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Partitions, Compositions and Divisibility

ABSTRACT. The paper deals with the connection between partitions, compositions and divisibility. The main result is derivation of an identity connecting these three notions. This identity enables us to compute the number of partitions of an integer n.

1. Introduction. The basic notions occuring in this paper are already classical at this time. The exact definition of them can be found for example in [1], [4], [5] or [9]. In what follows we shall use the following standard notation. We shall denote by $\sigma(n)$ the sum of all divisors of n and by $p(n)$ the number of partitions of n. As usual, we define $p(0) := 1$. Further, $\mathcal{C}_n$ denotes the set of all compositions of n and $\mathcal{C}_{n,k}$ the set of all compositions of n to exactly k summands. Finally, $N_n(k)$ represent the number of all numbers k which occur in all $p(n)$ partitions of n.

The structure of the paper is as follows. We first derive a recurrence formula for the number of partitions of an integer n. This formula is of infinite order and reveals the connection of this problem with the area of divisibility. However, the connection between partitions and divisibility has been known already from Euler's works (see e.g. the monograph of Mac Mahon, [8]). On the other hand, the aim of this paper is to express this connection in the form of

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the recurrence formula for $p(n)$. Moreover, we show that this connection can be clarified by means of simple and elementary methods. Furthermore, by means of the formula we derive a new expression of the generating function of the sequence of the number of partitions. It turns out that the final form of the generating function is exponential. Finally, from its development to the power series we obtain the principal result. The terminal identity, given by comparing the coefficients of power series, reveals the connection of partitions, compositions and divisibility.

We remark that in the proofs we shall use only standard and classical methods. In particular, we shall employ the machinery of power series, generating functions and elementary combinatorical techniques.

2. The recurrence formula for $p(n)$. The aim of this section is to prove the recurrence formula and its modification for the number $p(n)$ by means of elementary methods. We also demonstrate the computation by means of our formulas and compare with the well-known Euler's formula (see Theorem 2). For the proof of Euler's formula see e.g. [1] or [10].

Theorem 1. *For each positive integer n we have the formula*

$$p(n) = \frac{1}{n}\sum_{k=0}^{n-1} \sigma(n-k)p(k). \tag{1}$$

Proof. Clearly, we have the following identity

$$\sum_{k=1}^{n} k \cdot N_n(k) = n \cdot p(n). \tag{2}$$

We first determine the value $N_n(k)$ for $1 \le k \le n$. Let $n = m_k \cdot k + z_k$, where $z_k < k$ is the remainder after division of n by k and m_k is the partial quotient. Then we see that the number of all partitions of n, which contain at least r numbers k, $1 \le r \le m_k$, is $p(n - rk)$. Hence the number of all $k's$ in all $p(n)$ partitions is

$$N_n(k) = p(n-k) + \cdots + p(n - m_k k). \tag{3}$$

Further, the relations (2) and (3) together give the following formula for $p(n)$

$$(4) \qquad p(n) = \frac{1}{n}\sum_{k=1}^{n} k(p(n-k)+\cdots+p(n-m_k k)).$$

Now we simplify the formula (4). Let $0 \le s \le n-1$ be a natural number. Let us consider when $p(s)$ occurs among the members of the sum $p(n-k)+\cdots+p(n-m_k k)$, i.e. when $n - r\cdot k = s$ for some $1 \le r \le m_k$. Clearly, the relation $n - r\cdot k = s$ holds iff k divides $n-s$. Therefore $p(s)$ occurs in all the sums $p(n-k)+\cdots+p(n-m_k k)$ where k divides $n-s$. Then the number of all occurrances of $p(s)$ in the sum $\sum_{k=1}^{n} k(p(n-k)+\cdots+p(n-m_k k))$ is exactly $\sigma(n-s)$, since $p(s)$ occurs in the sum $\sum_{k=1}^{n} k(p(n-k)+\cdots+p(n-m_k k))$ k times for every natural divisor k of the number $n-s$. Thus we have

$$\sum_{k=1}^{n} k(p(n-k)+\cdots+p(n-m_k k)) = \sum_{s=0}^{n-1}\sigma(n-s)p(s).$$

This completes the proof of (1). □

Now we quite easily derive the following simple modification of the formula (1). Notice that the formula (1) and its modification (5) have the same form, but the coefficients are different.

Corollary 1. *For each positive integer n we have the formula*

$$(5) \qquad p(n) = \frac{1}{n}\sum_{k=0}^{n-1}(\sigma(1)+\cdots+\sigma(n-k)-k)p(k).$$

Proof. By addition of the first n formulas (1) we obtain

$$\sum_{k=1}^{n} kp(k) = \sum_{k=0}^{n-1} p(k)\sum_{i=1}^{n-k}\sigma(i).$$

Now an easy simplification yields (5). □

Remark 1. The relation (3) from the proof of Theorem 1 gives another possibility for enumeration of $p(n)$. Indeed, from (3) it follows that $N_{n+1}(1) = p(0) + \cdots + p(n)$ and $N_n(1) = p(0) + \cdots + p(n-1)$. Hence by subtraction we find

$$p(n) = N_{n+1}(1) - N_n(1). \tag{6}$$

Now we introduce important and practical Euler's recurrent relations for the functions $p(n)$ and $\sigma(n)$.

Theorem 2 (Euler). *For each positive integer n we have*

$$\begin{aligned} &p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) \\ &- p(n-12) - p(n-15) + \ldots \cdots + (-1)^m p(n - \frac{1}{2}m(3m-1)) \\ &+ (-1)^m p(n - \frac{1}{2}m(3m+1)) + \cdots = 0, \end{aligned} \tag{7}$$

where we define $p(k) = 0$ for every integer $k < 0$.

Theorem 3 (Euler). *For each positive integer n we have*

$$\begin{aligned} &\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) \\ &\quad - \sigma(n-15) + \ldots \cdots + (-1)^m \sigma\Big(n - \frac{1}{2}m(3m-1)\Big) \\ &\quad + (-1)^m \sigma\Big(n - \frac{1}{2}m(3m+1)\Big) + \ldots \\ &\quad = \begin{cases} (-1)^{k+1} n & \text{if } n = (3k^2 \pm k)/2, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{8}$$

where we define $\sigma(k) = 0$ for every integer $k \leqq 0$.

Example 1. We find the number of partitions of $n = 5$ by means of (1), (5) and (7). We first compute $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 7$, $\sigma(5) = 6$ and $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$ and $p(4) = 5$. By recurrence (1) we have $p(5) = \frac{1}{5}(6p(0) + 7p(1) + 4p(2) + 3p(3) + p(4)) = \frac{1}{5}(6 + 7 + 8 + 9 + 5) = 7$. Analogously, using

(5) we have $p(5) = \frac{1}{5}(21p(0) + 14p(1) + 6p(2) + p(3) - 3p(4)) = \frac{1}{5}(21 + 14 + 12 + 3 - 15) = 7$. Finally, by recurrence (7) we have $p(5) = p(4) + p(3) - p(0) = 5 + 3 - 1 = 7$.

Remark 2. We remark that the recurrences (1) and (5) are not practical for computation of the values $p(n)$. The best way to compute $p(n)$ is to use the well-known Euler's formula (7). We point out that our formulas (1) and (5) reveal the connection between the functions $p(n)$ and $\sigma(n)$, i.e. between partitions and divisibility. Recurrences for the computation of $\sigma(n)$ were studied by J. A. Ewell in [2] and [3]. But the best method for computing $\sigma(n)$ is applying the formula (8), which also comes from Euler. It turns out that there is an interesting analogy with $p(n)$ here (compare relations (7) and (8)). This is also proved in [9].

3. The generating function of the sequence $p(n)$. In this section we derive a new important form of the generating function of $p(n)$ by means of our recurrence (1). Let $P(x) = \sum_{n=0}^{\infty} p(n)x^n$ be the generating function of the sequence $p(n)$. Leonard Euler has proved that this function satisfies

$$P(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n}, \quad \text{where} \quad |x| < 1. \tag{9}$$

Remark 3. The series $\sum_{n=1}^{\infty} \sigma(n)x^n$ and $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n$ converge for every $|x| < 1$. To prove this, it suffices to use the comparison test and the root test.

Theorem 4. *The generating function $P(x)$ of the sequence $p(n)$ satisfies*

$$P(x) = e^{\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n}, \quad \textit{where} \quad |x| < 1. \tag{10}$$

Proof. Let $S(x) = \sum_{n=1}^{\infty} \sigma(n)x^n$ be the generating function of $\sigma(n)$. Multiplicating (1) by nx^n and then evaluating the sum for

$n = 1, 2, \ldots$ we get

$$\sum_{n=1}^{\infty} np(n)x^n = \sum_{k=0}^{\infty}\Big(p(k)\sum_{n=1}^{\infty}\sigma(n)x^{n+k}\Big) = \sum_{k=0}^{\infty}\Big(p(k)x^k\sum_{n=1}^{\infty}\sigma(n)x^n\Big)$$

$$= S(x)\sum_{k=0}^{\infty}p(k)x^k = S(x)P(x).$$

Furthermore,

$$\sum_{n=1}^{\infty} np(n)x^n = x\cdot\sum_{n=1}^{\infty} np(n)x^{n-1} = xP'(x).$$

This implies

$$xP'(x) = S(x)P(x), \tag{11}$$

Now (11) easily gives $\ln P(x) = \sum_{n=1}^{\infty}\frac{\sigma(n)}{n}x^n$, which is nothing else than (10). The theorem is proved. □

From the expression (10) of the generating function of $p(n)$ one deduces easily the following estimates.

Corollary 2. *Let $|x| < 1$. Then the generating function $P(x)$ satisfies the following inequalities*

$$e^{x/(1-x)} < P(x) < e^{x/(1-x)^2}. \tag{12}$$

Proof. We have $n \le \sigma(n) \le n^2$ for every natural number n. Then $\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}x^n \ge \sum_{n=1}^{\infty}\frac{n}{n}x^n \ge \frac{1}{1-x} - 1 = \frac{x}{1-x}$ and on the other hand $\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}x^n \le \sum_{n=1}^{\infty}\frac{n^2}{n}x^n = x\left(\frac{x}{1-x}\right)' = \frac{x}{(1-x)^2}$. This completes the proof. □

4. Development of the generating function of $p(n)$ into power series. The aim of this section is to prove a new assertion, which is the principal result of this paper. The following identity (13) reveals the connection between partitions, compositions and divisibility. This result turns out to be of theoretical importance most of all. Indeed, in one formula there are connected the number of partitions of an integer n, the set $\mathcal{C}_n$ of all compositions and the arithmetical multiplicative function σ. To prove the identity (13), we use the above deduced form (10) of the generating function $p(n)$, which we develop into power series.

Theorem 5. *We have the following identity for the number $p(n)$*

$$p(n) = \sum_{x_1+\cdots+x_k \in \mathcal{C}_n} \frac{1}{k!}\frac{\sigma(x_1)}{x_1}\cdots\frac{\sigma(x_k)}{x_k}. \tag{13}$$

Proof. Let $F(x) := \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n$. Then (10) yields $P(x) = e^{F(x)}$. Let us develop function P into the series of powers of the function F. We have

$$P(x) = \sum_{m=0}^{\infty} \frac{F^m(x)}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n\right)^m.$$

Hence after involution of the series F we obtain the development of function P into the series of powers of x. Now we determine the coefficient by x^n in the development of function P. Let k be a natural number. Clearly, each member of the series F^k arises as the product of k factors. The general member of the series F^k has the form

$$\left(\frac{\sigma(x_1)}{x_1}x^{x_1}\right)\cdots\left(\frac{\sigma(x_k)}{x_k}x^{x_k}\right) = \frac{\sigma(x_1)}{x_1}\cdots\frac{\sigma(x_k)}{x_k}x^{x_1+\cdots+x_k}.$$

If $k > m$, then there is no member with x^n or with the lower power of x in the series F^k, so that only a part of the series $\sum_{k=1}^{n} \frac{1}{k!}F^k$ contains members with x^n. Furthermore, in the series F^k the coefficient $\frac{\sigma(x_1)}{x_1}\cdots\frac{\sigma(x_k)}{x_k}$ is by x^n iff $x_1+\cdots+x_k = n$. Clearly, the number of coefficients with this property is equal to card$\mathcal{C}_{n,k}$. So in the series F^k the coefficient

$$\sum_{x_1+\cdots+x_k \in \mathcal{C}_{n,k}} \frac{\sigma(x_1)}{x_1}\cdots\frac{\sigma(x_k)}{x_k}$$

is by the member x^n and finally in the series $\sum_{m=0}^{\infty} \frac{F^m}{m!}$ the coefficient

$$\sum_{k=1}^{n} \frac{1}{k!} \sum_{x_1+\dots+x_k \in \mathcal{C}_{n,k}} \frac{\sigma(x_1)}{x_1} \dots \frac{\sigma(x_k)}{x_k} = \sum_{x_1+\dots+x_k \in \mathcal{C}_n} \frac{1}{k!} \frac{\sigma(x_1)}{x_1} \dots \frac{\sigma(x_k)}{x_k}$$

is by x^n. By comparison of the coefficients by x^n with the series $\sum_{n=0}^{\infty} p(n)x^n$ we obtain (13). This completes the proof. □

At the end we present an example, which demonstrates the identity (13) for the case of an integer n.

Example 2. We find the number of partitions of $n = 4$ by means our identity (13). We shall suppose that the values $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$ and $\sigma(4) = 7$ are known. For enumeration of these numbers we can use Euler's formula (8). Now by identity (13) it holds

$$p(4) = \frac{1}{1!} \frac{\sigma(4)}{4} + \frac{1}{2!} \left(\frac{\sigma(3)}{3} \frac{\sigma(1)}{1} + \frac{\sigma(1)}{1} \frac{\sigma(3)}{3} + \frac{\sigma(2)}{2} \frac{\sigma(2)}{2} \right) +$$

$$+ \frac{1}{3!} \left(\frac{\sigma(2)}{2} \frac{\sigma(1)}{1} \frac{\sigma(1)}{1} + \frac{\sigma(1)}{1} \frac{\sigma(2)}{2} \frac{\sigma(1)}{1} + \frac{\sigma(1)}{1} \frac{\sigma(1)}{1} \frac{\sigma(2)}{2} \right) +$$

$$+ \frac{1}{4!} \frac{\sigma(1)}{1} \frac{\sigma(1)}{1} \frac{\sigma(1)}{1} \frac{\sigma(1)}{1} = \frac{1}{1!} \frac{7}{4} + \frac{1}{2!} \frac{59}{12} + \frac{1}{3!} \frac{9}{2} + \frac{1}{4!} \frac{1}{1} = 5.$$

Remark 4. Applications of the results of this paper and especially of the ideas of proofs can be founded in our work [7]. Here we study the connection between partitions and posets from the point of view of enumeration of the numbers P_n of non-isomorphic posets.

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