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Methods of Optimization in Coefficient Estimates for Bounded Univalent Functions

Abstract. In some previous papers the author developed a method of tackling coefficient problems for univalent functions. This method was based on an optimal control system generated by the Loewner differential equation and on an algorithm involving Pontryagin's maximum principle for hamiltonian systems. This paper contains the solution of two extremal problems obtained by means of the author's method.

Let S be the class of holomorphic functions f,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1}$$

univalent in the unit disk $E = \{z : |z| < 1\}$. Let $S^M \subset S$, $M > 1$, be the subclass of bounded functions i.e. $S^M = \{f \in S : z \in E \Rightarrow |f(z)| < M\}$.

Recently the author used optimization methods to construct algorithms for building coefficient bodies in the class S^M. The problem was reduced to describing reachable set control systems generated by the Loewner differential equation. The algorithm is based on the Pontryagin maximum principle for hamiltonian systems. The general method provided solutions of some old and new extremal problems [2] – [5]. Different approaches were developed for the classes S^M , where M is either large or close to 1.

We denote by $P_\alpha^M \in S^M$ the Pick functions which map E onto the disk $E_M = \{w : |w| < M\}$ slit along a radial segment of argument α; $P_\alpha^\infty = K_\alpha$ are the Koebe functions. The k-symmetric Pick function denoted by $P_{\alpha,k}^M$, is defined by the formula

$$P_{\alpha,k}^M(z) = \left[P_\alpha^{M^k}(z^k)\right]^{1/k} .$$

Let

$$P_\pi^M(z) = \sum_{n=1}^{\infty} p_{n,M} z^n , \; p_{1,M} = 1 , \; p_{n,\infty} = n .$$

The author proved [4] a conjecture of Z. Jakubowski that for every even n there exists M_n such that for all $M \geq M_n$ and $f \in S^M$

$$|a_n| \leq p_{n,M} .$$

This estimate does not hold for odd n.

Besides, the author proved [5] a particular case of another conjecture of Z. Jakubowski: there exists $M_{2,3}$ such that for all $M \geq M_{2,3}$ and $f \in S^M$

$$|a_2 a_3| \leq p_{2,M}\, p_{3,M} .$$

The Pick functions are not extremal for the product $|a_k a_n|$ in the class S^M with odd k and n.

In this paper we show the efficiency of our methods by solving two extremal problems by means of reachable set methods.

Let us fix m arbitrary integers $k_1, \ldots, k_m$, $1 < k_1 < \cdots < k_m$, and consider the functional

$$I(k_1, \ldots, k_m; f) = \mathrm{Re} \sum_{j=1}^{\infty} a_{k_j} , \quad f \in S^M .$$

Theorem 1. *If there is at least one even number among $k_1, \ldots, k_m$, then there exists a finite $M(k_1, \ldots, k_m)$ such that for all $M \geq M(k_1, \ldots, k_m)$ and $f \in S^M$*

$$I(k_1, \ldots, k_m; f) \leq \sum_{j=1}^{\infty} p_{k_j, M} .$$

Let us consider a standard functional

$$I_4(p,q;f) = \mathrm{Re}\,(a_4 + p\,a_2a_3 + q\,a_2^3)\ ,$$

where p, q are real numbers.

Theorem 2. *For every* $p > -5/2$ *and arbitrary real* q *there exists* $M(p,q) > 1$ *such that for all* $M \in (1, M(p,q))$ *and* $f \in S^M$

$$I_4(p,q;f) \le I_4(p,q;P^M_{\pi,3})\ .$$

For every $p < -5/2$ *and arbitrary real* q *there exists* $M(p,q) > 1$ *such that for all* $M \in (1, M(p,q))$ *and* $f \in S^M$

$$I_4(p,q;f) \le I_4(p,q;P^M_{\pi})\ .$$

The author does not claim to have given the first proof of Theorem 2 but his proof is a good example of the efficiency of the method applied here.

Parametric method and control theory

Solutions $w = w(z,t)$ of the Loewner differential equation

$$\text{(2)}\qquad \frac{dw}{dt} = -\frac{e^{iu}+w}{e^{iu}-w}\ ,\quad w|_{t=0} = z\ ,$$

form a dense subclass of S^M consisting of functions $f(z) = Mw(z, \log M)$.

Let

$$w(z,t) = e^{-t}\Big(z + \sum_{n=2}^{\infty} a_n(t)z^n\Big)\ ,$$

$$a(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix},\quad A(t) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1(t) & 0 & \dots & 0 & 0 \\ a_2(t) & a_1(t) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1}(t) & a_{n-2}(t) & \dots & a_1(t) & 0 \end{pmatrix},$$

$a^0 = (1, 0, \dots, 0)^T, \; a_1(t) \equiv 1.$

The Loewner differential equation generates a control system [3]

(3) $$\frac{ds}{dt} = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} A^s(t) a(t) \, , \quad a(0) = a^0 \, .$$

The coefficient body $V_n^M = \{(a_2, \dots, a_n) : f \in S^M\}$ is a reachable set at the moment $t = \log M$ for the control system (3). Boundary points of V_n^M are reached only by optimal controls $u = u(t)$ maximizing the Hamilton function

$$H(t, a, \Phi, u) = -2 \sum_{s=1}^{n-1} \operatorname{Re} \left[e^{-s(t+iu)} (A^s a)^t \Phi \right] .$$

The vector $\Phi = (\Phi_1, \dots, \Phi_n)^T$ of complex-valued Lagrange multipliers satisfies the conjugate hamiltonian system

(4) $$\frac{d\Phi}{dt} = 2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (s+1) (A^T)^s \Phi \, , \quad \Phi(0) = \zeta \, .$$

The vector $(\Phi_2(\log M), \dots, \Phi_n(\log M))$ is orthogonal to a tangent or support plane (if it exists) of a boundary hypersurface ∂V_n^M at the point $(a_2, \dots, a_n) = (a_2(\log M), \dots, a_n(\log M))$.

Proof of Theorem 1

Let $k_m = n$. If $\operatorname{Re} \sum_{j=1}^m a_{k_j}$ attains its maximum at a point $X \in \partial V_n^M$, then there exists a conjugate function Φ such that $\Phi_{k_1}(\log M) = \dots = \Phi_{k_m}(\log M) = 1$, while other coordinates of the vector $\Phi(\log M)$ vanish.

Denote by $w'(z,t)$ the derivative of $w(z,t)$ with respect to z . Differentiating the Loewner equation with respect to z , we obtain

(5) $$\frac{dw'}{dt} = -w' \frac{\partial}{\partial w} \left(w \frac{e^{iu} + w}{e^{iu} - w} \right) \, , \quad w'|_{t=0} = 1 \, .$$

Put

$$h(z)=\sum_{j=1}^{m} z^{n-k_j}\,,\quad \frac{h(z)f'(z)e^{-t}}{w'(z,t)}=\sum_{q=0}^{\infty} b_q(t)z^q\,.$$

Equations (3) and (5) generate a differential equation for $b(t) = (b_0(t),\ldots,b_{n-1}(t))^T$ which differs from (4) only by replacing the matrix A by its transposition. Since $(\Phi_2(\log M),\ldots,\Phi_n(\log M)) = (b_{n-2}(\log M),\ldots,b_0(\log M))$ at the point x, then $(\Phi_2(t),\ldots,\Phi_n(t)) = (b_{n-2}(t),\ldots,b_0(t))$, $0\le t\le \log M$. Hence data values $\zeta = (\zeta_1,\ldots,\zeta_n)$ at $t=0$ are given by the following

$$\zeta_r=\sum_{j=q+1}^{m}(k_j-r+1)a_{k_j-r+1},\quad k_q+1\le r\le k_{q+1},\qquad q=0,1,\ldots,m-1,\quad k_0=1\,. \tag{6}$$

Let X denote a neighbourhood of the Koebe function K_π in S in the topology of uniform convergence on compact subsets of E, $X^M = X\cap S^M$. The Pick function P_π^M belongs to X^M for sufficiently large M. To prove the theorem it is sufficient to show that only P_π^M satisfy the necessary extremum conditions for $I(k_1,\ldots,k_m;f)$ in X^M.

Let the coordinates of $\zeta^0=(\zeta_1^0,\ldots,\zeta_n^0)$ be given by

$$\zeta_r^0=\sum_{j=q+1}^{m}(k_j-r+1)^2,\ k_q+1\le r\le k_{q+1},\ q=0,1,\ldots,m-1.$$

Then

$$H(0,a^0,\zeta^0,u)=-2\sum_{j=1}^{m}\sum_{s=1}^{k_j-1}(k_j-s)^2\cos(su)\,.$$

This is an algebraic polynomial with respect to $y=\cos u$. We have to ensure that it has a maximum on [-1,1] at $y=-1$ and this condition is stable under real variations of its coefficients. Various ways lead to the goal. For instance, evaluate that

$$H(0,a^0,\zeta^0,u)-H(0,a^0,\zeta^0,\pi)=\frac{-\sin u}{(1-\cos u)^2}\sum_{j=1}^{m}(k_j\sin u-\sin(k_ju))\,.$$

The function $k_j \sin u - \sin(k_j u)$ is positive on $(0,\pi)$ and vanishes at $u=\pi$. Hence for $u \in [0,2\pi]$ we have the inequality

$$H(0,a^0,\zeta^0,u) \le H(0,a^0,\zeta^0,\pi) \tag{7}$$

with the sign of equality only at $u=\pi$. Moreover,

$$\left.\frac{\partial H(0.a^0,\zeta^0,u)}{\partial(\cos u)}\right|_{u=\pi} = 2\sum_{j=1}^{m}\sum_{s=1}^{k_j-1}(-1)^s s^2(k_j-s)^2 . \tag{8}$$

This derivative is negative. Indeed, if k_j is odd, then $\delta_j = \sum_{s=1}^{k_j-1}(-1)^s s^2(k_j-s)^2$ vanishes, but if k_j is even, then $\delta_j < 0$ since it is equal to (k_j-2) – th coefficient of the function $-(1-z^2)^{-2}$. This proves the assertion.

Put $(\zeta_2,\dots,\zeta_n) = ((n-1)p_{n-1,M},\dots,2p_{2,M},1)$. For sufficiently large M the vector $\zeta = (\zeta_1,\dots,\zeta_n)$ belongs to a neighbourhood of ζ^0. The Hamilton function $H(0,a^0,\zeta^0,u)$ attains its maximum on $[0,2\pi]$ only at $u=\pi$. So the control $u=\pi$ satisfies the Pontryagin maximum principle for $t>0$ in a neighbourhood of the initial value $t=0$. As the corresponding solution $w(z,t)$ of the Loewner differential equation with $u=\pi$ on $[0,\log M]$, we receive the Pick function P_π^M.

It remains to show that necessary extremum conditions occur at a single point in X^M.

Boundary hypersurfaces depend analytically on M. Suppose that for every neighbourhood X and every large M there exist functions f^M in X^M, $f^M \ne P_\pi^M$, satisfying necessary extremum conditions in S^M. Hence it is possible to choose a function sequence converging to K_π as $M\to\infty$. Since P_π^M also tends to K_π as $M\to\infty$ we deduce the following property: There exists a direction l such that the hyperplane

$$\Pi = \left\{(a_2,\dots,a_n) : \operatorname{Re}\sum_{j=1}^{m} a_{k_j} = \sum_{j=1}^{m} k_j\right\}$$

has at least the second order tangency with the hypersurface $\partial V_n = \partial V_n^\infty$ at the point $N=(2,\dots,n)^T$ corresponding to K_π in the direction l.

Let Q_N be a neighbourhood of the point N on the hypersurface ∂V_n. This neighbourhood corresponds to a neighbourhood Q_ζ of the data value $\Lambda = (\zeta_2^0, \ldots, \zeta_n^0)^T$ in (4). The correspondence is not one – to – one. All points $\Lambda^* = (\zeta_2^*, \ldots, \zeta_n^*)^T \in Q_\zeta$ with real coordinates are mapped on N. But a correspondence between $(\Phi_2^*(\infty), \ldots, \Phi_n^*(\infty))$ and Λ^* is one – to –one. It means that the point N is angular for ∂V_n and there exists a family of support hyperplanes to ∂V_n at N. But Π and ∂V_n may have tangency along certain directions defined by imaginary parts of coordinates of phase vector or of the data value Λ^*. We show that this is at most the first order tangency.

Indeed this fact was realized by D. Bshouty [1] who completed the following results of E. Bombieri in the local coefficient problem (see bibliography in [1]): for even k there exist constants α_k and β_k such that if $|2 - a_2| \leq \beta_k$, then $\mathrm{Re}\,(2 - a_2) < \alpha_k \mathrm{Re}\,(k - a_k)$.
D. Bshouty added that for every $j > 1$ there exist constants c_j and d_j such that

$$\mathrm{Re}\,(j - a_j) \leq c_j \mathrm{Re}\,(2 - a_2), \quad j - |a_j| \leq d_j \mathrm{Re}\,(2 - a_2)\,.$$

Let $(a(t), \Phi(t))$ be a solution of the Cauchy problem for the hamiltonian system (3) – (4) with $u = \Pi$ and the data value Λ. Put $\Lambda^* = \Lambda + \epsilon(\delta_2, \ldots, \delta_n)^T$, where $\delta_2, \ldots, \delta_n$ are fixed complex numbers, $\epsilon > 0$, and $(a^*(t), \Phi^*(t))$ is a solution of the Cauchy problem for the hamiltonian system (3) – (4) with the data value Λ^* and an optimal control $u^* = u(t, a^*, \Phi^*)$.

Suppose that Π and ∂V_n have a high order tangency at the point N along a direction l defined by the vector $(\delta_2, \ldots, \delta_n)$. Let k be any even number among $k_1, \ldots, k_m$. The hypersurface ∂V_n in a neighbourhood Q_N is defined by $y = \varphi(x)$, where $y = \mathrm{Re} \sum_{j=1}^m a_{k_j}$ and $x \in \mathbb{R}^{2n-3}$ is obtained from $(a_2, \ldots, a_n)$ as $x_{2j-3} = \mathrm{Re}\, a_j$, $x_{2j-2} = \mathrm{Im}\, a_j$, $j = 2, \ldots, n$, with excluded $x_{2k-3} = \mathrm{Re}\, a_k$. Let x^0 correspond to N, $y^0 = \sum_{j=1}^m k_j$, $y^0 = \varphi(x^0)$. The high order tangency of Π and ∂V_n at N means that it $\Delta \in \mathbb{R}^{2n-3}$ corresponds to the direction l and $x = x^0 + \epsilon\Delta$, then $y = y^0 + O(\epsilon^3)$. Hence a representation $\mathrm{Re}\, a_{k_j}^* = k_j + \epsilon\gamma_j + O(\epsilon^2)$, $j = 1, \ldots, m$, involves that $\sum_{j=1}^m \gamma_j = 0$. If $\gamma_1^2 + \cdots + \gamma_m^2 > 0$, then there exists a positive γ_j. This contradicts the extremal property of the Koebe function in the coefficient problem. Similarly a representation

$\operatorname{Re} a^*_{k_j} = k_j + \epsilon^2\gamma_j + O(\epsilon^3)$, $j = 1, \dots, m$, involves the same equality $\sum_{j=1}^m \gamma_j = 0$ and hence $\gamma_1 = \dots = \gamma_m = 0$.

Thus the high order tangency involves a representation $\operatorname{Re} a^*_k = k + O(\epsilon^3)$. Then according to Bombieri's result $\operatorname{Re} a^*_2 = 2 + 0(\epsilon^3)$. Now Bshouty's result leads to the conditions

$$\operatorname{Re} a^*_j = j + O(\epsilon^3), \quad |a^*_j| = j + O(\epsilon^3)\ ,$$

and hence to

$$a^*_j = j + O(\epsilon^2)$$

which contradicts the equality $x = x^0 + \epsilon\Lambda$. So Π and ∂V_n have at most the first order tangency at N along each direction. This proves the theorem.

Obviously, the theorem may be generalized for a functional $\operatorname{Re}\sum_{j=1}^m \lambda_j a_{k_j}$, where all λ_j are positive. A boundary number M depends on $k_1, \dots, k_m$ and $\lambda_1, \dots, \lambda_m$.

If all $k_1, \dots, k_m$ are odd, then the Pick functions do not maximize $I(k_1, \dots, k_m; f)$ in S^M. Indeed, the derivative (8) vanishes, and it is easy to verify that the second order derivative at $u = \Pi$ is positive for sufficiently large M. Hence the control $u = \Pi$ does not satisfy the Pontryagin maximum principle.

Reachable set methods in Theorem 2

Let us examine a coefficient set

$$U(M) = \{(a_2, a_3, I_4(p, q; f)) : f \in S^M\}\ .$$

Points of its 4 – dimensional boundary hypersurface $\partial U(M)$ correspond to boundary functions $f(z) = Mw(z, \log M)$, where $w(z,t)$ are integrals of the Loewner differential equation (2). We may go to the generalized Loewner differential equation

$$\frac{dw}{dt} = -w\sum_{k=1}^{3}\lambda_k\frac{e^{iu_k} + w}{e^{iu_k} - w}\ , \quad 0 \le t \le \log M\ , \quad w(z, 0) = z\ , \tag{9}$$

with constant non–negative numbers λ_k, $\sum_{k=1}^3 \lambda_k = 1$, and continuous controls u_k (see e.g. [2], [3]).

Denote $X_1(t) = x_1(t) + ix_2(t) = a_2(t)$, $X_2(t) = x_3(t) + ix_4(t) = a_3(t)$, $x_5(t) = \operatorname{Re}(a_4(t) + p\, a_3(t)a_2(t) + q\, a_2^3(t))$. The generalized Loewner equation (9) produces a control system for $X(t) = (x_1(t), \ldots, x_5(t))$

(10)

$$\frac{dX}{dt} = \sum_{k=1}^{3} \lambda_k g(t, X, u_k), \quad X(0) = (0, \ldots, 0), \; g = (g_1, \ldots, g_5),$$

$$\begin{aligned} G_1(t, X, u) &= g_1 + ig_2 = -2e^{-t}e^{-iu}, \\ G_2(t, X, u) &= g_3 + ig_4 = -4e^{-t}e^{-iu}x_1 - 2e^{-2t}e^{-i2u}, \\ g_5(t, X, u) &= \operatorname{Re}[-2e^{-t}e^{-iu}((2+p)X_2 + (1 + 2p + 3q^2)X_1^2) \\ &\quad - 2(3+p)e^{-2t}e^{-i2u}X_1 - 2e^{-3t}e^{-i3u}] . \end{aligned}$$

The set $U(M)$ is a reachable set at $t = \log M$ for the control system (10). Optimal controls u_k corresponding to boundary functions f satisfy Pontryagin's maximum principle and maximize the Hamilton function

$$H(t, X, \Phi, u) = \sum_{k=1}^{5} g_k(t, X, u)\, \varphi_k$$

while a conjugate vector $\Phi = (\varphi_1, \ldots, \varphi_5)$, $\Phi_1 = \varphi_1 + i\varphi_2$, $\Phi_2 = \varphi_3 + i\varphi_4$, is an integral of the conjugate hamiltonian system

(11)

$$\begin{aligned} \frac{d\Phi_1}{dt} &= \sum_{k=1}^{3} \lambda_k [4e^{-t}e^{-iu}\Phi_2 \\ &\quad + [4(1 + 2p + 3q)e^{-t}e^{-iu_k}X_1 \\ &\quad + 2(3+p)e^{-2t}e^{-i2u_k}]\varphi_5], \qquad \Phi_1(0) = \zeta_1, \\ \frac{d\Phi_2}{dt} &= \sum_{k=1}^{3} \lambda_k 2(2+p)e^{-t}e^{-iu_k}\varphi_5, \quad \Phi_2(0) = \zeta_2, \\ \frac{d\varphi_5}{dt} &= 0 . \end{aligned}$$

The conjugate vector $\Phi(\log M)$ is orthogonal to a tangent hyperplane or to a support hyperplane (if they exist) of the boundary hypersurface $\partial U(M)$. If $I_4(p,q;f)$ attains its maximum at any point of $\partial U(M)$, then Φ may be normalized so that $\Phi(\log M) = (0,0,0,0,1)$. Hence we assume that $\varphi_5(\log M) = 1$. There remain only two complex - valued equations in (11) with free data values. Since M is close to 1, parameters (ζ_1, ζ_2) should be taken from a neighbourhood of $(0,0)$.

The hypersurface $\partial U(M)$ consists of parts Ω_k, $k = 1,2,3$, with different parametrizations. All parts are glued along their common borders.

Let $\Phi^0 = (\operatorname{Re}\zeta_1, \operatorname{Im}\zeta_1, \operatorname{Re}\zeta_2, \operatorname{Im}\zeta_2, 1)$.
Put

$$\begin{aligned}\mathcal{M}_k = \{(\zeta_1,\zeta_2) : & H(0,0,\Phi^0,u) \text{ has at least } k \\ & \text{maximum points } u_1,\dots,u_k \text{ on } [0,2\pi]\}, \quad k = 1,2,3.\end{aligned}$$

Evidently $\mathcal{M}_1 = \mathbb{C}^2$; $\mathcal{M}_2$ is a 3 - dimensional set; $\mathcal{M}_3$ is a 2 - dimensional set; $(0,0) \in \mathcal{M}_3$. The part Ω_1 is parametrized by (ζ_1,ζ_2):

$$\Omega_1 = \{X(\log M, \zeta_1, \zeta_2) : (\zeta_1,\zeta_2) \in \mathcal{M}_1\},$$

where $(X(t,\zeta_1,\zeta_2), \Phi(t,\zeta_1,\zeta_2))$ is a solution of the Cauchy problem (10) - (11) with $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$. The second part Ω_2 is parametrized by $(\zeta_1,\zeta_2) \in \mathcal{M}_2$ and $\lambda \in [0,1]$:

$$\Omega_2 = \{X(\log M, \zeta_1, \zeta_2, \lambda) : (\zeta_1,\zeta_2) \in \mathcal{M}_2,\ 0 \le \lambda \le 1\},$$

where $(X(t,\zeta_1,\zeta_2,\lambda), \Phi(t,\zeta_1,\zeta_2,\lambda))$ is a solution of the Cauchy problem (10) - (11) with $\lambda_1 = \lambda$, $\lambda_2 = 1-\lambda$, $\lambda_3 = 0$. Finally Ω_3 is parametrized by $(\zeta_1,\zeta_2) \in \mathcal{M}_3$ and λ_1, λ_2:

$$\begin{aligned}\Omega_3 = \{X(\log M, \zeta_1, \zeta_2, \lambda_1, \lambda_2) : & (\zeta_1,\zeta_2) \in \mathcal{M}_3,\ \lambda_1 \ge 0,\ \lambda_2 \ge 0, \\ & \lambda_1 + \lambda_2 \le 1\},\end{aligned}$$

where $(X(t,\zeta_1,\zeta_2,\lambda_1,\lambda_2), \Phi(t,\zeta_1,\zeta_2,\lambda_1,\lambda_2))$ is a solution of the Cauchy problem (10) - (11) with $\lambda_3 = 1 - \lambda_1 - \lambda_2$.

Notice that connected components of Ω_1 are locally convex for small $M-1$. Indeed, $(\partial\Phi_k/\partial\zeta_j)|_{t=0} = \delta_k^j$, where δ_k^j are the Kronecker symbols, and these derivatives vary continuously with respect to $t \ge 0$.

Let us find two linearly independent directions of the set $\mathcal{M}_3$ at $(0,0)$. We put $\zeta_k = \epsilon(\delta_{2k-1} + i\delta_{2k}), k = 1, 2$. Roots of the equation

$$(12)\quad \begin{aligned} H_u(0,0,\Phi^0,u) = 6\sin 3u + \epsilon\, 4\delta_3 \sin 2u + \epsilon\, 4\delta_4 \cos 2u \\ + \epsilon\, 2\delta_1 \sin u + \epsilon\, 2\delta_2 \cos u = 0 \end{aligned}$$

determine 3 branches: $u_1 = -\pi/3 + \epsilon\beta_1 + O(\epsilon^2)$, $u_2 = \pi/3 + \epsilon\beta_2 + O(\epsilon^2)$, $u_3 = \pi + \epsilon\beta_3 + O(\epsilon^2)$. The manifold $\mathcal{M}_3$ is defined by the condition

$$H(0,0,\Phi^0,u_1) = H(0,0,\Phi^0,u_2) = H(0,0,\Phi^0,u_3)$$

which leads to the following: $\delta_3 = \delta_1,\ \delta_4 = -\delta_2$. From (12) we have $\beta_1 = -\delta_1/(2\sqrt{3}) - \delta_2/6,\ \beta_2 = \delta_1/(2\sqrt{3}) - \delta_2/6,\ \beta_3 = (\delta_2 - 2\delta_4)/9$. So $\zeta_2 = \bar{\zeta}_1$.

Thus Ω_3 is locally parametrized by $\delta_1, \delta_2, \lambda_1, \lambda_2$ and it is locally convex in the directions defined by δ_1, δ_2.

Proof of Theorem 2, p > −5/2

We divide the proof into 3 parts.

1. Verify that $P^M_{\pi,3}$ satisfies the maximum principle. $P^M_{\pi,3}$ corresponds to $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$ and $u_1 = -\pi/3$, $u_2 = \pi/3$, $u_3 = \pi$ in the generalized Löewner differential equation (9). Thus $\zeta_1 = \zeta_2 = 0$ in (11). From (10) – (11) we obtain $X_1(t) = X_2(t) = \Phi_1(t) = \Phi_2(t) = 0$, $x_5(t) = (2/3)(1 - e^{-3t})$. Hence

$$H(t,X,\Phi,u) = -2e^{-3t}e^{-i3u}.$$

So the Hamilton function attains its maximum on $[0,2\pi]$ at u_1, u_2, u_3.

2. Verify that, if $\zeta_1 = \zeta_2 = 0$ then x_5 as a function of $\lambda = (\lambda_1, \lambda_2)$ attains its maximum at $\lambda_1 = \lambda_2 = 1/3$ for small $t > 0$. Branches u_1, u_2, u_3 maximizing the Hamilton function are smooth functions, $u_1(0) = -\pi/3$, $u_2(0) = \pi/3$, $u_3(0) = \pi$. From (10) we find $(dx_5/dt)|_{t=0} = 2$. Differentiating all the coordinate equations in (10) with respect to t, we find

$$\frac{d^2x_5}{dt^2} = -4(5+2p)(3\lambda_1^2 + 3\lambda_2^2 + 3\lambda_1\lambda_2 - 3\lambda_1 - 3\lambda_2 + 1) - 6 .$$

The right-hand side $Q(\lambda_1, \lambda_2)$ in this equation has a maximum at $\lambda_1 = \lambda_2 = 1/3$. The required conclusion follows from the expansion

$$x_5 = 2t + (1/2)Q(\lambda_1, \lambda_2)t^2 + O(t^3) . \tag{13}$$

3. Suppose there exist M_n, $\lim_{n\to\infty} M_n = 1$, and $f_n \in S^{M_n}$, $f_n \neq P^{M_n}_{\pi,3}$, such that f_n maximize $I_4(p,q;f)$ in S^{M_n}. Every f_n corresponds to data values $\zeta^n = (\zeta_1^n, \zeta_2^n)$ and parameters $\lambda^n = (\lambda_1^n, \lambda_2^n)$ in (10) - (11), $\lim_{n\to\infty} \zeta^n = (0,0)$. Taking a subsequence if necessary, we confirm that λ^n tends to a limit denoted by λ^*. The expansion (13) written for $\zeta_1 = \zeta_2 = 0$ slightly varies for (ζ_1, ζ_2) from an ϵ - neighbourhood of $(0,0)$, but the first - order term on the right-hand side of (13) is $O(\epsilon^2)$. Thus according to (13) $\lambda^* = (1/3, 1/3)$ since this point maximizes $Q(\lambda_1, \lambda_2)$.

The part Ω_3 is locally convex in the directions defined by free components of $(\zeta_1, \zeta_2) \in \mathcal{M}_3$. Together with (13) this requires that for each sufficiently small $t > 0$ the goal functional x_5 may have at most one point satisfying the necessary extremum conditions in a neighbourhood of $(\zeta_1, \zeta_2, \lambda_1, \lambda_2) = (0, 0, 1/3, 1/3)$. So we have no other f_n, except for $P^{M_n}_{\pi,3}$. This ends the proof of the first conclusion of Theorem 2.

Proof of Theorem 2, p < −5/2

Again we divide the proof into the same three parts.

1. Verify that P^M_π satisfies the maximum principle. P^M_π corresponds to $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$ and $u_1 = \pi$ in the generalized Loewner differential equation (9). Thus we have in (11)

$$\zeta_1 = 9 + 11p + 12q - 24(1+p+q)/M + (15 + 13p + 12q)/M^2 ,$$
$$\zeta_2 = 2(2+p)(1 - 1/M) .$$

From (10) – (11) we obtain

$$X_1(t) = 2(1 - e^{-t}), \quad X_2(t) = 2(1 - e^{-t})(3 - 5e-t) .$$

Hence

$$\begin{aligned} H(t, X, \Phi, u) = &-2e^{-t}[4e^{-2t}\cos^3 u - 4e^{-t}(e^{-t} + (2+p)/M \\ &- 3 - p)\cos^2 u + (e^{-2t}(6+5p) - 8e^{-t}(2+p) \\ &+ (15 + 13p + 12q)/M^2 - 24(1 + p + q)/M \\ &+ 16 + 14p + 12q)\cos u \\ &- 2e^{-t}(3 + p - e^{-t} - (2+p)/M)] . \end{aligned}$$

Therefore the Hamilton function attains its maximum on $[0, 2\pi]$ at $u = \pi$ for small $M - 1$, $0 \le t \le \log M$.

2. Verify that, if $\zeta_1 = \zeta_2 = 0$ then x_5 as a function of λ attains its maximum at $\lambda = (1, 0)$, $\lambda = (0, 1)$, or $\lambda = (0, 0)$ for sufficiently small $t > 0$. This conclusion follows from (13) because now $Q(\lambda_1, \lambda_2)$ attains its maximum at the angular points of λ - domain. Notice that the maximizing property of the angular points is preserved under slight variations of the data values, i.e. in a neighbourhood of $(\zeta_1, \zeta_2) = (0, 0)$.

3. Max x_5 is attained at the part Ω_1 of $\partial U(M)$ for small $M - 1$ and for (ζ_1, ζ_2) from a neighbourhood of $(0, 0)$ since λ is an angular point of the λ - domain. In every connected component of Ω_1 the goal functional x_5 may have at most one point satisfying the necessary extremum conditions in a neighbourhood of $((\zeta_1, \zeta_2) = (0, 0)$. Hence we have no other extremal functions, except for P^M_π, $P^M_{\pi/3}$, $P^M_{-\pi/3}$. This ends the proof of the second conclusion of Theorem 2.

It is interesting to notice that $P^M_{\pi,3}$ always satisfy the necessary extremum conditions. Hence, if $p < -5/2$ then this boundary function corresponds to a saddle point of $\partial U(M)$.

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