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Heinrich RENELT (Halle)

Generalized Powers and Extensions of Analytic Functions

ABSTRACT. Cauchy-Riemann systems possess, in particular, global solutions $w(z)$ of the topological structure $w(z) = (\chi(z))^n$ with n an integer and $\chi(\cdot)$ a quasiconformal mapping of $\mathbb{C}$ onto itself. If the coefficients of the systems satisfy (rather weak) additional smoothness properties, then at either of the branch points an asymptotic expansion (exists and) can be prescribed arbitrarily (with a well-defined degree of freedom), whereas an asymptotic expansion at the other branch point (also exists and) is then determined in a unique (but in general unknown) way. For a certain class of Cauchy-Riemann sustems we determine here this interdependence between both these expansions by means of a well-defined set of Taylor and Laurent coefficients. These ones depend (in an, in principle, computable way) only on the coefficients of the Caychy-Riemann system in question but are independent of our special problem. Moreover, they govern the continuability, of certain Taylor and Laurent series, to global solutions of Cauchy-Riemann systems.

I. Let a (generalized) Cauchy-Riemann system be given in complex notation

$$w_{\bar z} = \nu(z) w_z + \mu(z)\overline{w_z}, \tag{1}$$

where $\nu, \mu \in L_\infty$, $\| \, |\nu| + |\mu| \, \|_{L_\infty} < 1$. We assume without loss of generality that ν, μ are defined everywhere in $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$.

It proves to be natural to ask for special solutions of (1), which are analogues of the ordinary powers z^n. We have

Theorem 1 [1,2]. *Let ν, μ be Hölder continous at $z_0 \neq \infty$ and at ∞, n be any nonzero integer and a any constant $\neq 0$. There exists exactly one solution $w(z)$ of (1) in $\mathbb{C} \setminus \{z_0\}$ (in $\mathbb{C}$ if $n \geq 1$) such that*

(i) *$w(z) = (\chi(z))^n$, where χ is a quasiconformal mapping of $\mathbb{C}$ onto itself, $\chi(z_0) = 0$,*
and

(ii) *$w(z)$ admits the asymptotic expansion*

$$w(z) = a\left(z - z_0 + \mathfrak{b}(\overline{z - z_0})\right)^n - b\bar{a}\left(\overline{z - z_0} + \bar{\mathfrak{b}}(z - z_0)\right)^n + O\left(|z - z_0|^{n+\alpha}\right)$$

at z_0, where $\mathfrak{b}$, b are algebraic expressions in $\nu(z_0), \mu(z_0)$, and $\alpha > 0$.

We denote this unique $w(z)$ by $[a(z - z_0)^n]_{\nu,\mu}$ or simply by $[a(z - z_0)]$ if no misunderstanding is possible. Existence and uniqueness of $[a(z - z_0)^n]$ also hold under weaker conditions on ν, μ, cf. [1].

Every such generalized power $[a(z - z_0)^n]$ also admits an asymptotic expansion at ∞,

$$[a(z - z_0)^n] = \alpha_n(z + \mathfrak{b}_\infty \bar{z})^n - \overline{\alpha_n} b_\infty(\bar{z} + \bar{\mathfrak{b}}_\infty z)^n + O(|z|^{n+\alpha'})$$

where $\mathfrak{b}_\infty$, b_∞ are fixed algebraic expression in $\nu(\infty)$, $\mu(\infty)$, and $\alpha' > 0$. The correspondence between a and α_n is one-to-one for each n. Since (1) is real-linear, α_n runs through a (nondegenerate) ellipse around the origin if a runs through the unit circle.

We ask for the kind of this one-to-one correspondence under the additional condition that

(2) $\quad \nu(z) \equiv 0$ in $\mathbb{C}$, $\quad \mu(z) = 0$ for $|z| < r$ and for $|z| > R$,

r, R positive constants, and $z_0 = 0$.

II. For $n \geq 1$ we then have

$$(3) \quad [az^n] = \begin{cases} az^n + \sum_{j=n+1}^{\infty} \beta_j z^j & \text{for } |z| < r \\ \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0 + \sum_{j=1}^{\infty} \alpha_{-j} z^{-j} & \text{for } |z| > R. \end{cases}$$

We want to determine $\alpha_0, \alpha_1, \ldots, \alpha_n$ by a (in fact both a and α_n determine all the remaining coefficients in (3) in a unique way).

To start with we dispense with the condition $\mu(z) = 0$ for $|z| < r$. Let $Q(z)$ be any polynomial,

$$Q(z) = \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0,$$

and let $H(z)$ be the solution of

$$w_{\bar{z}} = \mu(z)\overline{w_z} \quad \text{in } \mathbb{C} \tag{1'}$$

such that $Q(z) - \alpha_0$ is the principal part of the Laurent expansion of H at ∞,

$$H(z) = Q(z) + \sum_{j=1}^{\infty} \alpha_{-j} z^{-j} \quad \text{for } |z| > R. \tag{4}$$

By Liouville's theorem (for solutions of (1)), H is uniquely determined by Q. Using the integral operator P defined by

$$Pg(z) = -\frac{1}{\pi}\int \frac{g(\zeta)}{\zeta - z} d\sigma_\zeta \quad \text{for } g \in L_p \cap L_q, \quad 1 < q < 2 < p < \infty,$$

and the two-dimensional Hilbert transformation T, symbolically

$$Tg(z) = -\frac{1}{\pi}\int \frac{g(\zeta)}{(\zeta - z)^2} d\sigma_\zeta \quad \text{for } g \in L_s, \quad 1 < s < \infty,$$

H admits the representation

$$H(z) = Q(z) + f(z) \tag{5}$$

where

$$f(z) = P(\mu\overline{Q'} + \mu\bar{h}), \tag{6}$$

and $h = f_z(z)$ is the unique solution of

$$h = T\mu\overline{Q'} + T\mu\bar{h} \quad \text{in } L_s, \tag{7}$$

s from a neighbourhood of 2. This follows easily by means of the well-known properties of P and T and the validity of (1′) for H.

Let

$$p_k(z) = k\bar{z}^{k-1}, \tag{8}$$

k a natural number for the time being, and let h_k be defined by

$$h_k = T\mu\overline{\alpha_k}p_k + T\mu\overline{h_k}. \tag{9}$$

Then

$$h = \sum_{k=1}^{n} h_k.$$

We put

$$(\overline{T\mu})^0 g = g, \quad (\overline{T\mu})^m g = \overline{T\mu((\overline{T\mu})^{m-1}g)} \quad \text{for } m \geq 1,$$

and

$$\begin{aligned} G_k(z;\mu) &= T\mu\left(\sum_{\nu=0}^{\infty}(\overline{T\mu})^{2\nu}p_k\right)(z), \\ F_k(z;\mu) &= T\mu\left(\sum_{\nu=0}^{\infty}(\overline{T\mu})^{2\nu+1}p_k\right)(z). \end{aligned} \tag{10}$$

Then the unique solution of (9) (by the Banach fixed point theorem) is

$$h_k = \alpha_k F_k(z;\mu) + \overline{\alpha_k}G_k(z;\mu).$$

Because $\mu(z) = 0$ for $|z| > R$, F_k, G_k are analytic there, and ∞ is a zero of second order,

$$F_k(z;\mu) = \sum_{j=2}^{\infty} A_{-j}^{(k)} z^{-j}, \quad G_k(z;\mu) = \sum_{j=2}^{\infty} B_{-j}^{(k)} z^{-j}. \tag{11}$$

Because

$$f_z(z) = f'(z) = \sum_{j=2}^{\infty}\left(\sum_{k=1}^{n}(\alpha_k A_{-j}^{(k)} + \overline{\alpha_k}B_{-j}^{(k)})\right)z^{-j} \quad \text{for } |z| > R, \tag{12}$$

a comparison of the coefficients in (4) and (12) together with Liouville's theorem gives, as a first consequence,

Extension criterion 1. *Let $H^*(z) = \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0 + \sum_{j=1}^{\infty} \alpha_{-j} z^{-j}$ be analytic for $R < |z| < \infty$. H^* is the restriction of a solution of $(1')$ to $\{|z| > R\}$ if and only if*

$$j\alpha_j = \sum_{k=1}^{n} (\alpha_k A_{j-1}^{(k)} + \overline{\alpha_k} B_{j-1}^{(k)}) \qquad \forall j = -1, -2, \dots .$$

Of course, a corresponding statement holds if ∞ is an essential singularity of H^*.

Now let μ also satisfy $\mu(z) = 0$ for $|z| < r$. Then F_k, G_k possess Taylor expansions at $z = 0$,

$$F_k(z;\mu) = \sum_{j=0}^{\infty} A_j^{(k)} z^j, \quad G_k(z;\mu) = \sum_{j=0}^{\infty} B_j^{(k)} z^j. \tag{13}$$

If we now require that $H(z) = [az^n]$, we must have

$$(az^n + \sum_{j=n+1}^{\infty} \beta_j z^j)' = Q'(z) + \sum_{j=0}^{\infty} \left(\sum_{k=1}^{n} (\alpha_k A_j^{(k)} + \overline{\alpha_k} B_j^{(k)}) \right) z^j$$

for $|z| < r$. This gives

$$j\alpha_j + \sum_{k=1}^{n} (\alpha_k A_{j-1}^{(k)} + \overline{\alpha_k} B_{j-1}^{(k)}) = 0 \quad \text{for } j = 1, \dots, n-1$$

and

$$n\alpha_n + \sum_{k=1}^{n} (\alpha_k A_{n-1}^{(k)} + \overline{\alpha_k} B_{n-1}^{(k)}) = na$$

as necessary and sufficient conditions that the $H(z)$ made up by $Q'(z)$ according to (5) – (7) is $[az^n]$; the missing condition for α_0 is simply

$$\alpha_0 = -P(\mu \overline{Q'} + \mu \bar{h})(0). \tag{14}$$

Here we have used the fact that a solution of $(1')$ having a zero of order $\geq n+1$ at 0 and a pole of order $\leq n$ at ∞ must be identically zero. Thus we have proved

Theorem 2. *$H(z) = [az^n]$ if and only if H is given by (5)–(7), where $\alpha_1, \ldots, \alpha_n$ satisfy*

$$(15) \qquad \sum_{k=1}^{n} [\alpha_k (A_{j-1}^{(k)} + \delta_{j,k} k) + \overline{\alpha_k} B_{j-1}^{(k)}] = j a \delta_{j,n}, \quad j = 1, \ldots, n,$$

and α_0 satisfies (14).

The equations (15) mean a $2n$–dimensional real system of linear equations for the $2n$ real unknowns $\Re\alpha_1, \ldots, \Re\alpha_n, \Im\alpha_1, \ldots, \Im\alpha_n$. Since we already know existence and uniqueness of $[az^n]$, we obtain

Corollary 1. *System* (15) *with any right–hand side $b_1, \ldots, b_n$ has always a unique solution $\alpha_1, \ldots, \alpha_n$.*

The determination of the coefficients β_j in (3) is contained in the following more general consideration: Let $H(z)$ be any solution of (1′) having a pole of order $\leq n$ at ∞, i. e.

$$H(z) = \sum_{j=0}^{\infty} \beta_j z^j \quad \text{for } |z| < r,$$

$$H(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 + \sum_{j=1}^{\infty} \alpha_{-j} z^{-j} \quad \text{for } |z| > R.$$

Then

$$\left(\sum_{j=0}^{\infty} \beta_j z^j \right)' = Q'(z) + f'(z) = \left(\sum_{j=0}^{n} \alpha_j z^j \right)' + \sum_{j=0}^{\infty} \left(\sum_{k=1}^{n} (\alpha_k A_j^{(k)} + \overline{\alpha_k} B_j^{(k)}) \right) z^j$$

for $|z| < r$. This implies

$$(16) \qquad \sum_{k=1}^{n} [\alpha_k (A_{j-1}^{(k)} + \delta_{j,k} k) + \overline{\alpha_k} B_{j-1}^{(k)}] = j \beta_j \quad \text{for } j = 1, \ldots, n$$

and

$$(17)\qquad \sum_{k=1}^{n}(\alpha_k A_{j-1}^{(k)} + \overline{\alpha_k} B_{j-1}^{(k)}) = j\beta_j \quad \text{for } j = n+1, n+2, \ldots .$$

By corollary 1, the system (16) has always a unique solution, for arbitrary $\beta_1, \ldots, \beta_n$. Thus we have proved the following

Extension criterion 2. *A function $H_*(z) = \sum_{j=0}^{\infty} \beta_j z^j$ analytic for $|z| < r$ can be extended to a solution $H(z)$ of $(1')$ with a pole of at most nth order at ∞ if and only if (17) holds, where $\alpha_1, \ldots, \alpha_n$ are the unique solution of (16).*

III. We now come to the case $n \leq -1$. We put $|n| = m$. Let

$$R(z) = \gamma_{-m} z^{-m} + \cdots + \gamma_{-1} z^{-1},$$

$H(z)$ be a solution of $(1')$ in $\mathbb{C} \setminus \{0\}$ with $R(z)$ being the principle part of the Laurent expansion of H at 0, and ∞ be a zero of H,

$$H(z) = \sum_{j=1}^{\infty} \beta_{-j} z^{-j} \quad \text{for } |z| > R.$$

Then $f(z) := H(z) - R(z)$ admits a Taylor expansion at 0,

$$f(z) = \sum_{j=0}^{\infty} \gamma_j z^j \quad \text{for } |z| < r.$$

Also $f(z)$ has a zero of at least the first order at ∞, hence

$$f_z \in L_p \cap L_q \quad \text{with } 1 < q < 2 < p < \infty.$$

Because of $(1')$ we have

$$f_{\bar z} = \mu \overline{f_z} + \mu \overline{R'},$$

thus

$$f(z) = P(f_{\bar z})(z) = P(\mu \overline{R'} + \mu \bar h)(z)$$

with

$$h = T\mu\overline{R'} + T\mu\bar{h}.$$

Let $p_k(z)$ be defined by (8), but now $k = -1, -2, \dots$. For

$$h_k = T\mu\overline{\gamma_k}p_k + T\mu\overline{h_k}, \quad k = -1, -2, \dots, -m$$

we have

$$h = \sum_{k=-1}^{-m} h_k = \sum_{k=-1}^{-m} (\gamma_k F_k(z;\mu) + \overline{\gamma_k}G_k(z;\mu))$$

with F_k, G_k as in (10), (11), (13), but where k is now a negative integer. Thus,

$$f_z(z) = h(z) = \begin{cases} \sum_{j=0}^{\infty}\left(\sum_{k=-1}^{-m}(\gamma_k A_j^{(k)} + \overline{\gamma_k}B_j^{(k)})\right)z^j & \text{for } |z| < r \\ \sum_{j=2}^{\infty}\left(\sum_{k=-1}^{-m}(\gamma_k A_{-j}^{(k)} + \overline{\gamma_k}B_{-j}^{(k)})\right)z^{-j} & \text{for } |z| > R, \end{cases}$$

and hence

$$\sum_{j=l}^{\infty}(-j)\beta_{-j}z^{-j-1} + \sum_{j=1}^{m} j\gamma_{-j}z^{-j-1} =$$

$$\sum_{j=1}^{\infty}\left(\sum_{k=-1}^{-m}(\gamma_k A_{-j-1}^{(k)} + \overline{\gamma_k}B_{-j-1}^{(k)})\right)z^{-j-1} \quad \text{for } |z| > R,$$

that is

$$(18) \qquad \begin{aligned} (-j)\beta_{-j} &= (-j)\gamma_{-j} \\ &+ \sum_{k=-1}^{-m}(\gamma_k A_{-j-1}^{(k)} + \overline{\gamma_k}B_{-j-1}^{(k)}) \quad \forall j = 1, 2, \dots, m \end{aligned}$$

and

$$(19) \qquad (-j)\beta_{-j} = \sum_{k=-1}^{-m}(\gamma_k A_{-j-1}^{(k)} + \overline{\gamma_k}B_{-j-1}^{(k)}) \quad \forall j \geq m+1.$$

We have $H(z) = [\gamma_{-m}z^{-m}]$ if and only if $H(z)$ has a zero of order m at ∞ , and this again holds if and only if $\beta_{-1} = \beta_{-2} = \cdots = \beta_{-(m-1)} = 0, \quad \beta_{-m} \neq 0$. Thus we have proved

Theorem 3. *$H(z) = [\gamma_{-m} z^{-m}]$ if and only if $\gamma_{-1}, \ldots, \gamma_{-m}$ satisfy*

$$\sum_{k=-1}^{-m} [\gamma_k (A_{j-1}^{(k)} + \delta_{j,k} k) + \overline{\gamma_k} B_{j-1}^{(k)}] = j \delta_{j,-m} \beta_{-m}, \quad j = -1, -2, \ldots, -m. \tag{20}$$

If we prescribe β_{-m}, then we again obtain $\gamma_{-1}, \ldots, \gamma_{-m}$ as the unique solution of system (20). Equally, $\gamma_{-m} = a$ can be prescribed. Then the first $m-1$ equations from (20) yield the unique $\gamma_{-1}, \ldots, \gamma_{-(m-1)}$, and the last equation then gives β_{-m} (and $\beta_{-m} \neq 0$ if $a \neq 0$). All this, in particular the unique solvability for both these variants, is again a consequence of the existence and uniqueness of $[az^{-m}]$.

As extension condition we here obtain analogously to extension criterion 1

Extension criterion 3. *The function*

$$_*H(z) = \sum_{j=-m}^{+\infty} \gamma_j z^j$$

analytic in $0 < |z| < r$ admits an extension to a solution of (1') in $\mathbb{C} \setminus \{0\}$, which is bounded at ∞, if and only if

$$j\gamma_j = \sum_{k=-1}^{-m} (\gamma_k A_{j-1}^{(k)} + \overline{\gamma_k} B_{j-1}^{(k)}) \quad \forall j = 1, 2, \ldots .$$

Of course, also extension criterion 2 has an analogue.

Remarks.

1. The situation for Beltrami systems

$$w_{\bar{z}} = \nu w_z$$

is analogous and simpler.

2. There arise some questions, for instance for

- necessary and/or sufficient conditions for a system of coefficients $(A_j^{(k)}, B_j^{(k)})$, $k = \pm 1, \pm 2, \ldots,\quad j = 0, 1, \pm 2, \ldots$ to belong to a system (1′),
- completions of these systems of coefficients such that these completions determine the corresponding $\mu(z)$ in a unique way,
- the behaviour of the $A_j^{(k)}, B_j^{(k)}$ or their expressions if $r \to 0$. (The corresponding questions of course also exist for Beltrami systems.)

References

[1] Renelt, H., *Generalized powers in the theory of (ν, μ)-solutions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **40** (1986), 217–235.

[2] ______, *An integral formula for the derivatives of solutions of certain elliptic systems*, Ann. Polon. Math. **54** (1991), 45–57.

Martin-Luther-Universität
Halle-Wittenberg,
Fachbereich Mathematik und Informatik,
D-06099 Halle/S Germany
e-mail renelt @mathematik.uni - halle.d 400.de

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