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# On Moments of a Class of Counting Distributions

ABSTRACT. Using two methods of determining moments of discrete distributions we give formulae and recurrence formulae for moments of a class of counting distributions.

**1. Introduction and preliminaries.** The aim of our note is to give formulae for moments of a class of discrete distributions. Let $p_T(n)$, $n = 0, 1, \dots$, be a probability function of a random variable $T$ defined by the following recurrence relation

$$p_T(n) := P_T(n; a, b) = (a + b/n)p_T(n-1), \; n = 1, 2, \dots, \tag{1.1}$$

where $a < 1$, $a + b \geq 0$, $p_T(0) \geq 0$. (cf. [1] and [2]).

We need to note that for a random variable $T$ obeying:

(i) binomial probability function

$$p_T(n) = \binom{N}{n} p^n q^{N-n}, \; n = 0, 1, \dots, N; \; 0 < p < 1, \; p + q = 1; \tag{1.2}$$

$$a = -p/q, \; b = [(N+1)/p]/q, \; p_T(0) = q^N,$$

(ii) Poisson probability function

$$p_T(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \dots; \quad \lambda > 0; \tag{1.3}$$

$$a = 0,\ b = \lambda,\ p_T(0) = e^{-\lambda},$$

(iii) geometrical probability function

$$p_T(n) = pq^n,\ n = 0, 1, ...,;\ 0 < p < 1,\ p + q = 1; \tag{1.4}$$

$$a = q,\ b = 0,\ p_T(0) = p,$$

(iv) negative binomial distribution function

$$p_T(n) = \binom{n+k-1}{n} p^k q^k,\ n = 0, 1, ...,\ k = 1, 2, ...; \quad 0 < p < 1,\ \ p + q = 1; \tag{1.5}$$

$$a = q,\ b = (k-1)q,\ p_T(0) = p^k,$$

(v) logaritmic distribution function

$$p_T(n) = -\frac{q^n}{n \ln p},\ n = 1, 2, ...;\ 0 < p < 1,\ p + q = 1; \tag{1.6}$$

$$a = q,\ b = -q,\ p_T(1) = -q/\ln p,$$

(vi) truncated Poisson distribution function

$$p_T(n) = \left(\frac{\lambda^n}{n!} e^{-\lambda}\right)/(1 - e^{-\lambda}),\ n = 1, 2, ...;\ \lambda > 0; \tag{1.7}$$

$$a = 0,\ b = \lambda,\ p_T(1) = (\lambda e^{-\lambda})/(1 - e^{-\lambda}).$$

**2. Moments of counting distributions established by the direct method.** Let $m_r(c) := E(T - c)^r$ denote moments about the point $c$ of a random variable $T$. Assume that $p_T(0) > 0$.

Now we see that elementary evaluations lead to formulae for moments $m_r(c)$ about the point $c$, ordinary moments $\alpha_r := m_r(0)$ and central moments $\mu_r := m_r(ET)$ of a discrete probability function $p(n) := p_T(n)$, $n = 0, 1, ...,$ obeying (1.1).

Taking into account that $p(n)$ satisfies (1.1) we get

$$m_r(c) := E(T-c)^r = \sum_{n=0}^{\infty}(n-c)^r p(n)$$

$$= (-c)^r p(0) + a\sum_{n=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j}(n-1-c)^j p(n-1)$$

$$+ b\sum_{n=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j}n^{j-1}(-c)^{r-j}p(n-1)$$

$$= (-c)^r p(0) + a\sum_{j=0}^{r}\binom{r}{j}m_j(c) + (-c)^r\sum_{n=1}^{\infty}\frac{b}{n}p(n-1)$$

$$+ b\sum_{n=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j}n^{j-1}(-c)^{r-1}p(n-1)$$

$$= (-c)^r p(0) + a\sum_{j=0}^{r}\binom{r}{j}m_j(c) + (-c)^r\sum_{n=1}^{\infty}[p(n) - ap(n-1)]$$

$$+ b\sum_{n=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j+1}(n-1-c+1+c)^j(-c)^{r-j-1}p(n-1)$$

$$= (1-a)(-c)^r + am_r(c) + a\sum_{j=0}^{r-1}\binom{r}{j}m_j(c)$$

$$+ b\sum_{n=1}^{\infty}\sum_{j=0}^{r-1}\binom{r}{j+1}(-c)^{r-j-1}\sum_{i=0}^{j}\binom{j}{i}(n-1-c)^i(1+c)^{j-i}p(n-1)$$

$$= (1-a)(-c)^r + am_r(c) + a\sum_{j=0}^{r-1}\binom{r}{j}m_j(c)$$

$$+ b\sum_{j=0}^{r-1}\binom{r}{j+1}(-c)^{r-j-1}\sum_{i=0}^{j}\binom{j}{i}(1+c)^{j-i}m_i(c)\,.$$

Hence we have

$$(1-a)m_r(c) = (1-a)(-c)^r + a\sum_{j=0}^{r-1}\binom{r}{j}m_j(c) \tag{2.1}$$

$$+b\sum_{j=0}^{r-1}\binom{r}{j+1}(-c)^{r-j-1}\sum_{i=0}^{j}\binom{j}{i}(1+c)^{j-i}m_i(c)\,.$$

By (2.1) after using the equivalence $\sum_{j=0}^{r-1}\sum_{i=0}^{j}(\cdot)_{ij}=\sum_{i=0}^{r-1}\sum_{j=i}^{r-1}(\cdot)_{ij}$ we are completing the proof of the following result. ■

**Theorem 1.** *Moments* $m_r(c)$, $\alpha_r$ *and* $\mu_r$ *satisfy the following equations:*

$$m_r(c)=(-c)^r+\frac{1}{1-a}\sum_{i=0}^{r-1}\Big[a\binom{r}{i}$$

$$+\,b\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}(-c)^{r-j-1}(1+c)^{j-i}\Big]m_i(c)\,,\tag{2.2}$$

$$r\ge 1,\qquad m_0(c)=1\,,$$

$$\alpha_r:=m_r(0)=\frac{1}{1-a}\sum_{j=0}^{r-1}\Big[a\binom{r}{j}+b\binom{r-1}{j}\Big]\alpha_j\,,\tag{2.3}$$

$$r\ge 1\,,\quad \alpha_0=1\,,$$

$$\mu_r:=m_r\Big(\frac{a+b}{1-a}\Big)=\Big(\frac{a+b}{a-1}\Big)^r+\frac{1}{1-a}\sum_{i=0}^{r-1}\Big[a\binom{r}{i}$$

$$+\,b\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\Big(\frac{a+b}{a-1}\Big)^{r-j-1}\Big(\frac{1+b}{1-a}\Big)^{j-i}\Big]\mu_i\,,\tag{2.4}$$

$$r\ge 1,\ \mu_0=1\,,$$

The moments $\alpha_r$ and $\mu_r$ for $r=1,2,3,4$ are as follows:

$$\alpha_0=1\,,\qquad \alpha_1=\frac{a+b}{1-a}\,,\qquad \alpha_2=\frac{(a+b)(a+b+1)}{(1-a)^2}\,,$$

$$\alpha_3=\frac{(a+b)[(a+b)^2+3(a+b)+a+1]}{(1-a)^3}\,,$$

$$\alpha_4=\frac{(a+b)[(a+b)^3+6(a+b)^2+(a+b)(4a+7)+a^2+4a+1]}{(1-a)^4}\,,$$

$$\mu_0 = 1\,,\qquad \mu_1 = 0\,,\qquad \mu_2 = \frac{a+b}{(1-a)^2}\,,\qquad \mu_3 = \frac{(a+b)(a+1)}{(1-a)^3}\,,$$
$$\mu_4 = \frac{(a+b)[3(a+b)+a^2+4a+1]}{(1-a)^4}\,.$$

**Corollary 1.** *The coeficient of skewness* $\gamma$ *and the kurtosis of probability functions* (1.1) *are given by*

$$\gamma = \frac{\mu_3}{\sigma^3} = \frac{a+1}{\sqrt{a+b}}\,,$$
$$\kappa = \frac{\mu_4}{\sigma^4} - 3 = \frac{a^2+4a+1}{a+b}\,.$$

*respectively.*

In the case when the first positive term in (1.1) is $p(n_0)$, $n_0 \geq 1$, the formulae (2.1)-(2.4) for moments need some modifications. They are given in the following theorem.

**Theorem 2.** *Moments* $m_r(c)$, $\alpha_r$, *and* $\mu_r$ *for a probability function* (1.1) *with* $p(n_0) > 0$, $n_0 \geq 0$, *satisfy the following equations:*

$$m_r(c) = (-c)^r + \frac{1}{1-a}\Big\{ n_0^r p(n_0) + p(n_0)\sum_{j=1}^{r-1}\binom{r}{j}(-c)^j n_0^{r-j}$$
$$+ \sum_{i=0}^{r-1}\Big[a\binom{r}{i} + b\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}(-c)^{r-j-1}(1+c)^{j-i}\Big] m_i(c)\Big\}\,,$$
$$r \geq 1\,,\ m_0(c) = 1\,,$$

$$\alpha_r := m_r(0) = \frac{1}{1-a}\Big\{ n_0^r p(n_0) + \sum_{j=0}^{r-1}\Big[a\binom{r}{j} + b\binom{r-1}{j}\Big]\alpha_j\Big\}\,,$$
$$r \geq 1,\ \alpha_0 = 1\,,$$

$$\mu_r := m_r\Big(\frac{n_0 p(n_0) + a + b}{1-a}\Big) = \Big(\frac{n_0 p(n_0) + a + b}{a-1}\Big)^r$$
$$+ \frac{1}{1-a}\Big\{ p(n_0)\sum_{j=0}^{r-1}\binom{r}{j}\Big(\frac{n_0 p(n_0) + a + b}{a-1}\Big)^j n_0^{r-j} + \sum_{i=0}^{r-1}\Big[a\binom{r}{i}$$

$$+\,b\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\Big(\frac{n_0p(n_0)+a+b}{a-1}\Big)^{r-j-1}\Big(\frac{n_0p(n_0)+1+b}{1-a}\Big)^{j-i}\Big]\mu_i\Big\},$$

$r \geq 1\,,\ \mu_0 = 1\,,$

*respectively.*

For $r = 1,2$ , we have

$$\alpha_1 = \frac{a+b+n_0p(n_0)}{1-a}\,,$$

$$\alpha_2 = \frac{(a+b)[a+b+1+n_0p(n_0)]+n_0p(n_0)[n_0-(n_0-1)a]}{(1-a)^2}\,,$$

$$\mu_2 = \frac{a+b-n_0p(n_0)(n_0a+b)+n_0^2p(n_0)[1-p(n_0)]}{(1-a)^2}\,.$$

**Corollary 2.** *Under the assumptions of* Theorem 2 *with* $n_0 = 1$ *we have*

(2.5)
$$\alpha_r = \frac{1}{1-a}\Big\{p(1)+\sum_{j=0}^{r-1}\Big[a\binom{r}{j}+b\binom{r-1}{j}\Big]\alpha_j\Big\}\,,$$

$r \geq 1\,,\ \alpha_0 = 1\,,$

(2.6)
$$\mu_r = \Big(\frac{p(1)+a+b}{a-1}\Big)^r + \frac{1}{1-a}\Big\{p(1)\sum_{j=0}^{r-1}\binom{r}{j}\Big(\frac{p(1)+a+b}{a-1}\Big)^j + \sum_{i=0}^{r-1}\Big[a\binom{r}{i}+b\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\Big(\frac{p(1)+a+b}{a-1}\Big)^{r-j-1} \times\Big(\frac{p(1)+1+b}{1-a}\Big)^{j-i}\Big]\mu_i\Big\}\,,\ r\geq 1\,,\ \mu_0 = 1\,.$$

**Remark.** The equations (2.5) and (2.6) allow us to give formulate for moments of probability functions (i)-(iv) truncated at the point $n_0 = 1$.

By Theorem 1 and 2 we obtain the following formulae for moments of the diserete distributions (i)-(iv).

**Corollary 3.** *The ordinary moments* $\alpha_r$ *and the central moments of the binomial distribution* (i), *the Poisson distribution* (ii), *the geometric distribution* (iii), *the negative binomial distribution* (iv), *the logaritmic distribution* (v) *and the truncated Poisson distribution* (vi) *satisfy the following equations:*

$$\alpha_r = p\sum_{j=0}^{r-1}\left[N\binom{r-1}{j} - \binom{r-1}{j-1}\right]\alpha_j\,,$$

(i) $$\mu_r = (-Np)^r - p\sum_{i=0}^{r-1}\left[\binom{r}{i} - (N+1) \times\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}(-Np)^{r-j-1}(1+Np)^{j-i}\right]\mu_i\,;$$

(ii) $$\alpha_r = \lambda\sum_{j=0}^{r-1}\binom{r-1}{j}\alpha_j\,,$$

$$\mu_r = (-\lambda)^r + \lambda\sum_{i=0}^{r-1}\left[\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}(-\lambda)^{r-j-1}(1+\lambda)^{j-i}\right]\mu_i\,;$$

(iii) $$\alpha_r = \frac{q}{p}\sum_{j=0}^{r-1}\binom{r}{j}\alpha_j, \quad \mu_r = \frac{q}{p}\left\{\sum_{j=0}^{r-1}\binom{r}{j}\mu_j - \left(-\frac{q}{p}\right)^{r-1}\right\};$$

$$\alpha_r = \frac{q}{p}\sum_{j=0}^{r-1}\left[k\binom{r-1}{j} + \binom{r-1}{j-1}\right]\alpha_j\,,$$

(iv) $$\mu_r = \left(-\frac{kq}{p}\right)^r + \frac{q}{p}\sum_{i=0}^{r-1}\left[\binom{r}{i} + (k-1)\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\left(-\frac{kq}{p}\right)^{r-j-1}\left(1+\frac{kq}{p}\right)^{j-i}\right]\mu_i\,;$$

$$\alpha_r = \frac{q}{p}\Big\{-\frac{1}{\ln p} + \sum_{j=1}^{r-1}\binom{r-1}{j-1}\alpha_j\Big\},$$

(v) $$\mu_r = \Big(\frac{q}{p\ln p}\Big)^r - \frac{q}{p}\Big\{\frac{1}{\ln p}\sum_{j=0}^{r-1}\binom{r}{j}\Big(\frac{q}{p\ln p}\Big)^j - \sum_{i=0}^{r-1}\Big[\binom{r}{i} - \sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\Big(\frac{q}{p\ln p}\Big)^{r-j-1}\Big(1-\frac{q}{p\ln p}\Big)^{j-i}\Big]\mu_i\Big\};$$

(vi)

$$\alpha_r = \lambda\Big\{\frac{e^{-\lambda}}{1-e^{-\lambda}} + \sum_{j=0}^{r-1}\binom{r-1}{j}\alpha_j\Big\},$$

$$\mu_r = \Big(-\frac{\lambda}{1-e^{-\lambda}}\Big)^r + \lambda\Big\{\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{j=0}^{r-1}\binom{r}{j}\Big(-\frac{\lambda}{1-e^{-\lambda}}\Big)^j + \sum_{i=0}^{r-1}\Big[\sum_{j=i}^{r-1}\binom{r}{j+1}\binom{j}{i}\Big(-\frac{\lambda}{1-e^{-\lambda}}\Big)^{r-j-1}\Big(1+\frac{\lambda}{1-e^{-\lambda}}\Big)^{j-i}\Big]\mu_i\Big\};$$

*respectively.*

**3. The method of the moment generating function.** Now, by the method of the moment generating function we give relations for moments of probability functions satisfying (1.1).

Using (1.1) we can write the generating function $M_{T-c}(u)$ of the moments of $T$ about $c$ as follows:

$$M_{T-c}(u) = \sum_{n=n_0}^{\infty} e^{u(n-c)}p(n) = e^{u(n_0-c)}p(n_0) + ae^u\sum_{n=n_0+1}^{\infty} e^{u(n-1-c)}p(n-1) + b\sum_{n=n_0+1}^{\infty}(e^{u(n-c)}/n)p(n-1),$$

with $p(n_0) > 0$ ($n_0$ is the smallest integer of that property). Hence we have

$$(1-ae^u)M_{T-c}(u) = e^{u(n_0-c)}p(n_0) + b\sum_{n=n_0+1}^{\infty}(e^{u(n-c)}/n)p(n-1)\ .$$

Derivation gives

$$(1-ae^u)M_{T-c}(u) - ae^u M_{T-c}(u) = (n_0-c)e^{u(n_0-c)}p(n_0)$$
$$+\, be^u \sum_{n=n_0+1}^{\infty} e^{u(n-1-c)}p(n-1) - bc \sum_{n=n_0+1}^{\infty} (e^{u(n-c)}/n)p(n-1)\,.$$

After putting

$$(b/n)p(n-1) = p(n) - ap(n-1)$$

we get

$$(1-ae^u)M_{T-c}(u) = (a+b+ac)e^u M_{T-c}(u) - cM_{T-c}(u) + n_0 e^{u(n_0-c)}p(n_0)\,.$$

According to Leibnitz' formula the derivative of order $r-1$ of both sides gives

$$(1-ae^u)M^{(r)}_{T-c} - a\sum_{j=1}^{r-1}\binom{r-1}{j}(e^u)^{(j)}M^{(r-j)}_{T-c}(u)$$
$$= (a+b+ac)\sum_{j=0}^{r-1}\binom{r-1}{j}(e^u)^{(j)}M^{(r-j-1)}_{T-c}(u)$$
$$-\, cM^{r-1}_{T-c}(u) + n_0(n_0-c)^{r-1}e^{u(n_0-c)}p(n_0)\,,$$

or

$$(1-ae^u)M^{(r)}_{T-c} = e^u\sum_{j=0}^{r-1}\left[a\binom{r}{j+1} + (b+ac)\binom{r-1}{j}\right]M^{(r-j-1)}_{T-c}(u)$$
$$-\, cM^{r-1}_{T-c}(u) + n_0(n_0-c)^{r-1}e^{u(n_0-c)}p(n_0)\,.$$

Hence setting $u=0$ we obtain the following moment formulae. ■

**Theorem 3.** *Moments $m_r(c)$, $\alpha_r$ and $\mu_r$ of discrete probability function* (1.1) *satisfy the following equations:*

$$m_r(c) = \frac{1}{1-a}\Big\{ n_0(n_0-c)^{r-1}p(n_0) + \sum_{j=1}^{r-1}\Big[a\binom{r}{j+1} + (b+ac)\binom{r-1}{j}\Big] m_{r-j-1}(c) + [ra+b-(1-a)c]m_{r-1}(c)\Big\},$$
$$r \ge 1; \qquad m_0(c) = 1,$$

$$\alpha_r = \frac{1}{1-a}\Big\{ n_0^r p(n_0) + \sum_{j=0}^{r-1}\Big[a\binom{r}{j+1} + b\binom{r-1}{j}\Big]\alpha_{r-j-1}\Big\},$$
$$r \ge 1,\ \alpha_0 = 1,$$

$$\mu_r = \frac{1}{1-a}\Big\{ n_0 p(n_0)\Big(n_0 - \frac{n_0p(n_0)+a+b}{1-a}\Big)^{r-1} + \sum_{j=1}^{r-1}\Big[a\binom{r}{j+1} + \frac{n_0p(n_0)a+a^2+b}{1-a}\binom{r-1}{j}\Big]\mu_{r-j-1} + [(r-1)a - n_0p(n_0)]\mu_{r-1}\Big\},\ r\ge 1,\ \mu_0 = 1 .$$

In the case when $n_0 = 0$ we get recurrence relations for moments $m_r(c)$, $\alpha_r$ and $\mu_r$ of counting distributions useful in many applications.

**Theorem 3′.** *Recurrence formulae for moments $m_r(c)$, $\alpha_r$ and $\mu_r$ of discrete distributions* (1.1) *with $p(0) > 0$ are as follows:*

$$m_r(c) = \frac{1}{1-a}\Big\{\sum_{j=1}^{r-1}\Big[a\binom{r}{j+1} + (b+ac)\binom{r-1}{j}\Big] m_{r-j-1}(c) + [ra+b-(1-a)c]m_{r-1}(c)\Big\},\ r\ge 1;\ m_0 = 1,$$

$$\alpha_r = \frac{1}{1-a}\sum_{j=0}^{r-1}\left[a\binom{r}{j+1} + b\binom{r-1}{j}\right]\alpha_{r-j-1}\,,\ r \geq 1\,,\ \alpha_0 = 1\,,$$

$$\mu_r = \frac{1}{1-a}\left\{\sum_{j=1}^{r-1}\left[a\binom{r}{j+1} + \frac{a^2+b}{1-a}\binom{r-1}{j}\right]\mu_{r-j-1} + (r-1)a\mu_{r-1}\right\},\ r \geq 1\,,\ \mu_0 = 1\,.$$

Moments of probability distribution functions (1.2)-(1.7) are contained in the following corollary.

**Corollary 4.** *Moments* $\alpha_r$ *and central moments* $\mu_r$ *of probability distribution functions* (i)-(iv) *satisfy the following equations:*

(i)
$$\alpha_r = p\sum_{j=0}^{r-1}\left[N\binom{r-1}{j} - \binom{r-1}{j+1}\right]\alpha_{r-j-1}\,,$$
$$\mu_r = p\left\{\sum_{j=1}^{r-1}\left[Nq\binom{r-1}{j} - \binom{r-1}{j+1}\right]\mu_{r-j-1} - (r-1)\mu_{r-1}\right\};$$

(ii) $$\alpha_r = \lambda\sum_{j=0}^{r-1}\binom{r-1}{j}\alpha_{r-j-1}\,,\qquad \mu_r = \lambda\sum_{j=1}^{r-1}\binom{r-1}{j}\mu_{r-j-1}\,;$$

(iii)
$$\alpha_r = \frac{q}{p}\sum_{j=0}^{r-1}\binom{r}{j+1}\alpha_{r-j-1}\,,$$
$$\mu_r = \frac{q}{p^2}\left\{\sum_{j=1}^{r-1}\left[p\binom{r-1}{j+1} + \binom{r-1}{j}\right]\mu_{r-j-1} + (r-1)q\mu_{r-1}\right\};$$

(iv)
$$\alpha_r = \frac{q}{p}\sum_{j=0}^{r-1}\left[k\binom{r-1}{j} + \binom{r-1}{j+1}\right]\alpha_{r-j-1}\,,$$
$$\mu_r = \frac{q}{p^2}\left\{\sum_{j=1}^{r-1}\left[p\binom{r-1}{j+1} + k\binom{r-1}{j}\right]\mu_{r-j-1} + (r-1)q\mu_{r-1}\right\};$$

$$\alpha_r = \frac{q}{p}\left\{\sum_{j=0}^{r-2}\binom{r-1}{j+1}\alpha_{r-j-1} - \frac{1}{\ln p}\right\},$$

(v) $$\mu_r = -\frac{q}{p}\left\{\frac{1}{\ln p}\left(1+\frac{q}{\ln p}\right)^{r-1} + \sum_{j=1}^{r-1}\left[\binom{r-1}{j}\frac{q}{p\ln p} - \binom{r-1}{j+1}\right]\mu_{r-j-1} - \left(r-1+\frac{1}{\ln p}\right)\mu_{r-1}\right\};$$

$$\alpha_r = \lambda\left\{\sum_{j=0}^{r-1}\binom{r-1}{j}\alpha_{r-j-1} + \frac{e^{-\lambda}}{1-e^{-\lambda}}\right\},$$

(vi) $$\mu_r = \lambda\left\{\sum_{j=1}^{r-1}\binom{r-1}{j}\mu_{r-j-1} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\mu_{r-1} + \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(1-\frac{\lambda}{1-e^{-\lambda}}\right)^{r-1}\right\}$$

$$r \geq 1, \qquad \alpha_0 = 1, \qquad \mu_0 = 1,$$

*respectively.*

**4. Relations between ordinary and central moments.** Using the formulae:

$$\alpha_r = \sum_{j=0}^{r}\binom{r}{j}\alpha_1^j\mu_{r-j}, \qquad \alpha_r = \sum_{j=0}^{r}\binom{r}{j}\alpha_1^{r-j}\mu_j;$$

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha_j(-\alpha_1)^{r-j}, \qquad \mu_r = \sum_{j=0}^{r}\binom{r}{j}(-\alpha_1)^j\alpha_{r-j},$$

we get the following results.

**Theorem 4.** *Relations between ordinary and central moments of the probability function* (1.1) *with* $p(0) > 0$ *describe the following equations:*

$$\alpha_r = \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a+b}{1-a}\right)^j\mu_{r-j} \text{ or } \alpha_r = \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a+b}{1-a}\right)^{r-j}\mu_j,$$

$$r \geq 1,\ \alpha_0 = 1,$$

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \left(\frac{a+b}{1-a}\right)^{r-j} \alpha_j \text{ or } \mu_r = \sum_{j=0}^{r} \binom{r}{j} \left(\frac{a+b}{a-1}\right)^{j} \alpha_{r-j},$$
$$r \geq 1, \; \alpha_0 = 1, \; \mu_0 = 1 .$$

Using Theorem 4 and the formulae $\alpha_1 = (a+b)/(1-a)$ and $\mu_2 = (a+b)/(1-a)^2$ we can state.

**Corollary 5.** *Under the assumptions of* Theorem 3′ *we have*

$$\frac{\alpha_r}{\alpha_1^r} = 1 + \sum_{j=2}^{r} \left(\frac{1-a}{a+b}\right)^{j} \mu_j, \qquad r \geq 1,$$

$$\frac{\mu_{2r}}{\mu_2^r} = \sum_{j=0}^{2r} \binom{2r}{j} (a+b)^{r-j} (a-1)^j \alpha_j .$$

In particular we see that

$$\frac{\alpha_2}{\alpha_1^2} = 1 + \frac{1}{a+b}, \quad \frac{\mu_4}{\mu_2^2} = 3 + \frac{a^2+4a+1}{a+b} .$$

## References

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