that the unknown parameters are: 1° $\mu_i = \mu + a_i$, 2° σ_e^2, 3° $\operatorname{Var}(b(v))$, 4° $\operatorname{Cov}(b(v), c_i(v))$ and 5° $\operatorname{Cov}(c_i(v), c_{i'}(v))$; $i, i' = 1, 2, \ldots, I$.

The unbiased estimates of the main effects are the means of the subclass means (unweighted means):

(8) $$\hat{\mu}_i = \tilde{y}_{i..} = \sum_j^J \overline{y}_{ij.}/J$$

In the orthogonal case we obtain (cf. [5], p. 268)

(9) $$\hat{a}_i = \overline{y}_{i..} - \overline{y}.$$

The remaining parameters are the known functions of

(10) $$\sigma_{ii'} = E(m_{ij}-\mu_i)(m_{i'j}-\mu_{i'}) = \mathrm{Cov}[m(i,v), m(i',v)],$$

$$i, i' = 1, 2, \ldots, I,$$

of the following forms (cf. [2]):

(11) $$\mathrm{Var}(b(v)) = \Big(\sum_{i=1}^{I}\sigma_{ii}\Big)\Big/I^2 + 2\Big(\sum_{\substack{i<i'\\ i,i'=1,2,\ldots,I}}\sigma_{ii'}\Big)\Big/I^2 = \sum_i^I\sum_{i'}^I \sigma_{ii'}/I^2 = \sigma_{..},$$

(12) $$\mathrm{Cov}[b(v), c_i(v)] = \sum_{r=1}^{I}\sigma_{ri}/I - \mathrm{Var}(b(v)) = \sigma_{.i} - \sigma_{..},$$

(13) $$\mathrm{Cov}[c_i(v), c_{i'}(v)] = \sigma_{ii'} - \sum_{r=1}^{I}(\sigma_{ri}+\sigma_{ri'})/I + \mathrm{Var}(b(v)).$$

Because of $E(MS_e) = \sigma_e^2$ (cf. [3], table 1) the MS_e is the unbiased estimator of σ_e^2, i. e.

(14) $$\hat{\sigma}_e^2 = MS_e.$$

From the relations (11), (12) and (13) we could estimate $\mathrm{Var}(b(v))$, $\mathrm{Cov}[b(v), c_i(v)]$ and $\mathrm{Cov}[c_i(v), c_{i'}(v)]$ if we knew the estimators of $\sigma_{ii'}$. Therefore, we are interested now in estimating $\sigma_{ii'}$. We remember that they are the elements of the covariance matrix Σ_m of vector random variable

(15) $$m = m(v) = [m(1,v), m(2,v), \ldots, m(I,v)]'.$$

From the assumptions we know that the J vector random variables $(m_{1j}, m_{2j}, \ldots, m_{Ij})'$ are independently distributed $N(\mu, \Sigma_m)$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_I)$ and are independent of the

(16) $$e_{ijl} = y_{ijl} - m_{ij}$$

which are independently $N(0, \sigma_e^2)$.

We find

$$\bar{y}_{ij.} = n_{ij}^{-1} \sum_{l=1}^{n_{ij}} y_{ijl} = m_{ij} + \bar{e}_{ij.} = \mu_i + b_j + c_{ij} + \bar{e}_{ij.} \tag{17}$$

where

$$n_{ij}\bar{e}_{ij.} = \sum_{l=1}^{n_{ij}} e_{ijl}. \tag{18}$$

Let us note that

$$E(\bar{y}_{ij.}) = E(m_{ij} + \bar{e}_{ij.}) = E(m_{ij}) = E(\mu + a_i + b_j + c_{ij}) = \tag{19}$$

$$= \mu + a_i = \mu_i, \quad i = 1, 2, \ldots, I,$$

and that

$$E(\tilde{y}_{i..}) = \mu_i = \mu + a_i. \tag{20}$$

Further, we have

$$\mathrm{Var}(\bar{y}_{ij.}) = \sigma_{ii} + n_{ij}^{-1}\sigma_e^2, \quad i = 1, 2, \ldots, I, \quad j = 1, 2, \ldots, J, \tag{21}$$

$$\mathrm{Cov}(\bar{y}_{ij.}, \bar{y}_{i'j.}) = \mathrm{Cov}(m_{ij}, m_{i'j}) = \sigma_{ii'}, \quad i, i' = 1, 2, \ldots, I; \quad i \neq i'. \tag{22}$$

If we find the estimate of $\sigma_{ii'}$ we obtain simultaneously the estimates of $\mathrm{Var}(\bar{y}_{ij.})$ and $\mathrm{Cov}(\bar{y}_{ij.}, \bar{y}_{i'j.})$.

In order to estimate $\sigma_{ii'}$ let us calculate as follows. We find the unweighted mean of i'th row:

$$\tilde{y}_{i..} = J^{-1}\sum_{j=1}^{J} \bar{y}_{ij.} = J^{-1}\sum_{j=1}^{J} (\mu_i + b_j + c_{ij} + \bar{e}_{ij.})$$

$$= \mu_i + J^{-1}\Big(\sum_{j}^{J} b_j + \sum_{j}^{J} c_{ij} + \sum_{j}^{J} \bar{e}_{ij.}\Big). \tag{23}$$

Because of (17) we obtain

$$E\sum_{j}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})^2 =$$

$$= \sum_{j}^{J} E\Big[\Big(b_{ij} - \sum_{j}^{J} b_j/J\Big) + \Big(c_{ij} - \sum_{j}^{J} c_{ij}/J\Big) + \Big(\bar{e}_{ij.} - \sum_{j}^{J} \bar{e}_{ij.}/J\Big)\Big]^2 \tag{24}$$

Let us note that when φ_j are random variables mutually independent, then we have

$$\mathrm{Var}\Big(\varphi_j - J^{-1}\sum_{j=1}^{J}\varphi_j\Big) = (J-1)J^{-1}\mathrm{Var}(\varphi_j). \tag{25}$$

It is easy to verify the formula

$$(26)\qquad \sum_{j=1}^{J} \mathrm{Var}(\bar{e}_{ij.} - \tilde{e}_{i..}) = (J-1)J^{-1}\Big(\sum_{j=1}^{J} n_{ij}^{-1}\Big)\sigma_e^2.$$

After transforming (23) and using the relations (25), (26) and $\mathrm{Var}(b_j + c_{ij}) = \mathrm{Var}(m_{ij}) = \sigma_{ii}$ we obtain

$$(27)\qquad E\Big[\sum_{j=1}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})^2\Big] = (J-1)\Big(\sigma_{ii} + \frac{\sigma^2}{J}\sum_{j}^{J} n_{ij}^{-1}\Big).$$

From the formula (27) it follows that

$$(28)\qquad \hat{\sigma}_{ii} = \hat{\mathrm{Var}}(m_{ij}) = \frac{1}{J-1}\sum_{j=1}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})^2 - J^{-1}\hat{\sigma}_e^2 \sum_{j=1}^{J} n_{ij}^{-1}$$

is the unbiased estimator of σ_{ii}, $i = 1, 2, \ldots, I$. Next, applying the well known formula (cf. [6], p. 69) $E\sum_{i=1}^{n}(x_i - \bar{x})(t_i - \bar{t}) = (n-1)\cdot\mathrm{Cov}(x, t)$ under the assumptions $E(x_i) = E(t_i) = 0$ we find

$$(29)\qquad E\sum_{j=1}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})(\bar{y}_{i'j.} - \tilde{y}_{i'..}) = (J-1)\cdot\mathrm{Cov}(\bar{y}_{ij.}, \bar{y}_{i'j.}) = (J-1)\sigma_{ii'}.$$

From (29) we see that

$$(30)\qquad \hat{\sigma}_{ii'} = \frac{1}{J-1}\sum_{j}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})(\bar{y}_{i'j.} - \tilde{y}_{i'..} = \widehat{\mathrm{Cov}}(m_{ij}, m_{i'j})$$

is the unbiased estimator of the parameter $\sigma_{ii'}$, $i \neq i'$; $i, i' = 1, 2, \ldots, I$.

For the orthogonal data $n_{ij} = k = \text{const}$ the particular case of (30) is the formula by H. Scheffé (cf. [5], 8.1.33).

Substituting the estimates $\hat{\sigma}_{ii}$ and $\hat{\sigma}_{ii'}$ from (28) and (30) instead of σ_{ii} and $\sigma_{ii'}$ into (21) and (22) we find the estimates

$$(31)\qquad \widehat{\mathrm{Var}}(\bar{y}_{ij.}) = \hat{\sigma}_{ii'} + n_{ij}^{-1}\hat{\sigma}_e^2 = \frac{1}{J-1}\sum_{j}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})^2 + \Big(n_{ij}^{-1} - J^{-1}\sum_{j}^{J} n_{ij}^{-1}\Big)\hat{\sigma}_e^2$$

and

$$(32)\qquad \widehat{\mathrm{Cov}}(\bar{y}_{ij.}, \bar{y}_{i'j.}) = \hat{\sigma}_{ii'} = (J-1)^{-1}\sum_{j}^{J} (\bar{y}_{ij.} - \tilde{y}_{i..})(\bar{y}_{i'j.} - \tilde{y}_{i'..});$$

$$i \neq i'; \quad i, i' = 1, 2, \ldots, I.$$

From (31) it follows that

$$\widehat{\mathrm{Var}}(\tilde{y}_{i..}) = \frac{1}{J}\hat{\sigma}_{ii} + \frac{1}{J^2}\hat{\sigma}_e^2 \sum_j^J n_{ij}^{-1} \tag{33}$$

where $J\tilde{y}_{i..} = \sum_j^J y_{ij.}$ and $\hat{\sigma}_{ii}$ is given in (28).

3. Estimates of Var $(b(v))$, Cov $[b(v), c_{i'}(v)]$ and Cov $[c_{i'}(v), c_i(v)]$

Substituting $\hat{\sigma}_{ii}$ and $\hat{\sigma}_{ii'}$ of (28) and (30) into the formulae (11), (12) and (13) we obtain the estimates of parameters $\mathrm{Var}(b(v))$, $\mathrm{Cov}[b(v), c_i(v)]$ and $\mathrm{Cov}[c_i(v), c_{i'}(v)]$.

Another estimate of $\mathrm{Var}(b(v))$ is obtained from Table 1, [2], in the form

$$\widehat{\mathrm{Var}}(b(v)) = (J-1)(MS_B - MS_e)\Big(\sum_j^J V_j - \sum_j V_j^2 \Big/ \sum_j V_j\Big)^{-1}. \tag{34}$$

Using this expression and $\hat{\sigma}_e^2$, $\hat{\sigma}_{ii}$, $\hat{\sigma}_{ii'}$ we obtain from (12) and (13) another set of estimates of the parameters $\mathrm{Cov}[b(v), c_i(v)]$ and $\mathrm{Cov}[c_i(v), c_{i'}(v)]$.

It is easy to show that the estimate

$$\widehat{\mathrm{Var}}(b(v)) = \hat{\sigma}_{..} \tag{35}$$

is identical with that of (34) if $V_j = I^2\big(\sum_{i'}^I n_{ij}^{-1}\big)^{-1} = \mathrm{const} = H$, then $\widehat{\mathrm{Var}}(b(v)) = (MS_B - MS_e)/H$ is the unbiased estimator of $\mathrm{Var}(b(v))$. In the orthogonal case $n_{ij} = k = \mathrm{const}$ we have $V_j = IK$ and $\widehat{\mathrm{Var}}(b(v)) = (MS_B - MS_e)/IK$ as it should be (cf. [5], p. 269, 8.1.27).

It is necessary to note that if Henderson's method 3 is used [1], unbiased estimates are obtained. However, the method can be applied when the number of different classes is small.

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Streszczenie

W pracy niniejszej, będącej kontynuacją prac [2] i [3], podano związek łączący problematykę weryfikowania hipotezy i oceny parametrów przy założeniach cytowanych prac i nieważonej restrykcji dla stałych elementów modelu, a. Wyznaczono dwa zbiory ocen parametrów $\mathrm{Var}(b(v))$, $\mathrm{Cov}[b(v), c_i(v)]$ i $\mathrm{Cov}[c_i(v), c_{i'}(v)]$; są one oparte na ocenach parametrów $\sigma_{ii'} = \mathrm{Cov}(m_{ij}, m_{i'j})$; $i, i' = 1, 2, \ldots, I$ i na dwóch metodach ocen parametru $\mathrm{Var}(b(v))$ oraz na tym, że $\mathrm{Cov}[b(v), c_i(v)]$ i $\mathrm{Cov}[c_i(v), c_{i'}(v)]$ można przedstawić jako funkcje parametrów $\sigma_{ii'}$ i $\mathrm{Var}(b(v))$.

Pierwsza z metod daje ocenę $\mathrm{Var}(b(v))$ jako funkcję parametrów $\sigma_{ii'}$ (por. wzór (11)) a druga pozwala określić ocenę $\mathrm{Var}(b(v))$ z wartości oczekiwanej MS_B w analizie wariancji (por. wzór (34)).

Oba zbiory parametrów są identyczne gdy $V_j = I^2(\sum^{I} n_{ij}^{-1})^{-1} = \mathrm{const.}$; $j = 1, 2, \ldots, J$; a w szczególnym przypadku gdy dane są ortogonalne ($n_{ij} = \mathrm{const.} = k$); w innych przypadkach oceny są różne.

Резюме

В этой работе, являющейся продолжением работ [2] и [3] дается связь проверки гипотезы с оценками параметров при предположениях цитированных работ и невзвешенного ограничения для постоянных элементов a модели. Определено два множества оценок параметров $\mathrm{Var}(b(v))$, $\mathrm{Cov}[b(v), c_i(v)]$ и $\mathrm{Cov}[c_i(v), c_{i'}(v)]$; они получаются из оценок параметров $б_{ii'} = \mathrm{Cov}(m_{ij}, m_{i'j})$; $i, i' = 1, 2, \ldots, I$ и при помощи двух методов оценки параметра $\mathrm{Var}(b(v))$ используя тот факт, что $\mathrm{Cov}[b(v), c_i(v)]$ и $\mathrm{Cov}[c_i(v), c_{i'}(v)]$ можно представить как функции параметров $б_{ii'}$ и $\mathrm{Var}(b(v))$.

Первый метод дает оценку $\mathrm{Var}(b(v))$ как функцию параметров $б_{ii'}$ (ср. форм. (11)), а второй позволяет определить оценку $\mathrm{Var}(b(v))$ из математического ожидания MS_B в дисперсионном анализе (ср. форм. (34)).

Оба множества оценок параметров тождественны когда $V_j = I^2(\sum^{I} n_{ij}^{-1})^{-1} = \mathrm{const.}$; $j = 1, 2, \ldots, J$; а в частном случае, когда данные являются ортогональными ($n_{ij} = \mathrm{const.} = к$); в других случаях оценки различны.