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Ramanujan Institute
University of Madras

Department of Mathematics
College of Engineering , Anna University
Madras

K.S.PADMANABHAN , R.PARAVATHAM , T.N.SHANMUGAM

α–convexity and α–close–to–convexity Preserving Integral Operators

Operatory całkowe zachowujące α–wypukłość oraz α–prawie wypukłość

Let E be the open unit disc in C and $H(E)$ be the class of all functions f holomorphic in E. Let $g \in H(E)$ be such that $g(0) = g'(0) - 1 = 0$ and $\frac{g(z)g'(z)}{z} \neq 0$ in E. We consider the integral operator $A_g(f) = F$, defined by

$$F(z) = A_g(f) = \left\{ \frac{(c+1/\alpha)}{g^c(z)} \int_0^z g^{c-1}(t) g'(t) f^{1/\alpha}(t)\, dt \right\}^\alpha . \tag{1}$$

For $c = 1$ and $\alpha = 1$, this reduces to an integral operator introduced by P.T.Mocanu [5], wherein he has determined conditions on g so that A_g is an convexity or close–to–convexity preserving integral operator. For $g(z) = z$, our operator reduces to the operator introduced by St.Ruscheweyh [9] and further $\alpha = 1$ yields Bernardi's operator [1]. These two operators have been extensively studied by several authors in the field. In this paper we first consider the class M_α of α–convex functions defined by P.T.Mocanu [4] as follows:

Definition 1. A function $f(z) \in H(E)$ with $f(0) = f'(0) - 1 = 0$ is called an α–convex function if $\frac{f(z)f'(z)}{z} \neq 0$ in E and for some non–negative real number α,

$$\operatorname{Re}\left\{ (1-\alpha)\frac{z f'(z)}{f(z)} + \alpha\left(1 + \frac{z f''(z)}{f'(z)}\right) \right\} > 0 , \qquad z \in E .$$

In Theorem 1 we obtain sufficient conditions on g so that A_g is an α–convexity preserving operator. Next, we consider the class $P(\alpha)$ of α–close–to–convex functions studied by K.S.Padmanabhan and R.Bharati [7].

Definition 2. Let $f \in H(E)$ with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)f'(z)}{z} \neq 0$ in E. f is said to be in $P(\alpha)$ provided

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\alpha\left(1 + \frac{z f''(z)}{f'(z)}\right) + (1-\alpha)\frac{z f'(z)}{f(z)}\right\} d\theta > -\pi$$

whenever $0 < \theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $z = re^{i\theta}$, $r < 1$, $\alpha \geq 0$. They also established a characterization for functions in $P(\alpha)$, that is $f \in P(\alpha)$ if and only if there exists an $\varphi \in S^*$ such that

$$\mathrm{Re}\frac{z^\alpha f'^\alpha(z) f^{1-\alpha}(z)}{\varphi(z)} \geq 0 \qquad \text{in } E .$$

In Theorem 2, we establish conditions on g so that $A_g(f) \in P(\alpha)$ whenever $f \in P(\alpha)$. Also we determine conditions on g so that $A_g(f) \in S^*$ whenever $f \in S^*$. Finally we give an application of Theorem 3.

To prove our main results, we make use of the following theorems.

Theorem A. [6] *Let $\alpha > 0$, $\alpha + \gamma > 0$, and consider the integral operator $I_{\alpha,\gamma}(f) = \{\frac{\alpha+\gamma}{z^\gamma}\int_0^z f^\alpha(t)\, t^{\gamma-1}\, dt\}^{1/\alpha}$. If $\beta \in [-\frac{\gamma}{\alpha}, 1)$ then the order of starlikeness of the class $I_{\alpha,\gamma}(S^*(\beta))$ is given by $\delta(\alpha;\beta;\gamma) = \inf_{|z|<1} \mathrm{Re}\, q(z)$ where $S^*(\beta)$ is the class of starlike functions of order β. Moreover, if $\beta \in [\beta_0, 1)$ where $\beta_0 = \max\{\frac{\alpha-\gamma-1}{2\alpha}, -\frac{\gamma}{\alpha}\}$ and $g = I_{\alpha,\gamma}(f)$ for $f \in S^*(\beta)$ then*

$$\mathrm{Re}\frac{z g'(z)}{g(z)} \geq q(-r) = \frac{1}{\alpha}\left[\frac{\alpha+\gamma}{F(1, 2\alpha(1-\beta), \alpha+\gamma+1; \frac{r}{1+r})} - \gamma\right]$$

for $|z| = r < 1$ and

$$\delta(\alpha;\beta;\gamma) = q(-1) = \frac{1}{\alpha}\left[\frac{\alpha+\gamma}{F\left(1, 2\alpha(1-\beta), \alpha+\gamma+1; \frac{1}{2}\right)} - \gamma\right] \tag{2}$$

where $q(z) = \frac{1}{\alpha Q(z)} - \frac{\gamma}{\alpha}$ with $Q(z) = \int_0^1 \left(\frac{1-z}{1+zt}\right)^{2\alpha(1-\beta)} t^{\alpha+\gamma-1}\, dt$, $z \in E$ and $F(a,b,c;z)$ is the hypergeometric function. The extremal function is given by $g = I_{\alpha,\gamma}(k)$ where $k(z) = z(1-z)^{2(\beta-1)}$.

Theorem B. [2] *Let $f \in S^*$, let Φ and φ be regular functions in E with $\Phi(0) = \varphi(0) = 1$ and $\Phi(z)\varphi(z) \neq 0$ in E and let $\alpha, \beta, \gamma, \delta$ be real numbers satisfying $\alpha \geq 0$, $\beta \geq 0$, $\delta \geq 0$, $\alpha + \delta > 0$ and $\beta + \gamma = \alpha + \delta$. If there exists a real number $J \geq 0$ such that*

$$J \geq \gamma + \mathrm{Re}\frac{z\,\Phi'(z)}{\Phi(z)}, \qquad z \in E , \tag{3}$$

$$\beta + \gamma > J \tag{4}$$

$$\delta + \mathrm{Re}\frac{z\,\varphi'(z)}{\varphi(z)} \geq \max(0, J - \lambda(J)) , \qquad z \in E \text{ where} \tag{5}$$

$$\lambda(J) = \tfrac{1}{2}\min\left[\frac{\beta+\gamma-J}{J}, \frac{J}{\beta+\gamma-J}\right] , \qquad \lambda(0) = 0 , \tag{6}$$

then there exists a unique function $F(z) = z + A_2 z^2 + \cdots$, satisfying

$$(7)\qquad F(z) = \left(\frac{\beta+\gamma}{z^\gamma \Phi(z)} \int_0^z f^\alpha(t)\varphi(t) t^{\delta-1}\, dt\right)^{1/\beta}, \qquad z \in E$$

such that $F \in S^*$.

We also need the following result which is a slight modification of a lemma due to K.S.Padmanabhan and R.Paravatham [8].

Lemma 1. *Let $\beta, \gamma \in \mathbb{C}$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and let $q \in H(E)$ with $q(0) = 1$ and $\operatorname{Re}(\beta q(z) + \gamma) > 0$, $z \in E$. If $p(z) = 1 + p_1 z + \cdots$ is analytic in E, then*

$$p(z) + \frac{z p'(z)}{\beta q(z) + \gamma} < h(z) \Longrightarrow p(z) < h(z) .$$

Since the proof of this lemma is essentially in same as the one in [8], we omit the details. Now, we proceed to prove our main results.

Theorem 1. *Suppose $g \in H(E)$ with $g(0) = g'(0) - 1 = 0$ and $\frac{g'(z)g(z)}{z} \neq 0$ in E. Let $\alpha > 0$, $c > 0$, $(c+1)\alpha > 1 > (c-1)\alpha$ and $\beta \in \mathbb{R}$ such that $\beta \in [\beta_0, 0)$ where $\beta_0 = \max\{\frac{1-(c+1)\alpha}{2}, -\alpha c, \frac{c\alpha - (\alpha+1)}{2\alpha c}\}$*

$$(8)\qquad \operatorname{Re}\left\{c\,\frac{z g'(z)}{g(z)}\right\} \geq c + \beta|_\alpha$$

and

$$(9)\qquad \operatorname{Re}\left\{(c+1)\frac{z g'(z)}{g(z)} - \left(1 + z\frac{g''(z)}{g'(z)}\right)\right\} \leq c + \delta|_\alpha$$

where $\delta = \delta(\frac{1}{\alpha}; \beta; c)$ is given by (2). Then $F = A_g(f)$ defined by (1) is in M_α whenever $f \in M_\alpha$.

Proof. It is clear that F is homomorphic in a neighbourhood of $z = 0$ and satisfies $F(0) = 0 = F'(0) - 1$. Thus there exists an $R > 0$ such that $F(z) \neq 0$ for $0 < |z| < R$. We begin by showing that F is α–convex and hence univalent in $|z| < R$ and the proof will be complete if we establish that $R \geq 1$. Indeed, if $F(z_0) = 0$, $|z_0| = R < 1$ then for any given $\varepsilon > 0$, $\exists$ a neighbourhood of z_0 in which $|F(z)| < \varepsilon$. This is a contradiction because F is univalent and so $|F(z)| > \frac{|z|}{(1-|z|)^2}$ in $|z| < R$. Thus $F(z) \neq 0$ in E.

From the definition of $F = A_g(f)$, we have

$$(10)\qquad \frac{g^c(z) F^{1/\alpha}(z)}{z^{c+1/\alpha}} = \frac{(c + 1/\alpha)}{z^{c+1/\alpha}} \int_0^z g^{c-1}(t) g'(t) f^{1/\alpha}(t)\, dt .$$

Differentiating with respect to z and simplifying,

$$\frac{g^{c+1}(z)F^{1/\alpha-1}(z)F'(z)}{g'(z)} = (c+1/\alpha)\int_0^z g^c(t)f^{1/\alpha-1}(t)f'(t)\,dt\,.$$

Putting $k(z) = \dfrac{g^c(z)f^{1-\alpha}(z)f'^{\alpha}(z)}{z^{\alpha c-\alpha}}$ and $H(z) = \dfrac{g^{(c+1)\alpha}(z)F^{1-\alpha}(z)F'^{\alpha}(z)}{g'^{\alpha}(z)z^{\alpha c}}$ we have

$$H(z) = \left\{\frac{(c+1/\alpha)}{z^c}\int_0^z t^{c-1}k^{1/\alpha}(t)\,dt\right\}^{\alpha}. \tag{11}$$

Now

$$\mathrm{Re}\frac{z\,k'(z)}{k(z)} = \alpha c\,\mathrm{Re}\left(\frac{z\,g'(z)}{g(z)}\right) + \mathrm{Re}\left\{(1-\alpha)\frac{z\,f'(z)}{f(z)} + \alpha\left(1+z\frac{f''(z)}{f'(z)}\right)\right\} - \alpha c \geq \beta$$

by condition (8) and the fact that $f \in M_\alpha$. Now an application of Theorem A to H yields that if $\beta \in [\beta_0, 0)$ where $\beta_0 = \max\{\dfrac{1/\alpha - c - 1}{2/\alpha}, -\alpha c\}$, then

$$\mathrm{Re}\left\{z\frac{H'(z)}{H(z)}\right\} \geq \alpha\left[\frac{\frac{1}{\alpha}+c}{F(1,\frac{2}{\alpha}(1-\beta),\frac{1}{\alpha}+c+1;\frac{1}{2})} - c\right] = \delta\left(\frac{1}{\alpha};\beta;c\right) = \delta \tag{12}$$

where $F(a, b, c; z)$ is the hypergeometric function. We proceed to verify that $\delta > 0$. Consider

$$F\left(1,\tfrac{2}{\alpha}(1-\beta),\tfrac{1}{\alpha}+c+1;\tfrac{1}{2}\right) = 1 + \frac{\frac{2}{\alpha}(1-\beta)}{\frac{1}{\alpha}+c+1}\cdot\frac{1}{2} + \frac{\frac{2}{\alpha}(1-\beta)\left(\frac{2}{\alpha}(1-\beta)+1\right)}{(\frac{1}{\alpha}+c+1)(\frac{1}{\alpha}+c+2)}\cdot\left(\frac{1}{2}\right)^2 + \cdots$$

Since $\beta > \dfrac{1-(c+1)\alpha}{2}$ we have $\dfrac{2(1-\beta)}{1+(c+1)\alpha} < 1$ and so

$$F\left(1,\tfrac{2}{\alpha}(1-\beta),\tfrac{1}{\alpha}+c+1;\tfrac{1}{2}\right) < 1 + \frac{2(1-\beta)}{1+(c+1)\alpha}$$

and

$$\delta = \frac{1+\alpha c}{F\left(1,\frac{1}{2}(1-\beta),\frac{1}{\alpha}+c+1;\frac{1}{2}\right)} - \alpha c > \frac{(1+\alpha c)(1+(c+1)\alpha)}{1+(c+1)\alpha+2-2\beta} - \alpha c =$$

$$= \frac{1+\alpha-\alpha c+2\beta\alpha c}{3+(c+1)\alpha-2\beta} > 0$$

provided $c\alpha < \dfrac{1+\alpha}{1-2\beta}$ which is true because $\beta > \dfrac{c\alpha-(\alpha+1)}{2\alpha c}$. From

$$\mathrm{Re}\left\{(1-\alpha)z\frac{F'(z)}{F(z)} + \alpha\left(1+z\frac{F''(z)}{F'(z)}\right)\right\} = \mathrm{Re}\left(z\frac{H'(z)}{H(z)}\right) + \alpha c + \alpha\,\mathrm{Re}\left\{1+z\frac{g''(z)}{g'(z)} - (c+1)z\frac{g'(z)}{g(z)}\right\}$$

using (9) and (12) we get

$$\mathrm{Re}\Big\{(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\big(1+z\frac{F''(z)}{F'(z)}\big)\Big\}\ge\delta+\alpha c-\delta-\alpha c=0$$

which completes the proof of the theorem.

Remark 1. When $\alpha=1$, $c=1$ this gives Theorem 1 of P.T.Mocanu [5]. When $\alpha=1$ and $g(z)=z$, we get the well–known result that the class K of convex univalent functions is closed under Bernardi's operator.

Theorem 2. *Let $g\in H(E)$ with $g(0)=g'(0)-1=0$ and $\frac{g(z)g'(z)}{z}\neq 0$ in E. If $\exists\ \alpha\ge 1$, $c>0$, $(c-1)\alpha<1$ and $\beta\in\mathbb{R}$ such that $\beta\in[\beta_0,0)$ where $\beta_0=\max\{\frac{1-(c+1)\alpha}{2},-\alpha c,\frac{c\alpha-(\alpha+1)}{2\alpha c}\}$, $\mathrm{Re}\{c\frac{zg'(z)}{g(z)}\}>c+\beta/\alpha$ and*

$$0\le\mathrm{Re}\Big\{(1+c)z\frac{g'(z)}{g(z)}-\big(1+z\frac{g''(z)}{g'(z)}\big)\Big\}\le c+\frac{\delta}{\alpha}\tag{13}$$

where $\delta=\delta\big(\frac{1}{\alpha};\beta;c\big)$, then $F=A_g(f)\in P(\alpha)$ whenever $f\in P(\alpha)$.

Proof. We remark that $\alpha\ge 1$ and $c>0\Longrightarrow(c+1)\alpha>1$. The existence and analyticity of F in E follow in the same way as in Theorem 1.

Since $f\in P(\alpha)$, $\exists$ an $\varphi_1\in S^*$ such that $\mathrm{Re}\{\frac{z^\alpha f'^\alpha(z)f^{1-\alpha}(z)}{\varphi_1(z)}\}>0$ in E. P.T.Mocanu [4] showed that $\varphi\in M_\alpha$ if and only if $\exists$ an $\varphi_1\in S^*$ such that $z^\alpha\varphi'^\alpha(z)\varphi^{1-\alpha}(z)=\varphi_1(z)$. Hence if $f\in P(\alpha)$ then $\exists$ an $\varphi\in M_\alpha$ such that $\mathrm{Re}\frac{f'^\alpha(z)f^{1-\alpha}(z)}{\varphi'^\alpha(z)\varphi^{1-\alpha}(z)}>0$ in E. Also from the definition of F, (10) holds, whence we get by differentiation with respect to z,

$$\frac{g^c(z)F^{1/\alpha-1}(z)F'(z)}{\alpha}+cg^{c-1}(z)g'(z)F^{1/\alpha}(z)=\big(c+\tfrac{1}{\alpha}\big)g^{c-1}(z)g'(z)f^{1/\alpha}(z)\ .\tag{14}$$

Putting $p^\alpha(z)=\frac{F'^\alpha(z)F^{1-\alpha}(z)}{\Phi'^\alpha(z)\Phi^{1-\alpha}(z)}$ where $\Phi(z)=A_g(\varphi)$, (14) becomes

$$\frac{g^c(z)\Phi'(z)\Phi^{1/\alpha-1}(z)p(z)}{\alpha}+cg^{c-1}(z)g'(z)F^{1/\alpha}(z)=\big(c+\tfrac{1}{\alpha}\big)g^{c-1}(z)g'(z)f^{1/\alpha}(z)$$

or

$$\frac{g^{c+1}(z)\Phi'(z)\Phi^{1/\alpha-1}(z)p(z)}{\alpha g'(z)}=\big(c+\tfrac{1}{\alpha}\big)g^c(z)f^{1/\alpha}(z)-cg^c(z)F^{1/\alpha}(z)=$$

$$=\frac{(c+\frac{1}{\alpha})}{\alpha}\int_0^z g^c(t)f^{1/\alpha-1}(t)f'(t)\,dt\ .$$

Thus

$$\frac{g^{c+1}(z)p(z)\Phi'(z)\Phi^{1/\alpha-1}(z)}{g'(z)}=\big(c+\tfrac{1}{\alpha}\big)\int_0^z g^c(t)f^{1/\alpha-1}(t)f'(t)\,dt\ .$$

Now differentiating again and simplifying,

$$\frac{\alpha z p'(z)}{\alpha G(z)+q(z)}+p(z)=\frac{\alpha\left(c+\frac{1}{\alpha}\right)g^{c}(z)zf^{1/\alpha-1}(z)f'(z)}{\left(\alpha G(z)+q(z)\right)\dfrac{g^{c+1}(z)\Phi'(z)\Phi^{1/\alpha-1}(z)}{g'(z)}}$$

where $q(z)=(1-z)\dfrac{z\Phi'(z)}{\Phi(z)}+\alpha\left(1+z\dfrac{\Phi''(z)}{\Phi'(z)}\right)$ and $G(z)=(c+1)z\dfrac{g'(z)}{g(z)}-\left(1+z\dfrac{g''(z)}{g'(z)}\right)$. Now,

$$\left(\alpha G(z)+q(z)\right)\frac{g^{c+1}(z)\Phi'(z)\Phi^{1/\alpha-1}(z)}{g'(z)}=\alpha\left(c+\tfrac{1}{\alpha}\right)zg^{c}(z)\varphi^{1/\alpha-1}(z)\varphi'(z)$$

and hence

$$\frac{zp'(z)}{G(z)+\dfrac{q(z)}{\alpha}}+p(z)=\frac{f^{1/\alpha-1}(z)f'(z)}{\varphi^{1/\alpha-1}(z)\varphi'(z)}\,. \tag{15}$$

Theorem 1 asserts that under the given conditions $\operatorname{Re}q(z)\geq 0$ and from (13) it follows that $\operatorname{Re}G(z)\geq 0$. $G(0)=c$ and $q(0)=1$. Let $Q(z)=G(z)+\dfrac{q(z)}{\alpha}-c-\dfrac{1}{\alpha}+1$. Then $Q(0)=1$. Then (15) can be written as

$$\frac{zp'(z)}{Q(z)+c+1/\alpha-1}+p(z)=\frac{f^{1/\alpha-1}(z)f'(z)}{\varphi^{1/\alpha-1}(z)\varphi'(z)}$$

with $\operatorname{Re}\left(Q(z)+c+1/\alpha-1\right)\geq 0$. As $f\in P(\alpha)$ we have

$$\left|\arg\left(\frac{zp'(z)}{Q(z)+c+1/\alpha-1}+p(z)\right)\right|\leq\frac{\pi}{2\alpha}\quad\text{where}\quad\operatorname{Re}\left(Q(z)+c+1/\alpha-1\right)\geq 0.$$

Since $\alpha\geq 1$, this with Lemma 1 implies that $|\arg p(z)|\leq\dfrac{\pi}{2\alpha}$ which in turn shows that

$$\left|\arg\left(\frac{F'^{\alpha}(z)F^{1-\alpha}(z)}{\Phi'^{\alpha}(z)\Phi^{1-\alpha}(z)}\right)\right|=|\arg p^{\alpha}(z)|\leq\frac{\pi}{2}$$

where $\Phi\in M_{\alpha}$. This shows that $F\in P(\alpha)$.

Remark 3. For $\alpha=1$ we get Theorem 2 of P. T. Mocanu [5] and for $g(z)=z$ we get Theorem 1 of K. S. Padmanabhan and R. Bharati [7] for real c such that $\alpha(c-1)\leq 1$.

Theorem 3. *Let $g\in H(E)$ with $g(0)=g'(0)-1=0$, $\dfrac{g(z)g'(z)}{z}\neq 0$ in E and let $f\in S^{*}$. Define $F=A_g(f)$ where $c>0$, $\alpha>0$ and $(c+1)\alpha>1$. If $\exists\ J\in\mathbb{R}$ such that $J>J_2=\max\left(\dfrac{1/\alpha+c}{2},J_1,c\right)$ where J_1 is the positive root of the equation*

$$2J^2+J-\left(c+\tfrac{1}{\alpha}\right)=0\ , \tag{16}$$

and

$$c\,\mathrm{Re}\left(\frac{z g'(z)}{g(z)}\right) \le J < c + \frac{1}{\alpha}\,, \tag{17}$$

$$\mathrm{Re}\left\{(c-1)\frac{z g'(z)}{g(z)} + \left(1 + z\frac{g''(z)}{g'(z)}\right)\right\} \ge J - \lambda(J)\,, \qquad z \in E \tag{18}$$

where $\lambda(J) = \left(c + \frac{1}{\alpha} - J\right)/2J$, *then* $F \in S^*$.

Proof. The hypothesis (17) implies $c \le J$ and since $J > J_2 = \max\left(\frac{1/\alpha + c}{2}, J_1, c\right)$ we have $\frac{c + 1/\alpha}{2} \le J < c + \frac{1}{\alpha}$ which implies $0 < c + \frac{1}{\alpha} - J \le J$. So

$$\tfrac{1}{2}\min\left\{\frac{c + 1/\alpha - J}{J}, \frac{J}{c + 1/\alpha - J}\right\} = \frac{c + 1/\alpha - J}{2J} = \lambda(J) \tag{19}$$

Again $J - \lambda(J) = \dfrac{2J^2 + J - (c + 1/\alpha)}{2J} > 0$ provided $J > J_1$ – the positive root of $2J^2 + J - (c + 1/\alpha) = 0$. Let J_0 be the positive root of the equation $2J^2 + (1-2c)J - -(c + 1/\alpha) = 0$. (18) implies $J - \lambda(J) \le c$ or $2J^2 + (1-2c)J - (c + 1/\alpha) \le 0$ which will hold if $J \le J_0$. Since $2J_0{}^2 + J_0 - (c + 1/\alpha) = 2c J_0 > 0$, it follows that $J_0 < J_1$. Also clearly $J_0 \ge c$. If $J_2 = \max(c, \frac{c + 1/\alpha}{2}, J_1)$ then $J_0 > J_2$. Further, $(c+1)\alpha > 1$ implies $J_0 > (c + 1/\alpha)/2$. Thus $J_0 > J_2$. Let $J \in [J_2, J_0]$. Then $0 < J - \lambda(J) \le c$. Set of $g(z) = z\phi(z)$, $\Phi(z) = \phi^c(z)$, $\Phi \in H(E)$ with $\Phi(0) = \phi(0) = 1$. Since $\frac{g(z)}{z} \ne 0$ in E we have $\phi(z) \ne 0$ in E which implies $\phi^{c-1}(z) \in H(E)$. $\Phi(z) \ne 0$ in E, since $\phi(z) \ne 0$ in E.

$$F(z) = \left(\frac{c + 1/\alpha}{z^c\Phi(z)}\int_0^z g^{c-1}(t)g'(t)f^{1/\alpha}(t)\,dt\right)^\alpha = \left(\frac{c + 1/\alpha}{z^c\Phi(z)}\int_0^z t^{c-1}Q(t)f^{1/\alpha}(t)\,dt\right)^\alpha$$

where $Q(z) = \phi^{c-1}(z)g'(z) \in H(E)$, $Q(z) \ne 0$ in E with $Q(0) = 1$. Also

$$\frac{z\Phi'(z)}{\Phi(z)} = cz\frac{\phi'(z)}{\phi(z)} = c\left(\frac{z g'(z)}{g(z)} - 1\right);$$

$$\frac{zQ'(z)}{Q(z)} = (c-1)\frac{z\phi'(z)}{\phi(z)} + \frac{z g''(z)}{g'(z)}\,.$$

In Theorem B, if we put $\beta = 1/\alpha$, $\gamma = \delta = c$ and change α to $1/\alpha$, the operator (7) yields our operator for our choice of Φ and Q. Then

$$\begin{aligned} c + \mathrm{Re}\frac{zQ'(z)}{Q(z)} &= c + \mathrm{Re}\left\{z\frac{g''(z)}{g'(z)} + (c-1)z\frac{\phi'(z)}{\phi(z)}\right\} = \\ &= c + \mathrm{Re}\left\{z\frac{g''(z)}{g'(z)} + (c-1)\left(z\frac{g'(z)}{g(z)} - 1\right)\right\} = \\ &= \mathrm{Re}\left\{1 + z\frac{g''(z)}{g'(z)} + (c-1)z\frac{g'(z)}{g(z)}\right\} \\ &\ge J - \lambda(J) = \max\{0, J - \lambda(J)\} \end{aligned}$$

from (18) ; (5) is satisfied. From (17) we obtain

$$c + \mathrm{Re}\left(z\frac{\Phi'(z)}{\Phi(z)}\right) = \mathrm{Re}\left(cz\frac{g'(z)}{g(z)}\right) \leq J \leq c + \frac{1}{\alpha}$$

whence (3) and (4) are satisfied. (19) shows that (16) is fulfilled. Thus conditions in Theorem B are all satisfied and the conclusion follows as an application of Theorem B.

Remark 4. If we put $c = \alpha = 1$ we obtain Theorem 1 of S.S.Miller, P.T.Mocanu, and M.O.Reade [3].

We now prove a theorem which serves as an example to Theorem 3.

Theorem 4. *Suppose* $f \in S^*$ *and* $|\lambda| \leq \varrho_0$ *where* ϱ_0 *is the positive root of* $(c+1/\alpha)\varrho^3 + (2c - 1/\alpha)\varrho^2 - (4c^2 + 3c + 1/\alpha)\varrho + 1/\alpha = 0$, *lying in* $(0,1)$. *Then* F *defined by* $F(z) = \left\{\frac{(c+1/\alpha)(1+\lambda z)^c}{z^c}\int_0^z \frac{t^{c-1}}{(1+\lambda t)^{c+1}} f^{1/\alpha}(t)\,dt\right\}^\alpha$, $z \in E$ *belongs to* S^*.

Proof. Choose $g(z) = \frac{z}{1+\lambda z}$, $|\lambda| = \varrho < 1$ in Theorem 3. The condition $c\,\mathrm{Re}\, z\frac{g'(z)}{g(z)} \leq J$ becomes

$$\mathrm{Re}\frac{c}{1+\lambda z} \leq \frac{c}{1-\varrho} \leq J \tag{20}$$

and the condition $\mathrm{Re}\left\{(c-1)z\frac{g'(z)}{g(z)} + 1 + z\frac{g''(z)}{g'(z)}\right\} \geq J - \lambda(J)$ becomes

$$\begin{aligned}\mathrm{Re}\left(\frac{c-1}{1+\lambda z} + \frac{1-\lambda z}{1+\lambda z}\right) \geq \frac{c-1}{1+\varrho} + \frac{1-\varrho}{1+\varrho} = \frac{c-\varrho}{1+\varrho} \geq J - \lambda(J) = \\ = \frac{2J^2 + J - (c+1/\alpha)}{2J}\end{aligned} \tag{21}$$

If we take $J = \frac{c}{1-\varrho}$, then the above inequality will hold if

$$\frac{2c^2}{(1-\varrho)^2} + \frac{c}{1-\varrho} - (c+1/\alpha) \leq \frac{2c(c-\varrho)}{1-\varrho^2}\ ;$$

or $T(\varrho) \equiv (c+1/\alpha)\varrho^3 + (2c-1/\alpha)\varrho^2 - (4c^2 + 3c + 1/\alpha)\varrho + 1/\alpha \geq 0$. Let ϱ_0 be the positive root of $T(\varrho) = 0$, in $(0,1)$. Then for $0 \leq \varrho \leq \varrho_0$ the inequality $T(\varrho) \geq 0$ holds and so (20) and (21) hold. Also

$$\varrho_0 < \frac{1}{c\alpha + 1}$$

This implies that for $\varrho \in [0, \varrho_0]$, $\varrho < \frac{1}{c\alpha+1} \Longrightarrow J < c + \frac{1}{\alpha}$. Hence Theorem 4 follows from Theorem 3.

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STRESZCZENIE

W pracy tej autorzy zajmują się operatorem całkowym $A_g(f) = F$, określonym wzorem (1), i działającym na funkcje f holomorficzne w kole jednostkowym E. Znajdują warunki na funkcję g i stałą c we wzorze (1) zapewniające prawdziwość implikacji : $f \in M_\alpha \Rightarrow F \in M_\alpha$, gdzie M_α jest klasą Mocanu. Analogiczny wynik otrzymują dla klasy S^* funkcji gwiaździstych.

SUMMARY

The authors deal with the integral operator $A_g(f) = F$ defined by the formula (1) and acting on f holomorphic in the unit disk E. They find conditions on the function g and the constant c in the formula (1) for the implication : $f \in M_\alpha \Rightarrow F \in M_\alpha$ to be satisfied, where M_α denotes the Mocanu class. An analogous result for the class S^* of starlike functions was obtained.