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A New Approach to the Krzyż Conjecture

Abstract. It has been conjectured by Krzyż [15] that if $0<|a_0+a_1z+a_2z^2+\ldots|\leqslant 1$ for $|z|<1$, then $|a_n|\leqslant 2/e$ for all $n\geqslant 1$. The aim of this paper is to present some new related problems. In particular, solving a moment problem, we find a simple proof of the Krzyż conjecture for $n\leqslant 4$.

1. Introduction

Let $\mathcal{H}(\Delta)$ denote the set of complex functions f analytic on the unit disc $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $a_n(f) = f^{(n)}(0)/n!$, $n = 0, 1, \ldots$. For $\mathcal{W} \subset \mathcal{H}(\Delta)$ we define

$$A_n(\mathcal{W}) = \sup\{|a_n(f)| : \ f \in \mathcal{W}\}, \quad n = 0, 1, 2, \ldots .$$

We will consider the following classes of bounded functions:

$$(1) \quad \mathcal{B} = \left\{f \in \mathcal{H}(\Delta) : \ f(\Delta) \subset \overline{\Delta}\right\} \ \text{ and } \mathcal{B}_0 = \{f \in \mathcal{B} : \ 0 \notin f(\Delta)\}.$$

The Krzyż conjecture [15], still remaining open, asserts that

$$A_n(\mathcal{B}_0) = 2/e \ \text{ for all } \ n \geqslant 1$$

with equality only for the functions

$$z \mapsto \xi \exp\left[-\left(1 + \eta z^n\right)/(1 - \eta z^n)\right], \ |\xi| = |\eta| = 1.$$

This coefficient problem has attracted the attention of many mathematicians, see e.g. [3, 4, 6, 8, 10, 11, 13, 15–18], and it is known that

$$\text{(I)} \qquad A_1(\mathcal{B}_0) = A_2(\mathcal{B}_0) = 2/e \text{ (easy to prove)},$$

$$\text{(II)} \qquad |a_3(f)| \leqslant \Phi(a_0(f)) \leqslant 2/e \text{ for all } f \in \mathcal{B}_0,$$

where the expressions for Φ, depending on several cases, can be found in [8, 11, 18].

Furthermore, D. Bshouty, J. E. Brown, Delin Tan, R. Ermers and others claim they have proved

$$\text{(III)} \qquad A_4(\mathcal{B}_0) = 2/e,$$

but not all of them give full details. Also, the use of computers in their calculations is too extensive.

A uniform bound

$$\text{(IV)} \qquad A_n(\mathcal{B}_0) \leqslant 1 - \frac{1}{3\pi} + \frac{4}{\pi}\sin\frac{1}{12} = 0.9998\ldots, \quad n \geqslant 1,$$

is due to Horowitz [1] and

$$\text{(V)} \qquad A_n(\mathcal{B}_0) \leqslant \frac{4}{5} + \frac{4}{\pi}\sin\frac{\pi}{20} = 0.9991\ldots, \quad n \geqslant 1,$$

was obtained by Ermers [8]. Both bounds are far away from $2/e$.

The standard calculus seems to be useless in the Krzyż conjecture. By a simple variational technique we get that

$$\begin{aligned} A_n(\mathcal{B}_0) &= \sup\Big\{\operatorname{Re} a_n\Big(\exp\Big[-\sum_{j=1}^{n}\lambda_j p(e^{i\theta_j}, \cdot\,)\Big]\Big)\Big\} \\ &= \max\Big\{\exp\Big[-\sum_{j=1}^{n}\lambda_j\Big]\operatorname{Re}\left[U\left(\ldots,\lambda_j,\ldots,e^{i\theta_s},\ldots\right)\right]\Big\}, \end{aligned}$$

where $p(\xi, z) \equiv (1+\xi z)/(1-\xi z)$ and the maximum is taken over all $\lambda_j > 0$ and $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ (here U is a polynomial of several variables). Hence the equations for critical points and the shape of boundary surfaces of various dimensions are very involved.

This way, subordination techniques [7, 19, 20] seem to be the main tool in solving the coefficient problem. Neglecting rotations it is sufficient to consider the Krzyż conjecture within the class

$$\widehat{\mathcal{B}}_0 = \bigcup_{t\geqslant 0}\{f \in \mathcal{H}(\Delta) : \ f \prec h_t \text{ in } \Delta\},$$

where

$$h_t \equiv \exp\left[-t\frac{1-z}{1+z}\right] = e^{-t} + 2\sum_{j=1}^{\infty} b_j(t) z^j$$

$$(2) \qquad = e^{-t} + e^{-t}\sum_{j=1}^{\infty}(-1)^j [L_j(2t) - L_{j-1}(2t)] z^j$$

$$= e^{-t} + \sum_{j=1}^{\infty}\left\{(-1)^j e^{-t}\sum_{k=1}^{j}\binom{j-1}{k-1}(-2t)^k/k!\right\} z^j$$

and L_j is the j-th Laguerre polynomial. Observe that h_t is a non-vanishing inner function so that

$$4\sum_{j=1}^{\infty} b_j^2(t) = 1 - e^{-2t}.$$

The relation $f \prec h_t$ in Δ means that for some $\omega \in \mathcal{B}$ with $\omega(0) = 0$ we have

$$f(z) \equiv h_t(\omega(z)) = e^{-t} + 2\sum_{j=1}^{\infty} b_j(t) \sum_{n=j}^{\infty} c_n^{(j)} z^n$$

$$= e^{-t} + 2\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n} c_n^{(j)} b_j(t)\right] z^n,$$

where the coefficients $c_s^{(j)}$, $s \geqslant j$, are generated by

$$(3) \qquad \omega^j(z) \equiv \sum_{s=j}^{\infty} c_s^{(j)} z^s, \ j = 1, 2, \ldots.$$

Thus

$$(4)\qquad a_n(f) = 2\sum_{j=1}^{n} c_n^{(j)} b_j(t) \quad \text{for } f \in \widehat{\mathcal{B}}_0 \quad \text{and} \quad n \geqslant 1.$$

By the subordination principle,

$$\frac{1}{2}\left|a_n\left(\frac{1+\xi\omega}{1-\xi\omega}\right)\right| = \left|a_n\left(\frac{\xi\omega}{1-\xi\omega}\right)\right| \leqslant 1 \quad \text{for all } |\xi| \leqslant 1,\ n \geqslant 1,$$

so the Schwarz lemma gives

$$(5)\qquad \left|\sum_{j=1}^{n} c_n^{(j)} \zeta^{j-1}\right| \leqslant 1 \quad \text{for all } |\zeta| \leqslant 1, \text{ and } n \geqslant 1.$$

Let us mention that the famous de Branges theorem [1] implies

$$(6)\qquad \left|\sum_{j=1}^{n} c_n^{(j)} a_j(F)\right| \leqslant n \quad \text{for all } F \in \mathcal{S} \quad \text{and } n \geqslant 1,$$

where $\mathcal{S} \subset \mathcal{H}(\Delta)$ is the well-known class of univalent functions F on Δ, normalized by $F(0) = F'(0) - 1 = 0$. In the past the inequality (6) was considered as the Rogosinski conjecture, see [7].

Where is the difficulty in estimating the coefficients (4) situated? Well, this problem is related to a hard non-linear problem concerning one of the two homeomorphic classes:

$$(7)\qquad \begin{aligned} &\Omega = \{\omega \in \mathcal{B} :\ \omega(0) = 0\} \quad \text{or} \\ &\mathcal{P} = \{f \in \mathcal{H}(\Delta) :\ f(0) = 1,\ \operatorname{Re} f > 0 \text{ on } \Delta\}. \end{aligned}$$

Even the $(2/e)$-bound for coefficients of the superordinate functions (2) needs some hard numerical calculations. The authors of [13] have just calculated that

$$2|b_n(t)| = |a_n(h_t)| \leqslant 2/e \text{ for all } t > 0 \text{ and } n \geqslant 21139,$$

so it remains to check a finite (but not small) number of initial functions $\{b_j\}$. Thus the Krzyż problem is one of great difficulty.

Ten and more years ago the Krzyż conjecture looked considerably easier than that of Bieberbach. During a meeting of the Krzyż seminar at the Maria Curie–Skłodowska University, I proposed to estimate (4) just by means of the relation (5). In other words, our problem lies in calculating

(8)
$$d_n(t) = 2\sup\left\{\left|\sum_{j=1}^{n} c_{j-1} b_j(t)\right| : \ c_j \in \mathbb{C}, \left|\sum_{j=0}^{n-1} c_j \zeta^j\right| \leqslant 1 \text{ for } |\zeta| \leqslant 1\right\}, \ t \geqslant 0$$

and

(9) $$D_n = \sup d_n((0,\infty)), \ n \geqslant 1.$$

Clearly, $A_n(\mathcal{B}_0) \leqslant D_n$ and the sequence (D_n) is non-decreasing. Unfortunately, an information on extreme points of the closed unit ball in the space of polynomials of degree at most $n-1$ is not sufficient to estimate (8)–(9), see [2]. Moreover, we have

Theorem 1.
$$\lim_{n\to\infty} D_n = 1.$$

Hence we cannot get more than $A_n(\mathcal{B}_0) \leqslant D_n < 1$. By the Horowitz uniform bound (IV), the set of polynomials $\zeta \mapsto \sum_{j=1}^{n} c_n^{(j)} \zeta^{j-1}$ created by means of relations (3) and its convex hull differ essentially from the set of polynomials of degree at most $n-1$, bounded by 1 on Δ, provided n is large. Fortunately, like for the Krzyż conjecture, we have

Theorem 2.
$$D_1 = D_2 = D_3 = D_4 = 2/e.$$

What could be obtained by studying (8)–(9)? By definition, we have

(10) $$\{D_n \leqslant c\} \Rightarrow \{D_m \leqslant c \quad \text{for all } \ 1 \leqslant m \leqslant n\}$$

and

(11) $$\{D_n = 2/e\} \Rightarrow \{D_m = 2/e \quad \text{for all} \quad 1 \leqslant m \leqslant n\}.$$

For the Krzyż conjecture we know only that

(10′) $$\{A_n(\mathcal{B}_0) \leqslant c\} \Rightarrow \{A_m(\mathcal{B}_0) \leqslant c \quad \text{for all} \quad m \mid n\}$$

and

(11′) $$\{A_n(\mathcal{B}_0) = 2/e\} \Rightarrow \{A_m(\mathcal{B}_0) = 2/e \quad \text{for all} \quad m \mid n\}.$$

Also the least upper bound in (8) may be taken over polynomials with real coefficients, since we have

Theorem 3. *For $n \geqslant 1$ and $t \geqslant 0$,*

(i)

$$d_n(t) = 2\sup\left\{\sum_{j=1}^{n} c_{j-1}b_j(t) : \; c_j \in \mathbb{R}, \; \left|\sum_{j=0}^{n-1} c_j\zeta^j\right| \leqslant 1 \text{ for } |\zeta| \leqslant 1\right\}$$

and

(ii)

$$d_n(t) =$$

$$2\sup\left\{\sum_{j=1}^{n} c_{j-1}\frac{t^j e^{-t}}{j!} : \; c_j \in \mathbb{R}, \; \left|\sum_{j=0}^{n-1} c_j\zeta^j\right| \leqslant 1 \;\; \textit{for} \;\; |2\zeta - 1| \leqslant 1\right\} =$$

$$2\sup\left\{\left|\sum_{j=1}^{n} c_{j-1}\frac{t^j e^{-t}}{j!}\right| : \; c_j \in \mathbb{C}, \; \left|\sum_{j=0}^{n-1} c_j\zeta^j\right| \leqslant 1 \text{ for } \; |2\zeta - 1| \leqslant 1\right\}.$$

2. Open questions

Problem 1. Up to what number n does $D_n = 2/e$?

Problem 2. Does the estimate

(12) $$\left|\sum_{j=1}^{n} c_j a_j(F)\right| \leqslant n \quad \text{for all} \quad F \in \mathcal{S}$$

hold whenever

$$\left|\sum_{j=1}^{n} c_j\zeta^{j-1}\right| \leqslant 1 \quad \text{for all } |\zeta| \leqslant 1? \tag{13}$$

Observe that by Bernstein's inequality [7] we have an equivalent form of (13):

$$\left|\sum_{j=1}^{n} c_j(j-1+k)\zeta^{j-1}\right| \leqslant n-1+k \tag{13'}$$

for all $|\zeta| \leqslant 1$ and $k = 0, 1, 2, \ldots$.

Thus (13) $\Rightarrow$ (12), if we replace $\mathcal{S}$ by its subset $\mathcal{S}^*$ consisting of functions starlike with respect to the origin, or by the closed convex hull of $\mathcal{S}^*$, for the form of the closed convex hull of $\mathcal{S}^*$ see [9, 20].

Problem 3. If the answer for the Problem 2 is 'No', determine or estimate

$$\widetilde{d}_n(t) = 2\sup\left\{\sum_{j=1}^{n} c_j b_j(t) : \ c_j \in \mathbb{R} \text{ and (12)–(13) hold}\right\}, \ t \geqslant 0,$$

and

$$\widetilde{D}_n = \sup \widetilde{d}_n((0,\infty)), \ n = 1, 2, \ldots,$$

where $\{b_j\}$ are given by (2). Obviously , $A_n(\mathcal{B}_0) \leqslant \widetilde{D}_n \leqslant D_n$ for $n \geqslant 1$, see the proof of Theorem 3.

3. Related problems and the proof of Theorems 1 and 3

In this section we shall discuss more general extremal problems than those considered in Theorems 1 and 3. We begin with some notation.

For any $u : G \mapsto \mathbb{C}$ we define $\|u\|_G = \sup\{|u(\alpha)| : \ \alpha \in G\}$.

The class C_0 consists of all complex functions h continuous on $[0, \infty)$ with $h(+\infty) = 0$ so that

$$\|h\|_{[0,\infty)} = \max\{|h(t)| : \ 0 \leqslant t < \infty\}.$$

For $r > 0$ we write $\Delta_r = \{z : |z| < r\}$ so that $\Delta_1 = \Delta$, and let $K = \{z : |2z - 1| < 1\}$.

We will work within the classes

$$\mathcal{H}^2 = \{f \in \mathcal{H}(\Delta) : \|f\| < \infty\}, \tag{14}$$

where $\|f\| = \left(\sum_{j=0}^{\infty} |a_j(f)|^2\right)^{1/2}$, and

(15)
$$\mathcal{P}_n = \left\{f \in \mathcal{H}(\Delta) : f^{(j)}(z) \equiv 0 \text{ for all } j > n\right\}, \ n = 0, 1, 2, \ldots .$$

Consider now the following two linear operators:

$$H : \mathcal{H}^2 \to C_0 \quad \text{and} \quad V : \mathcal{H}^2 \to C_0$$

defined by

$$(Hf)(t) \equiv f * h_t \quad \text{and} \quad (Vf)(t) \equiv f * v_t,$$

where the operation $*$ is given by

$$f * g = \sum_{j=1}^{\infty} a_{j-1}(f) a_j(g),$$

the functions h_t are specified in (2) and

$$v_t(z) \equiv 2e^{t(z-1)}.$$

Both operators H and V are well-defined. Indeed, for every $f \in \mathcal{H}^2$ we have

$$\|Hf\|_{[0,\infty)} \leqslant \|f\| \text{ and } \|Vf\|_{[0,\infty)} \leqslant 2\|f\|,$$

so the series Hf and Vf converge absolutely and uniformly on $[0, \infty)$. Moreover,

$$\limsup_{t\to\infty} |f * h_t| \leqslant 2 \limsup_{t\to\infty} \left|\sum_{j=1}^{M} a_{j-1}(f) b_j(t)\right| + 2\sqrt{\varepsilon} = 2\sqrt{\varepsilon}$$

whenever $\sum_{j=M}^{\infty} |a_j(f)|^2 < \varepsilon$, which means that $Hf(+\infty) = 0$. Similarly, $Vf(+\infty) = 0$.

Let $\mathcal{A}$ be one of the classes (14)–(15) and let $\mathcal{A}^{\mathbb{R}} = \{f \in \mathcal{A} : f((-1,1)) \subset \mathbb{R}\}$. We are interested in the following bounds

$$\begin{aligned} d(t,\mathcal{A}) &= \sup\{|Hf(t)| : \ f \in \mathcal{A} \text{ and } \|f\|_{\Delta} \leqslant 1\}, \\ d(t,\mathcal{A}^{\mathbb{R}}) &= \sup\{Hf(t) : \ f \in \mathcal{A}^{\mathbb{R}} \text{ and } \|f\|_{\Delta} \leqslant 1\}, \\ D(\mathcal{W}) &= \sup\{d(t,\mathcal{W}) : \ 0 \leqslant t \leqslant \infty\}, \ \mathcal{W} = \mathcal{A} \text{ or } \mathcal{A}^{\mathbb{R}}, \end{aligned}$$

and, analogously,

$$\begin{aligned} q(t,\mathcal{A}) &= \sup\{|Vf(t)| : \ f \in \mathcal{A} \text{ and } \|f\|_{K} \leqslant 1\}, \\ q(t,\mathcal{A}^{\mathbb{R}}) &= \sup\{Vf(t) : \ f \in \mathcal{A}^{\mathbb{R}} \text{ and } \|f\|_{K} \leqslant 1\}, \\ Q(\mathcal{W}) &= \sup\{q(t,\mathcal{W}) : \ 0 \leqslant t < \infty\}, \ \mathcal{W} = \mathcal{A} \text{ or } \mathcal{A}^{\mathbb{R}}. \end{aligned}$$

Observe that $d(t,\mathcal{P}_{n-1}) \equiv d_n(t)$, $D(\mathcal{P}_{n-1}) = D_n$, see (8)–(9), and Theorem 3 can be turned into

(i) $d(t,\mathcal{P}_{n-1}) = d\left(t,\mathcal{P}_{n-1}^{\mathbb{R}}\right)$,
(ii) $d(t,\mathcal{P}_{n-1}) = q\left(t,\mathcal{P}_{n-1}^{\mathbb{R}}\right) = q(t,\mathcal{P}_{n-1})$, $n = 1,2,\ldots$.

Lemma 1. *For all $n \geqslant 1$ and $t \geqslant 0$ we have $d(t,\mathcal{P}_{n-1}) = q(t,\mathcal{P}_{n-1}) \leqslant q(t,\mathcal{P}_n) \leqslant q\left(t,\mathcal{H}^2\right) \leqslant d\left(t,\mathcal{H}^2\right) \leqslant \sqrt{1-e^{-2t}}$. In particular, $2/e \leqslant D(\mathcal{P}_{n-1}) = Q(\mathcal{P}_{n-1}) \leqslant Q(\mathcal{P}_n) \leqslant Q\left(\mathcal{H}^2\right) \leqslant D\left(\mathcal{H}^2\right) \leqslant 1$.*

Proof. We first prove the relations

$$d(t,\mathcal{P}_{n-1}) = q(t,\mathcal{P}_{n-1}), \ n = 1,2,\ldots \text{ and } t \geqslant 0. \tag{16}$$

Let $f, g \in \mathcal{P}_{n-1}$ be interrelated by the identity $g(z) \equiv f(2z-1)$. Then $\|f\|_{\Delta} \leqslant 1$ iff $\|g\|_K \leqslant 1$. Moreover, for any $t > 0$,

$$Hf(t) = 2\sum_{j=1}^{n} a_{j-1}(f)b_j(t) = \sum_{j=1}^{n} a_{j-1}(f)(-1)^j e^{-t} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{(-2t)^k}{k!}$$

$$= \sum_{k=1}^{n}\left[(-2)^k \sum_{j=k}^{n}\binom{j-1}{k-1}(-1)^j a_{j-1}(f)\right]\frac{e^{-t}t^k}{k!}$$
$$= 2\sum_{k=1}^{n} a_{k-1}(g)\frac{e^{-t}t^k}{k!} = Vg(t).$$

Hence (16) holds. Since $\mathcal{P}_{n-1} \subset \mathcal{P}_n \subset \mathcal{H}^2$, we get $d(t,\mathcal{P}_{n-1}) = q(t,\mathcal{P}_{n-1}) \leqslant d(t,\mathcal{P}_n) = q(t,\mathcal{P}_n) \leqslant \min\{d(t,\mathcal{H}^2), q(t,\mathcal{H}^2)\}$. Moreover,

$$d(t,\mathcal{H}^2) \leqslant \left(\sum_{j=1}^{\infty} a_j^2(h_t)\right)^{1/2} = 2\left(\sum_{j=1}^{\infty} b_j^2(t)\right)^{1/2} = \sqrt{1-e^{-2t}}$$

and

$$D(\mathcal{P}_0) = 2/e\ ,$$

so it suffices to show that

$$q(t,\mathcal{H}^2) \leqslant d(t,\mathcal{H}^2) \quad \text{for all} \quad t \geqslant 0.$$

The proof requires the following elementary formulas

$$\sum_{j=k}^{\infty}\binom{j-1}{k-1}\zeta^{j-1} = \zeta^{k-1}\sum_{s=0}^{\infty}\binom{k+s-1}{s}\zeta^s$$
$$= \frac{\zeta^{k-1}}{(1-\zeta)^k} \ \text{for}\ |\zeta|<1,\ k=1,2,\ldots, \tag{17}$$

$$\sum_{j=k}^{\infty}\binom{j-1}{k-1}\frac{1}{2^j} = 2,\ k=1,2,\ldots, \tag{18}$$

$$\binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k} \ \text{for}\ 0\leqslant k\leqslant j\leqslant n, \tag{19}$$

$$\phi(k,n) \stackrel{\text{def}}{=} \sum_{j=k}^{n}(-1)^j\binom{n}{j}\binom{j}{k}$$
$$= \begin{cases} 0 & \text{if } k=0,1,\ldots,n-1, \\ (-1)^n & \text{if } k=n, \end{cases} \tag{20}$$

$$\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k} = \binom{n}{k}2^{n-k} \ \text{for}\ k=0,1,\ldots,n. \tag{21}$$

Let $g \in \mathcal{H}^2$, $\|g\|_K \leqslant 1$ and $0 < r < 1$. The functions $f(z) \equiv g((1+z)/2)$ and $f_r(z) \equiv g(r(1+z)/2)$ are in $\mathcal{B} \subset \mathcal{H}^2$ and

$$\lim_{r \to 1^-} (Hf_r)(t) = (Hf)(t) \text{ for all } t \geqslant 0. \tag{22}$$

Indeed, fix $t \geqslant 0$ and observe that $f_r \to f$ as $r \to 1$ uniformly on compact subsets of Δ, and

$$|(Hf_r)(t) - (Hf)(t)| \leqslant 2 \sum_{j=1}^{M} |a_{j-1}(f_r - f)||b_j(t)| + 8\sqrt{\varepsilon}$$

whenever $\sum_{j=M+1}^{\infty} |b_j(t)|^2 < \varepsilon$. Moreover,

$$\begin{aligned}\sum_{k=1}^{\infty} a_{k-1}(f_r) z^{k-1} &= f_r(z) = g(r(1+z)/2) \\ &= \sum_{j=1}^{\infty} a_{j-1}(g)(r/2)^{j-1} \sum_{k=1}^{j} \binom{j-1}{k-1} z^{k-1} \\ &= \sum_{k=1}^{\infty} \left[\sum_{j=k}^{\infty} \binom{j-1}{k-1} a_{j-1}(g) r^{j-1} / 2^{j-1} \right] z^{k-1}.\end{aligned}$$

Hence for arbitrary $t > 0$,

$$\begin{aligned}(Hf_r)(t) = \sum_{k=1}^{\infty} \Bigg[\sum_{j=k}^{\infty} &\binom{j-1}{k-1} a_{j-1}(g)\, r^{j-1} / 2^{j-1} \\ &\times \sum_{s=1}^{k} e^{-t} (-1)^k \binom{k-1}{s-1} (-2t)^s / s! \Bigg].\end{aligned}$$

But

$$e^{-t} \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \sum_{s=1}^{k} \binom{j-1}{k-1} \binom{k-1}{s-1} |a_{j-1}(g)| \frac{r^{j-1}}{2^j} \frac{(2t)^s}{s!}$$

$$\leqslant e^{-t}\sum_{s=1}^{\infty}\sum_{j=s}^{\infty}\left[\sum_{k=s}^{j}\binom{j-1}{k-1}\binom{k-1}{s-1}\right]\frac{r^{j-1}}{2^{j}}\frac{(2t)^{s}}{s!}$$

$$\overset{(21)}{=} e^{-t}\sum_{s=1}^{\infty}\sum_{j=s}^{\infty}\binom{j-1}{s-1}2^{j-s}\frac{r^{j-1}}{2^{j}}\frac{2^{s}t^{s}}{s!}$$

$$\overset{(17)}{=} e^{-t}\sum_{s=1}^{\infty}\frac{r^{s-1}}{(1-r)^{s}}\frac{t^{s}}{s!}<\frac{1}{r}e^{tr/(1-r)-t}<\infty,$$

so the triple series is commutative and

$$(Hf_r)(t)=\sum_{s=1}^{\infty}\left[\sum_{j=s}^{\infty}\left\{\sum_{k=s}^{j}\binom{j-1}{k-1}\binom{k-1}{s-1}(-1)^{k}\right\}\right.$$
$$\left.\times\frac{a_{j-1}(g)r^{j-1}}{2^{j-1}}\frac{e^{-t}(-2t)^{s}}{2s!}\right]$$
$$=\sum_{s=1}^{\infty}\sum_{j=s}^{\infty}\{-\phi(s-1,j-1)\}\frac{a_{j-1}(g)r^{j-1}}{2^{j-1}}\frac{e^{-t}(-2t)^{s}}{2s!}$$
$$\overset{(20)}{=}\sum_{s=1}^{\infty}a_{s-1}(g)r^{s-1}\frac{e^{-t}t^{s}}{s!}\to (Vg)(t) \text{ as } r\to 1.$$

By (22) we obtain

$$Hf(t)=Vg(t)\quad\text{for all } t\geqslant 0.$$

Thus

$$q\left(t,\mathcal{H}^2\right)=\sup\left\{|Hf(t)| : f(z)\equiv g\left(\frac{1+z}{2}\right)\text{ for some } g\in\mathcal{H}^2\right.$$
$$\left.\text{with } \|g\|_K\leqslant 1\right\}\leqslant d\left(t,\mathcal{H}^2\right),$$

which completes the proof.

Lemma 2. *For all* $t\geqslant 0$,

$$1-e^{-t}\leqslant\lim_{n\to\infty}d_n(t)\leqslant\sqrt{1-e^{-2t}}.$$

In particular,

$$\lim_{n\to\infty} D(\mathcal{P}_{n-1}) = \lim_{n\to\infty} Q(\mathcal{P}_{n-1}) = Q\left(\mathcal{H}^2\right) = D\left(\mathcal{H}^2\right) = 1 .$$

Proof. Because of Lemma 1, it is sufficient to prove that for any $t > 0$ we have

$$1 - e^{-t} \leqslant \lim_{n\to\infty} d(t, \mathcal{P}_{n-1}) = \lim_{n\to\infty} d_n(t).$$

For $f \in \mathcal{H}(\Delta)$ denote

$$(f)_n(z) \equiv \sum_{j=1}^{n} a_{j-1}(f) z^{j-1} ,$$

the n-th partial section of f. Fix $t \geqslant 0$ and put $f_t(z) \equiv [h_t(z) - e^{-t}] / [(1 + e^{-t}) z]$. Since $\|f_t\|_\Delta \leqslant 1$ and since $(h_t)_n \to h_t$ and $(f_t)_n \to f_t$ as $n \to \infty$ uniformly on compact subsets of Δ, for every positive integer k there exists a positive integer $n_k \geqslant k$ such that

$$\|(f_t)_n\|_{\Delta_{1-1/k}} \leqslant \|(f_t)_n - f_t\|_{\Delta_{1-1/k}} + \|f_t\|_{\Delta_{1-1/k}} \leqslant 1 + 1/k \text{ for } n \geqslant n_k.$$

Consider the functions

$$g_{k,t}(z) \equiv \frac{k}{k+1} (f_t)_{n_k}((1 - 1/k)z), \;\; k = 1, 2, \ldots .$$

Obviously, all the $g_{k,t}$ are in $\mathcal{B}$ and

$$\begin{aligned} \lim_{n\to\infty} d_n(t) &\geqslant (H g_{k,t})(t) = 2 \sum_{j=1}^{n_k} a_{j-1}(g_{k,t}) b_j(t) \\ &= 4 \sum_{j=1}^{n_k} \frac{b_j^2(t)(1 - 1/k)^{j-1}}{(1 + e^{-t})(1 + 1/k)} \\ &\geqslant \frac{4}{(1 + e^{-t})(1 + 1/k)} \sum_{j=1}^{s} b_j^2(t)(1 - 1/k)^{j-1}, \end{aligned}$$

whenever $n_k \geqslant s$. Hence

$$\lim_{n\to\infty} d_n(t) \geqslant \frac{4}{1+e^{-t}} \sum_{j=1}^{s} b_j^2(t) \quad \text{because of } k \to \infty$$

and

$$\lim_{n\to\infty} d_n(t) \geqslant \frac{4}{1+e^{-t}} \sum_{j=1}^{\infty} b_j^2(t) = 1 - e^{-t} \text{ because of } s \to \infty.$$

Thus we have actually proved Theorem 1.

Lemma 3. *For all $n \geqslant 1$ and $t \geqslant 0$ we have*

$$d\left(t, \mathcal{P}_{n-1}^{\mathbb{R}}\right) = d(t, \mathcal{P}_{n-1}) = q(t, \mathcal{P}_{n-1}) = q\left(t, \mathcal{P}_{n-1}^{\mathbb{R}}\right)$$

and

$$q\left(t, (\mathcal{H}^2)^{\mathbb{R}}\right) = q\left(t, \mathcal{H}^2\right) \leqslant d\left(t, \mathcal{H}^2\right) = d\left(t, (H^2)^{\mathbb{R}}\right).$$

Thus all the classes occurring in Lemmas 1–2 can be replaced by their subclasses consisting of functions with real coefficients.

Proof. Let $\mathcal{A}$ be one of the classes (14)–(15), and let denote $\|f\|_*$ either $\|f\|_\Delta$, or $\|f\|_K$ for $f \in \mathcal{H}(\Delta)$. If we put $\widetilde{f}(z) \equiv [f(z) + \overline{f(\overline{z})}]\big/ 2$, then

$$\mathcal{A}^{\mathbb{R}} = \left\{\widetilde{f} :\ f \in \mathcal{A}\right\} \subset \mathcal{A} = \{e^{i\alpha} f :\ \alpha \in \mathbb{R},\ f \in \mathcal{A}\},$$
$$\|\widetilde{f}\|_* \leqslant \|f\|_* \text{ for all } f \in \mathcal{A}$$

and

$$\mathcal{L} \stackrel{\text{def}}{=} \sup\left\{\sum a_{j-1}(f) b_j(t) :\ f \in \mathcal{A}^{\mathbb{R}},\ \|f\|_* \leqslant 1\right\}$$
$$\leqslant \sup\left\{\left|\sum a_{j-1}(f) b_j(t)\right| :\ f \in \mathcal{A},\ \|f\|_* \leqslant 1\right\} \stackrel{\text{def}}{=} \mathcal{R}.$$

Observe now that

$$\text{(23)} \qquad \operatorname{Re} \sum a_{j-1}(f) b_j(t) \leqslant \mathcal{L} \text{ for any } f \in \mathcal{A}.$$

Indeed, if $f \in \mathcal{A}$ with $\|f\|_* \leqslant 1$, then $\widetilde{f} \in \mathcal{A}^{\mathbb{R}}$, $\|\widetilde{f}\| \leqslant 1$ and

$$\operatorname{Re} \sum a_{j-1}(f) b_j(t) = \sum a_{j-1}\left(\widetilde{f}\right) b_j(t) \leqslant \mathcal{L}.$$

Therefore, for any $f \in \mathcal{A}$ with $\|f\|_* \leqslant 1$ there is a suitable real θ such that

$$\left|\sum a_{j-1}(f) b_j(t)\right| = \operatorname{Re} \sum a_{j-1}\left(e^{i\theta} f\right) b_j(t),$$

$$e^{i\theta} f \in \mathcal{A} \text{ and } \|e^{i\theta} f\|_* \leqslant 1.$$

By (23) we obtain that

$$\left|\sum a_{j-1}(f) b_j(t)\right| \leqslant \mathcal{L} \text{ for all } f \in \mathcal{A} \text{ with } \|f\|_* \leqslant 1.$$

Hence $\mathcal{R} \leqslant \mathcal{L}$, and the proof is complete.

Thereby we have proved Theorem 3.

4. A finite moment problem

Lemma 4. *Let $t \geqslant 0$ and let $b_1, \ldots, b_n$ be given as in (2). If for a Borel measure μ_t on $\overline{\Delta}$ (nonnegative, signed or complex) we have*

$$\text{(24)} \qquad \int_{\overline{\Delta}} \zeta^{j-1} \, d\mu_t(\zeta) = b_j(t)/b_1(t) \quad \textit{for} \quad j = 1, \ldots, n,$$

then

$$\text{(25)} \qquad d_n(t) \leqslant 2te^{-t} |\mu_t| \left(\overline{\Delta}\right),$$

where $|\mu_t|$ is the total variation of μ_t.

Proof. From (24) it follows that

$$2\left|\sum_{j=1}^{n} c_j b_j(t)\right| = 2b_1(t)\left|\int_{\overline{\Delta}}\left(\sum_{j=1}^{n} c_j\zeta^{j-1}\right)d\mu_t(\zeta)\right| \leqslant 2b_1(t)|\mu_t|\left(\overline{\Delta}\right),$$

whenever $\left|\sum_{j=1}^{n} c_j\zeta^{j-1}\right| \leqslant 1$ for $|\zeta| \leqslant 1$. By the definition of $d_n(t)$, see (8), the conclusion (25) follows.

According to Theorem 3, we have also an equivalent form of Lemma 4.

Lemma 4′. *If $t \geqslant 0$ and if for a Borel measure μ_t on $\overline{K}$ (non-negative, signed or complex) we have*

$$(24') \qquad \int_{\overline{K}} \zeta^{j-1} d\mu_t(\zeta) = t^{j-1}/j! \quad for \quad j = 1,\ldots,n,$$

then

$$(25') \qquad d_n(t) \leqslant 2te^{-t}|\mu_t|\left(\overline{K}\right).$$

Remark. For any subset $T \subset \overline{\Delta}$ (resp. $T \subset \overline{K}$) with card$(T) \geqslant n$ there is a collection $\{\mu_t : \ t \geqslant 0\}$ of complex measures supported on T and satisfying (24) (resp. (24′)) for all $t \geqslant 0$. To construct it, consider purely atomic measures with atoms in T. If we associate some elements of T with the parameter t, the cardinality of T can be less than n. By Lemma 4 (resp. Lemma 4′) we have

$$(26) \qquad A_n(\mathcal{B}_0) \leqslant D_n \leqslant \sup_{t>0}\left[2te^{-t}|\mu_t|(T)\right].$$

Hence the optimal choices of measures μ_t are desired, especially those for which the right-hand side of (26) is less or equal to $2/e$.

Application 1. *According to* Lemma 4, *the choice of probability measures gives:*

(i) $\qquad d_2(t) \leqslant 2b_1(t)$ *for* $0 \leqslant t \leqslant t_2 = 2$,

(ii) $\qquad d_3(t) \leqslant 2b_1(t)$ *for* $0 \leqslant t \leqslant t_3 = 3/2$,

(iii) $d_4(t) \leqslant 2b_1(t)$ *for* $0 \leqslant t \leqslant t_4 = 3 - \sqrt{3}$,

(iv) $d_5(t) \leqslant 2b_1(t)$ *for* $0 \leqslant t \leqslant t_5 = 1.12$,

(v) $d_6(t) \leqslant 2b_1(t)$ *for* $0 \leqslant t \leqslant t_6 = 1.03$.

In particular,

(vii) $d_j(t) \leqslant 2/e$ *for* $0 \leqslant t \leqslant 3/2$ *and* $j = 1, 2, 3$,

(viii) $d_j(t) \leqslant 2/e$ *for* $0 \leqslant t \leqslant 3 - \sqrt{3}$ *and* $j = 1, 2, 3, 4$,

and

(ix) $d_j(t) \leqslant 2/e$ *for* $0 \leqslant t \leqslant 1.03$ *and* $j = 1, 2, \ldots, 6$.

Proof. The trigonometric moment problem or, equivalently, the coefficient sequences for analytic functions having positive real parts on Δ were characterized by Carathéodory, see [5,9,12,14,19,21]. If

$$\delta_m(t) \stackrel{\text{def}}{=} \det[b_{|j-k|+1}(t)/b_1(t)]_{j,k=1,\ldots,m} \geqslant 0 \quad \text{for } m = 2, \ldots, n,$$

then there exists an analytic function of the form

$$f_n(z) \equiv 1 + 2\sum_{j=2}^{n}[b_j(t)/b_1(t)]z^{j-1} + \sum_{j=n+1}^{\infty} c_j z^{j-1}$$

with positive real part on Δ. By the Riesz-Herglotz representation formula we have then connections:

$$b_j(t)/b_1(t) = \int_{\partial\Delta} \zeta^{j-1}\,d\mu(t), \; j = 1, \ldots, n,$$

for a probability measure μ, so the conclusion of Lemma 4 holds. Elementary calculations show that

$$\delta_2 = (2-t)t, \quad 9\delta_3(t) = 4(-3+t)t^2(-3+2t),$$
$$81\delta_4(t) = (-6+t)t^3(6-6t+t^2)(-18+6t+t^2),$$
$$6075\delta_5(t) = t^4 u_5(t)v_5(t) \;\text{ and }\; 4100625\delta_6(t) = t^5 u_6(t)v_6(t),$$

where

$$u_5(t) = -540 + 720t - 240t^2 + 24t^3 - t^4,$$
$$v_5(t) = -180 + 120t - 8t^3 + t^4,$$
$$u_6(t) = -8100 + 5400t + 900t^2 - 780t^3 + 75t^4 + t^6$$

and

$$v_6(t) = -16200 + 29700t - 18000t^2 + 4860t^3 - 570t^4 - 15t^5 + 12t^6 - t^7.$$

The standard calculus, or the rule of sign of Fourier, shows that the polynomials u_5, v_5, u_6, v_6 have exactly one zero in the interval $[0,2]$. Therefore, $\delta_m(t) \geqslant 0$ for $0 \leqslant t \leqslant t_m$ and $m = 2,3,4,5,6$. Since $t_2 > t_3 > t_4 > t_5 > t_6$, the desired conclusions follow.

Application 2. $D_1 = D_2 = D_3 = D_4 = 2/e$.

Proof. It suffices to show that $D_4 = 2/e$. By Application 1 (iii) we may assume $t \geqslant 3 - \sqrt{3}$, and according to Lemma 4 consider two Borel measures on $\partial\Delta$:

$$\mu_t = x_t\delta_{-1} + y_t\delta_i + z_t\delta_{-i} + u_t\delta_1$$

and

$$\nu_t = \alpha_t\delta_{-1} + \beta_t\delta_\eta + \gamma_t\delta_{\overline{\eta}},$$

where

$$x_t = -(t - 3 + \sqrt{3})(t-2)(t - 3 - \sqrt{3})/12,$$
$$u_t = t\left[(t-2)^2 + 2\right]/12,$$
$$y_t = t\left[6 - 2t + i\left(6 - 6t + t^2\right)\right]/12 = \overline{z}_t,$$
$$\alpha_t = t/3 - 1,$$
$$\beta_t = 1 - t/6 - i\frac{t^2 + 6t - 18}{6\sqrt{(t-2)(6-t)}} = \overline{\gamma}_t$$

and

$$\eta = \frac{t^2 - 4t + 6 + it\sqrt{(t-2)(6-t)}}{4t - 6}.$$

Obviously,

$$\int_{\partial\Delta} (1,\zeta,\zeta^2,\zeta^3)\, d\mu_t(\zeta) = (1, b_2(t)/b_1(t), b_3(t)/b_1(t), b_4(t)/b_1(t))$$

for all $t \geqslant 0$, and

$$\int_{\partial\Delta} (1,\zeta,\zeta^2,\zeta^3)\, d\nu_t(\zeta) = (1, b_2(t)/b_1(t), b_3(t)/b_1(t), b_4(t)/b_1(t))$$

for $2 < t < 6$. Moreover,

$$2b_1(t)|\mu_t|(\partial\Delta) = \begin{cases} \frac{t}{3}e^{-t}[t^3 - 6t^2 + 12t - 6 + t\sqrt{(2-t)^2(4-t)^2+8}] \\ \text{if } 3-\sqrt{3} \leqslant t \leqslant 2 \text{ or } t \geqslant 3+\sqrt{3}, \\ \frac{t}{3}e^{-t}\left[2t^2 - 6t + 6 + t\sqrt{(2-t)^2(4-t)^2+8}\right] \\ \text{if } 2 \leqslant t \leqslant 3+\sqrt{3}, \end{cases}$$

and

$$2b_1(t)|\nu_t|(\partial\Delta) = \frac{t}{3}e^{-t}\left[|t-3| + \frac{2(2t-3)\sqrt{2t-3}}{\sqrt{(t-2)(6-t)}}\right] \quad \text{if } 2 < t < 6.$$

We are looking for all t with $2b_1(t)|\mu_t|(\partial\Delta) \leqslant 2/e$ or $2b_1(t)|\nu_t|(\partial\Delta) \leqslant 2/e$. Both inequalities are equivalent to

$$\begin{aligned} &6e^{t-1} - t^4 + 6t^3 - 12t^2 + 6t \\ (27)\qquad &\geqslant t^2\sqrt{(2-t)^2(4-t)^2+8} \text{ and } t \notin \left(2, 3+\sqrt{3}\right) \\ &6e^{t-1} - 2t^3 + 6t^2 - 6t \\ (28)\qquad &\geqslant t^2\sqrt{(2-t)^2(4-t)^2+8} \text{ and } t \in \left[2, 3+\sqrt{3}\right], \\ &[3e^{t-1} - t|t-3|]\sqrt{(t-2)(6-t)} \\ (29)\qquad &\geqslant 2t(2t-3)\sqrt{2t-3} \text{ and } 2 < t < 6. \end{aligned}$$

If we prove that the solutions of (27)–(29) cover the whole interval $[3-\sqrt{3},\infty)$, the assertion follows. To deduce it, consider some stronger inequalities

(30)
$$l(t) \stackrel{\text{def}}{=} \left[6 \sum_{j=0}^{7} \frac{(t-1)^j}{j!} - t^4 + 6t^3 - 12t^2 + 6t \right]^2 - t^4 \left[(2-t)^2 (4-t)^2 + 8 \right] \geqslant 0 \quad \text{for } t \in \left[3-\sqrt{3}, 2\right] \cup \left[3+\sqrt{3}, \infty\right),$$

(31)
$$m(t) \stackrel{\text{def}}{=} \left[6 \sum_{j=0}^{7} \frac{(t-1)^j}{j!} - 2t^3 + 6t^2 - 6t \right]^2 - t^4 \left[(2-t)^2 (4-t)^2 + 8 \right] \geqslant 0 \quad \text{for } t \in \left[2, 3+\sqrt{3}\right],$$

(32)
$$n(t) \stackrel{\text{def}}{=} \left[3 \sum_{j=0}^{7} \frac{(t-1)^j}{j!} - t|t-3| \right]^2 \times (t-2)(6-t) - 4t^2(2t-3)^3 \geqslant 0 \quad \text{for } 2 < t < 6.$$

Indeed, we have

$$6 \sum_{j=0}^{7} \frac{(t-1)^j}{j!} - t^4 + 6t^3 - 12t^2 + 6t \geqslant 5 \text{ for } t \geqslant 1$$

$$\left((\text{l.h.s.})''' > 0 \Rightarrow (\text{l.h.s.})'' \geqslant 1 \text{ for } t \geqslant 1 \Rightarrow (\text{l.h.s.})' \geqslant 2 \text{ for } t \geqslant 1 \right),$$

$$6 \sum_{j=0}^{7} \frac{(t-1)^j}{j!} - 2t^3 + 6t^2 - 6t \geqslant 4 \text{ for } t \geqslant 1$$

$$\left((\text{l.h.s.})'' \geqslant 6 \left[t + \frac{(t-1)^2}{2} \right] - 12t + 12 = 3\left(t^2 - 4t + 5\right) > 0 \Rightarrow (\text{l.h.s.})' \geqslant 6 \text{ for } t \geqslant 1 \right),$$

and

$$3\sum_{j=0}^{1}\frac{(t-1)^j}{j!}-t|3-t|=t^2 \quad \text{for } 2<t<3,$$

$$3\sum_{j=0}^{2}\frac{(t-1)^j}{j!}-t|3-t|\geqslant 15 \quad \text{for } 3\leqslant t<6.$$

Using any version of *Mathematica*, we get easily solutions of (30)–(32), that is

1) the polynomial l has exactly two real zeros, both negative; thus $l(t)>0$ for all $t>0$ and, in particular, $l(t)>0$ for $t\in[3-\sqrt{3},2]\cup[3+\sqrt{3},\infty)$,

2) the polynomial m has exactly four real zeros at the points: $0.68\ldots$, $1.12\ldots$, $2.44\ldots$ and $3.17\ldots$; hence $m(t)>0$ for $t\in[2,2.4]\cup[3.2,\infty)$,

3) the extension of $n\mid(2,3]$ has exactly six real zeros, from which four are positive: $0.28\ldots$, $0.96\ldots$, $2.21\ldots$ and $5.80\ldots$; thus $n(t)>0$ for $t\in[2.3,3]$,

4) the extension of $n\mid[3,6)$ has exactly four real zeros and its positive ones are $2.05\ldots$ and $5.74\ldots$; hence $n(t)>0$ for $t\in[3,5.7]$.

Thus we have proved Theorem 2.

Is it possible to avoid a computer calculation? Since *Mathematica* does exact computations on integers and rational numbers, we may mainly reduce its use only to the rules of sign. Regarding the exact computations, even the classical method of Sturm sequences is applicable. Observe first that for $x\geqslant 0$ we have

$$\sqrt{x+8}\leqslant\sqrt{2}(x+16)/8\leqslant 71(x+16)/400. \tag{33}$$

Applying the method of Sturm to the polynomial

$$k(t)\stackrel{\text{def}}{=}6\sum_{j=0}^{7}\frac{(t-1)^j}{j!}-t^4+6t^3-12t^2+6t$$
$$-\frac{71}{400}t^2\left[(2-t)^2(4-t)^2+16\right],$$

we obtain that k has no zeros in the interval $[0,5]$. The rule of sign of Fourier shows that l has no zeros in $[5,\infty)$ so that $l(t)>0$ for all $t\geqslant 1$.

The same rule shows that m has no zeros in the intervals $[2, 12/5]$ and $[16/5, \infty)$. Also $n \mid [2,3]$ has no zeros in $[23/10, 3]$ and $n \mid [3,6]$ has no zeros in $[3, 57/10]$.

Remark. Distributions of measures on $\overline{\Delta}$ giving the proof of the Krzyż conjecture for $n \leqslant 3$ can be a little simpler. This means that the optimal choices of measures in Lemma 4 are not uniquely determined, see the list below.

(i) For $n = 2$ we need only one measure $\mu = (1-t/2)\delta_{-1} + (t/2)\delta_1$.

(ii) For $n \leqslant 3$ it suffices to consider

(a)
$$\mu_t = \frac{3-2t}{3+t}\delta_{-1} + \frac{3t}{2(3+t)}[\delta_\eta + \delta_{\overline{\eta}}],$$
$$\eta = \exp[i \arccos(t/3)], \text{ if } 0 < t \leqslant 1.7 \text{ or } 2.4 \leqslant t \leqslant 3;$$

(b)
$$\mu_t = \left(1 - t + \frac{t^2}{6}\right)\delta_{-1} + \left(\frac{t}{2} - \frac{t^2}{6}\right)[\delta_i + \delta_{-i}] + \left(\frac{t^2}{6}\right)\delta_1, \text{ if } t \geqslant 3;$$

(c)
$$\mu_t = \frac{b_2\eta - b_3}{b_1\xi(\eta - \xi)}\delta_\xi + \frac{b_2\xi - b_3}{b_1\eta(\xi - \eta)}\delta_\eta, \eta = \overline{\xi} \in \partial\Delta,$$
$$\mathrm{Re}\xi = \frac{b_1 + b_3}{2b_2} \in (-1, 1),\ b_j = b_j(t),$$

whenever $1.7 \leqslant t \leqslant 3$.

Indeed, let $w(t) = b_1(t)|\mu_t|(\partial\Delta)$. In the case (a), we have $w(t) = te^{-t} \leqslant 1/e$ for $0 \leqslant t \leqslant 3/2$ and $w(t) = te^{-t}(5t-3)/(3+t)$ for $3/2 \leqslant t \leqslant 3$, so the function w strictly increases on $[3/2, t_0]$ and strictly decreases on $[t_0, 3]$, where $t_0 = 2.02\ldots$ is the zero of the polynomial $p(t) \equiv -9 + 39t - 7t^2 - 5t^3$ (p has zeros in $(-4, -3)$, $(0, 1)$ and $(2, 3)$). Since

$$\left.\frac{t(5t-3)}{3+t}\right|_{t=1.7} < 2 < \sum_{j=0}^{3} \frac{(0.7)^j}{j!} < e^{0.7}$$

and

$$\left.\frac{t(5t-3)}{3+t}\right|_{t=2.4} = 4 < \sum_{j=0}^{5} \frac{(1.4)^j}{j!} < e^{1.4},$$

we get $w(t) \leqslant 1/e$ for $t \in [0, 1.7] \cup [2.4, 3]$.

In the case (b), we observe that

$$e^{t-1} \geqslant \sum_{j=0}^{5} \frac{(t-1)^j}{j!} > \left(\frac{t^2}{3} - 1\right) t \text{ for } t \geqslant 3$$

(apply the method of Sturm). Hence $w(t) \leqslant 1/e$ for $3 \leqslant t \leqslant 3+\sqrt{3}$. If $t \geqslant 3+\sqrt{3}$, then $w(t) = b_3(t) \leqslant 1/e$.

In the last case, $w(t) = t\sqrt{3}(t-1)e^{-t} \Big/ \sqrt{6t-6-t^2}$ and

$$w'(t) = \frac{\sqrt{3}e^{-t}(t-t_1)(t-t_2)(t-2)(t-t_3)}{(6t-6-t^2)^{3/2}},$$

where $t_1 = 0.4\ldots$, $t_2 = 1.6\ldots$, $t_3 = 3.8\ldots$ are all the zeros of the polynomial $t \mapsto t^3 - 6t^2 + 9t - 3$. Thus w strictly increases on $[1.7, 2]$ and strictly decreases on $[2, 3]$ so that $w(t) \leqslant w(2) = \sqrt{6}/e^2 < 1/e$ for $1.7 \leqslant t \leqslant 3$.

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