ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLIX, 11 SECTIO A 1995

Małgorzata KRÓLIKOWSKA (Lublin)

Nonlinear Mappings with an Almost Sparse Jacobian Matrix

ABSTRACT. In this article we consider a class of nonlinear mappings with an almost sparse Jacobian matrix.We propose two methods for solving nonlinear equations $F(x) = 0$ for mappings F with an almost sparse Jacobian matrix . We establish some results about Q-linear and Q-superlinear convergence and discuss several least-change updates for which local Q-superlinear convergence is proved. We also give numerical results which compare the classical methods with their proposed modification.

1. Introduction. Let $\mathbb{R}^n$ denote the n-dimensional real linear space and $\mathbb{R}^{n\times n}$ denote the $n \times n$ - dimensional space of matrices. Let $\|\cdot\|$ be an arbitrary norm. Let $F : \mathbb{R}^n \supseteq D \to \mathbb{R}^n$ be a nonlinear mapping, where D is an open, convex set. We are interested in finding a solution $x^* \in D$ to the equation

$$F(x) = 0. \tag{1.1}$$

A traditional method for solving nonlinear systems of equations is Newton's method. Newton's method is attractive because it converges rapidly from any sufficiently close initial guess x_0, but sometimes it is difficult to describe all formulae for the partial derivatives. Quasi-Newton's methods are used in practice; the Jacobian matrix $F'(x_k)$ of the mapping F is approximated by a matrix B_k with a similar structure.

In this paper we introduce the definition of a nonlinear mapping with an almost sparse Jacobian matrix:

Definition 1.1. An operator $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a mapping with the almost sparse Jacobian matrix at x^* if:

A1. There exist mappings F_1 and F_2 such that $F(x) = F_1(x) + F_2(x)$, $F_1'(x)$ is the sparse matrix for $x \in D$
and

A2. The Jacobian matrix $F_1'(x^*)$ is nonsingular;

A3. There exists a constant $\gamma \in (0,1)$ such that

$$\|[F_1'(x^*)]^{-1} F_2'(x^*)\| \le \gamma,$$

where x^* is the solution of the equation $F(x) = 0$.

This paper is an attempt of investigating Q-linear and superlinear convergence of mappings with an almost sparse Jacobian matrix.

The optimization problems with the small parameter belong to the class of mappings with an almost sparse Jacobian matrix. These problems are given by the formula

$$\min_{x \in \mathbb{R}^n} h(x) = \min_{x \in \mathbb{R}^n} [h_1(x) + t h_2(x)],$$

where the matrix of second partial derivatives $\nabla^2 h_1(x)$ is nonsingular, sparse and the parameter $t \in (0,1]$. We assume that

$$\|[\nabla^2 h_1(x^*)]^{-1} \nabla^2 h_2(x^*)\| < t_0^{-1}.$$

J. Jayakuma and N. Ramanujan consider in their paper [5] the differential equation

$$L_u(x) = \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$$

with initial conditions

$$u(0) - \varepsilon u'(0) = A, u'(1) = B, x \in [0,1],$$

$$a(x) > a > 0, b(x) > b > 0, A, B - \text{constants}$$

for a small parameter ε , $0 < \varepsilon < \varepsilon_0 < 1$.

We propose for the problems with an almost sparse Jacobian matrix the method defined by recurrention formula

$$(1.2) \qquad x_{k+1} = x_k - [F_1'(x_k)]^{-1} F(x_k),\ k = 0, 1, \ldots$$

or

$$(1.3) \qquad x_{k+1} = x_k - B_k^{-1} F(x_k),\ k = 0, 1, \ldots,$$

where $B_k \approx F_1'(x_k)$.

Here it is necessary to get the sparse Jacobian matrix $F_1'(x_k)$ (for example p-diagonal), or the matrix B_k with a similar structure.

Moreover we show in this paper that the proposed methods are locally Q-linearly convergent to the solution x^*, i.e. for the sequence $\{x_k\}$, $\lim_{k\to\infty} x_k = x^*$, there exists a constant q such that

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq q < 1,$$

and we give conditions which guarantee the methods are locally Q-superlinearly convergent to x^*, i.e. :

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0$$

The proposed methods (1.2),(1.3) need almost the same number of iterations like the Newton's to get a required approximation of the solution x^*, but the Jacobian matrix $F_1'(x_k)$ is a sparse matrix (for example p -diagonal) so solving the equation $F_1'(x_k)\ s_k = -F(x_k)$, where $s_k = x_{k+1} - x_k$, needs less computing operations than solving the equation $F'(x_k)\ s_k = -F(x_k)$ with full-rank Jacobian matrix $F'(x_k)$. That's why it is useful proposed method of solving some kinds of problems, like problems with a small parameter.

2. Modification of Newton's method for nonlinear systems with an almost sparse Jacobian matrix. Consider a system of equations

$$(2.1) \qquad F(x) = 0,$$

where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a nonsingular mapping with following properties:

A4. There exists a point $x^* \in D$ such that $F(x^*) = 0$;

A5. F is continously differentiable in an open neighborhood of x^*;

A6. The Jacobian matrix $F'(x^*)$ is nonsingular;

A7. There exists a constant $L > 0$ such that:

$$\forall x, y \in D \qquad \|F'(x) - F'(y)\| \leq L\|x - y\|; \tag{2.2}$$

A8. There exists a constant $L_1 > 0$ such that:

$$\forall x, y \in D \qquad \|F_1'(x) - F_1'(y)\| \leq L_1\|x - y\|. \tag{2.3}$$

Let the initial guess $x_0 \in D$ be given. Consider the method given by the formula (1.2):

$$x_{k+1} = x_k - [F_1'(x_k)]^{-1} F(x_k), k = 0, 1, ...$$

for the mapping F with the properties **A1 - A8** .

Now we show that the sequence $\{x_k\}$ described by (1.2) is locally Q-linearly convergent to x^*, where x^* is the solution of the problem (2.1).

Theorem 2.1. *Let a mapping F satisfy the properties* **A1 - A8**. *Then the sequence $\{x_k\}$ defined by the formula (1.2) is locally Q-linearly convergent to x^*.*

To prove this theorem we will use bounded deterioration

Theorem 2.2. *Let F satisfy the assumption* **A1 - A8.** *Consider the sequence $\{x_k\}$ defined by the formula*

$$x_{k+1} = x_k - B_k{}^{-1} F(x_k),$$

where $B_k \approx F_1'(x_k)$.

Assume that there are nonnegative constants q_1 and q_2 such that:

$$\|B_{k+1} - F_1'(x^*)\| < (1 + q_1 r_k)\|B_k - F_1'(x^*)\| + q_2 r_k,$$

for k = 0,1,..., where $r_k = \max(\|x_k - x^*\|, \|x_{k+1} - x^*\|)$.

Then there exist positive constants ε and δ such that for

$$\|x_0 - x^*\| \le \varepsilon, \qquad \|B_0 - F_1'(x^*)\| \le \delta$$

the sequence $\{x_k\}$ is Q-linearly convergent to x^*. Furthermore the sequences $\{\|B_k\|\}, \{\|B_k^{-1}\|\}$ are uniformly bounded.

Proof Theorem 2.2. This theorem is a particular case of the Theorem 1 given in [4] for $B^* = F_1'(x^*)$.

Proof Theorem 2.1. Note that

$$\|F_1'(x_{k+1}) - F_1'(x^*)\| \le$$

$$\le \|F_1'(x_{k+1}) - F_1'(x_k)\| + \|F_1'(x_k) - F_1'(x^*)\|$$

$$\le L_1\|x_{k+1} - x_k\| + \|F_1'(x_k) - F_1'(x^*)\| \le$$

$$\le 2L_1 r_k + \|F_1'(x_k) - F_1'(x^*)\|.$$

Now the proof directly follows from the bounded deterioration Theorem 2.2 for $B_k = F_1'(x_k)$. □

3. On local Q-linear and Q-superlinear convergence of least-change secant method. In [3] the author considers the class of methods

$$x_{k+1} = x_k - B_k^{-1} F(x_k),$$

such that $B_k \approx F'(x_k)$ and $B_k = C(x_k) + A_k$ where $C(x_k)$ denotes a "computer part" of $F'(x)$.

He defines B_{k+1} as orthogonal projection B_k on the convex set $Q \cap L_k$, where

$$L_k = \{X \in \mathbb{R}^{n\times n} : X(x_{k+1} - x_k) = F(x_{k+1}) - F(x_k)\}$$

and Q is a convex closed set such that $F'(x) \in Q$ for $x \in D$ and B_k should be in Q.

In this section we present the method:

$$x_{k+1} = x_k - B_k^{-1} F(x_k), k = 0, 1, ...,$$

where $B_k \approx F_1'(x_k)$ and we give an algorithm for computing the matrices B_k with a similar structure as the matrix $F_1'(x_k)$.

Consider the set of matrices

$$M_k = \{X \in \mathbb{R}^{n\times n} : X(x_{k+1} - x_k) = F_1(x_{k+1}) - F_1(x_k)\}. \tag{3.1}$$

Define the matrix H_k as

$$H_k = \int_0^1 F_1'(x_k + t(x_{k+1} - x_k))dt. \tag{3.2}$$

Denote $s_k = x_{k+1} - x_k$, $y_k = F_1(x_{k+1}) - F_1(x_k)$. Note that $H_k s_k = y_k$:

$$F_1(x_{k+1}) - F_1(x_k) = \int_0^1 F_1'(x_k + t(x_{k+1} - x_k))s_k dt = H_k s_k,$$

so $H_k \in M_k$.

In [3] the assumption $F'(x) \in Q \subset \mathbb{R}^{n\times n}$ was introduced. In this section we consider the assumption of the form:

A9. Assume that $F_1'(x) \in Q \subset \mathbb{R}^{n\times n}$ for $x \in D$, where Q is a closed, convex set.

Q may be the whole space $\mathbb{R}^{n\times n}$ of matrices or in particular: (a) a set of symmetric matrices,

(b) a set of sparse matrices,

(c) a set of symmetric and sparse matrices,

(d) a set of p-diagonal matrices.

Now we give the Theorem 3.1 and two properties used in proving some properties of matrices B_k, H_k.

Theorem 3.1 [3]. *Let H denote a Hilbert space and Q be a convex closed set, $Q \subset H$. Let $P_Q x$ denote the orthogonal projection $x \in H$ onto Q. Then*

$$\|x - P_Q x\|^2 + \|P_Q x - y\|^2 \le \|x - y\|^2 \text{ for all } y \in Q. \tag{3.3}$$

Property 1. *If F satisfies the assumptions* **A1, A2, A4 - A8** *then*

$$\|H_k - F_1'(x^*)\| \leq L_1 r_k. \tag{3.4}$$

The proof is an analogue of the proof of Property 1 in [3].

Property 2. *If F satisfies the assumptions* **A1, A2, A5 - A8** *and $H_k \in Q \subset \mathbb{R}^{n\times n}$, where Q is a closed convex set, then*

$$H_k \in Q. \tag{3.5}$$

The proof is an analogue of the proof of Property 2 in [3].

It follows from Property 2 that if the mapping F satisfies the assumptions **A1 - A8** and **A9**, then $H_k \in M_k \cap Q$. Now we give an Algorithm U1 to compute the matrix B_{k+1} which is an analogue of the Algorithm 1 in [3]; the only difference is that the definition L_k in paper [3] is replaced by the definition M_k in this paper:

Algorithm U1. *Suppose that $B_k \in \mathbb{R}^{n\times n}$ is given. Then*

$$B_{k+1} = P_{Q\cap M_k} B_k, \quad k = 0, 1, ..., \tag{3.6}$$

where $P_A X$ denotes the orthogonal projection X onto the closed, convex set A.

Theorem 3.2. *The sequence $\{x_k\}$ defined by (1.3) with the update* U1 *is locally Q-linearly convergent to x^*.*

Proof. Since $H_k \in M_k \cap Q$ we get the inequality

$$\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 \leq \|B_k - H_k\|^2$$

from Theorem 3.1, thus

$$\|B_{k+1} - F_1'(x^*)\| \leq \|B_{k+1} - H_k\| + \|H_k - F_1'(x^*)\|$$
$$\leq \|B_k - H_k\| + \|H_k - F_1'(x^*)\| \leq \|B_k - F_1'(x^*)\| + 2\|H_k - F_1'(x^*)\|$$
$$\leq \|B_k - F_1'(x^*)\| + 2L_1 r_k$$

from Property 1.

Now the bounded deterioration Theorem 2.2 for the mapping F with the almost sparse Jacobian matrix implies local and Q-linear convergence of the sequence $\{x_k\}$. □

Now we give two lemmas used in proving the superlinear convergence of the sequence $\{x_k\}$.

Lemma 3.3 [4]. *If* $\{A_k\}$,$\{B_k\}$ *and* $\{C_k\}$ *are sequences of matrices in* $\mathbb{R}^{n\times n}$ *such that*

$$\sum_{k=0}^{\infty} \|A_{k+1} - C_k\| < \infty, \qquad \|B_k - A_k\| \leq M < \infty$$

and

$$\|B_k - A_k\|^2 \geq \|B_{k+1} - C_k\|^2 + a_k$$

then $\sum\limits_{k=0}^{\infty} a_k < \infty$.

Lemma 3.4 [4]. *Let* $F : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ *be a mapping with properties* **A4 - A7.** *Assume that the sequence* $\{x_k\}$ *defined by* (1.3) *is Q-linearly convergent to* x^*. *Then*

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0 \Leftrightarrow \lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\|s_k\|} = 0$$
$$\Leftrightarrow \lim_{k\to\infty} \frac{\|[B_k - F'(x^*)]s_k\|}{\|s_k\|} = 0.$$

Theorem 3.5. *Let* F *satisfy the assumptions* **A1 - A9.** *Moreover assume that* $F_2'(x^*) = 0$. *Then the sequence* $\{x_k\}$ *described by* (1.3) *with the update* U1 *is locally Q-superlinearly convergent to* x^*.

Proof. By Lemma 3.4 it is sufficient to prove that

$$\lim_{k\to\infty} \frac{\|[B_k - F'(x^*)]s_k\|}{\|s_k\|} = 0.$$

Note that $H_k, B_{k+1} \in M_k$ so $H_k s_k = y_k$ and $B_{k+1}s_k = y_k$. Now

$$\frac{\|[B_k - F'(x^*)]s_k\|}{\|s_k\|} \leq \frac{\|[B_k - H_k]s_k\|}{\|s_k\|} + \|H_k - F_1'(x^*)\|$$
$$\leq \|B_{k+1} - B_k\| + L_1 r_k.$$

Note

$$\|B_k - H_k\| \leq \|B_k - F_1'(x^*)\| + \|F_1'(x^*) - H_k\|$$
$$\leq \|B_k - F_1'(x^*)\| + L_1 r_k \leq M$$

because the sequence $\{\|B_k\|\}$ is bounded (by Th. 2.2). From the Property 1 of orthogonal projection we also have

$$\|B_k - H_k\|^2 \geq \|B_{k+1} - H_k\|^2 + \|B_{k+1} - B_k\|^2.$$

Since $\sum_{k=0}^{\infty} \|H_{k+1} - H_k\| \leq \sum_{k=0}^{\infty} L_1\|s_k\| < \infty$, it follows from the Lemma 3.3 that

$$\sum_{k=0}^{\infty} \|B_{k+1} - B_k\|^2 < \infty$$

and $\lim_{k\to\infty} \|B_{k+1} - B_k\| = 0$. Thus

$$\lim_{k\to\infty} \frac{\|[B_k - F'(x^*)]s_k\|}{\|s_k\|} = 0.$$

□

The problem $\min_{B \in M_k \cap Q} \|B - B_k\|$ may be difficult to solve. S. M. Grzegórski in [3] proposed two level update of the form:

$$R_k = P_{Q_1 \cap L_k} P_{Q_2} B_k$$

In such a case we propose an analogous Algorithm U2 to compute B_{k+1} .

Algorithm U2. *Suppose $B_k \in \mathbb{R}^{n\times n}$ is given. Define B_{k+1} as*

$$B_{k+1} = P_{Q_1} P_{Q_2 \cap M_k} B_k, k = 0, 1, \ldots \tag{3.7}$$

where $Q = Q_1 \cap Q_2$ and Q_1, Q_2 are convex, closed sets.

In the case $Q_2 = \mathbb{R}^{n\times n}$ the orthogonal projection $P_{Q_2 \cap M_k} B_k$ denotes the projection onto M_k.

Theorem 3.6. *The sequence $\{x_k\}$ defined by* (1.3) *with the update* U2 *is locally Q-linearly convergent to x^*.*

Proof. Put $B'_k = P_{Q_2 \cap M_k} B_k$ and $B_{k+1} = P_{Q_1} B'_k$. Since $H_k \in Q \cap M_k$ and $Q = Q_1 \cap Q_2$, where H_k is defined by (3.3) and M_k is defined by (3.2) ,

$$\|B'_k - B_k\|^2 + \|B'_k - H_k\|^2 \leq \|B_k - H_k\|^2$$

and
$$\|B_{k+1}-B_k'\|^2+\|B_{k+1}-H_k\|^2\le\|B_k'-H_k\|^2.$$
Then, from these inequalities and from PROPERTY 1 :
$$\|B_{k+1}-F_1'(x^*)\|\le\|B_{k+1}-H_k\|+\|H_k-F_1'(x^*)\|\le$$
$$\|B_k-H_k\|+L_1r_k\le\|B_k-F_1'(x^*)\|+2L_1r_k.$$
Now the bounded deterioration Theorem 2.2 implies local and Q-linear convergence of the sequence $\{x_k\}$.

4. Numerical examples. We now give some numerical examples which are calculated on TURBO PASCAL v.7.0 by the IBM PC 386 SX. Define the norm
$$\|A\|=\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{1/2},$$
where $A\in\mathbb{R}^{n\times n}$. First we compare our modification of Newton's method, given by (1.2) with the classical Newton method, i.e. we compare the number of iterations and time (taken in hundredths of a second) of calculations needed to get a required approximation of the solution x^*.

Consider the mapping $F:\mathbb{R}^7\to\mathbb{R}^7$such that:
$$F=(f_1,f_2,f_3,\dots,f_7),$$
$$f_i=2x_i+x_{i-1}+x_{i+1}-d_i\quad\text{for}\quad i\ne4\quad\text{and}\quad i\ne7$$
$$f_7=2x_7+x_6-d_7,\qquad f_4=2x_4+x_3+x_5+tx_1x_7-d_4,$$
where t is a small parameter and $d_i=[0.3,\,0.4,\,0.4,\,0.4+0.01t,\,0.4,\,0.4,\,0.3]^T$.

Put
$$F_1=\begin{bmatrix}2x_1+x_2-d_1\\x_1+2x_2+x_3-d_2\\x_2+2x_3+x_4-d_3\\x_3+2x_4+x_5-d_4\\x_4+2x_5+x_6-d_5\\x_5+2x_6+x_7-d_6\\x_6+2x_7-d_7\end{bmatrix}\quad\text{and}\quad F_2=\begin{bmatrix}0\\0\\0\\t\cdot x_1x_7\\0\\0\\0\end{bmatrix}$$

Note that $x^* = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)^T$ is the solution of the equation $F(x) = 0$.

Example 1. Let $t = 10^{-2}$. Since $\|[F_1'(x^*)]^{-1}F_2'(x^*)\| = \sqrt{18} \cdot 10^{-3}$, F is a mapping with an almost sparse Jacobian matrix. Let's take the following initial guesses x_0 :

a) $(0, 0, 0, 0, 0, 0, 0)^T$

b) $(1, 1, 1, 1, 1, 1, 1)^T$.

Our results are given below (TABLE 1):

Table 1. The number of iterations and the time needed to approximate the solution x^* with the accuracy $\varepsilon = \|F(x_k) - F(x^*)\|$.

initial guess	accuracy ε	Newton's method		modif. of Newton's method	
		time	number of iter.	time	number of iter.
a)	10^{-2}	2.47	2	1.09	2
b)	10^{-2}	2.28	2	1.7	3
a)	10^{-8}	2.37	2	1.92	4
b)	10^{-8}	5.11	4	2.48	5

Example 2. Let $t = 10^{-5}$. The norm $\|[F_1'(x^*)]^{-1}F_2'(x^*)\| = \sqrt{18} \cdot 10^{-6}$. The start points are the same as in Example 1.

Table 2. The number of iterations and the time needed to approximate the solution x^* with the accuracy $\varepsilon = \|F(x_k) - F(x^*)\|$.

initial guess	accuracy ε	Newton's method		modif. of Newton's method	
		time	number of iter.	time	number of iter.
a)	10^{-2}	3.37	3	1.15	2
b)	10^{-2}	2.8	2	1.27	2
a)	10^{-8}	3.63	3	1.54	3
b)	10^{-8}	4.01	3	1.82	3

Now we compare our method (1.3) with update defined by Algorithm U1 with the classical Broyden method, i.e. analogously as in the above examples we are interested in the number of iterations and time (in hundredths of a second) needed to get the required approximation x^*. Consider the mapping $F: R^5 \to R^5$ such that:

$$F = (f_1, f_2, ..., f_5), \quad f_i = 2x_i + 0.5{x_{i+1}}^2 - d_i \quad \text{for} \quad i \neq 1, i \neq 5$$

$$f_1 = 2x_1 + 0.5{x_2}^2 + tx_5 - d_1, \qquad f_5 = 2x_5 + tx_1 - d_5,$$

where t is a small parameter and $d = [2.5 + t,\ 2.5,\ 2.5,\ 2.5,\ 2 + t]^T$.

Put

$$F_1 = \begin{bmatrix} 2x_1 + 0.5{x_2}^2 - d_1 \\ 2x_2 + 0.5{x_3}^2 - d_2 \\ 2x_3 + 0.5{x_4}^2 - d_3 \\ 2x_4 + 0.5{x_5}^2 - d_4 \\ 2x_5 - d_5 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ t \end{bmatrix}.$$

Note that $x^* = (1,1,1,1,1)^T$ is the solution of the equation $F(x) = 0$.

Example 3. Let $t = 10^{-2}$. Since $\|[F_1'(x^*)]^{-1}F_2'(x^*)\| = 10^{-2}\frac{\sqrt{341}}{32} < 1$, F is a mapping with an almost sparse Jacobian matrix. Let's take the following initial guesses x_0 :

a) $(0,0,0,0,0)^T$

b) $(1.2, 1.2, 1.2, 1.2, 1.2)^T$.

Our results are given below (Table 3):

Table 3. The number of iterations and the time needed to approximate the solution x^* with the accuracy $\varepsilon = \|F(x_k) - F(x^*)\|$.

initial guess	accuracy ε	Broyden method		method 1.3 with update U1.	
		time	number of iter	time	number of iter
a)	10^{-2}	9.17	6	3.98	5
b)	10^{-2}	2.02	2	1.81	2
a)	10^{-8}	18.49	12	8.51	11
b)	10^{-8}	10.73	7	5.49	7

Example 4. Now let $t = 10^{-5}$. The norm $\|[F_1'(x^*)]^{-1}F_2(x^*)\| = 10^{-5} \cdot \frac{\sqrt{341}}{32} < 1$. The start points are the same as in Example 3.

Table 4. The number of iterations and the time needed to approximate x^* with the accuracy $\varepsilon = \|F(x_k) - F(x^*)\|$.

initial guess	accuracy ε	Broyden method		method 1.3 with update U1.	
		time	number of iter.	time	number of iter.
a)	10^{-2}	9.20	6	4.01	5
b)	10^{-2}	3.04	2	1.78	2
a)	10^{-8}	17.00	11	7.09	9
b)	10^{-8}	10.48	7	4.88	6

Our results suggest that for some kinds of problems like problems with a small parameter the proposed methods (1.2) and (1.3) with update U1 need almost the same number of iterations to get a required approximation of the solution x^* as the classical Newton's or Broyden method. Because the Jacobian matrix F_1' is 3-diagonal (in the Example 1 and 2) matrix or 2-diagonal (in Example 3 and 4), the modification of Newton's method needs less computing operations than Newton's or Broyden methods, so the times of calculations are shorter than in classical methods.

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Institute of Mathematics,
Maria Curie-Skłodowska University,
20-031 Lublin, Poland;

received August 16, 1994