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Two – Slit Harmonic Mappings

ABSTRACT. We consider the class $\mathcal{S}_H(\Delta, \Omega_{a,b})$ of complex functions f which are univalent, harmonic, orientation preserving on the open unit disk Δ, satisfy $f(0) = f_{\overline{z}}(0) = 0 < f_z(0)$, and have the fixed range $f(\Delta) = \Omega_{a,b}$, where $a < 0 < b$ and $\Omega_{a,b} = \mathbb{C} \setminus \{(-\infty, a] \cup [b, +\infty)\}$. In particular, we describe the closure $\overline{\mathcal{S}_H(\Delta, \Omega_{a,b})}$ and characterize its extreme points. Also, an auxiliary class $\mathcal{S}_0$ of univalent harmonic orientation preserving functions f on Δ with $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$ and $f(\Delta \setminus \mathbb{R}) = \mathbb{C} \setminus \mathbb{R}$ has been examined.

1. Introduction. Let $\mathcal{H}(\Delta)$ be the linear space of all analytic functions on the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, with the topology of locally uniform convergence.

There has been recently interest [1–3, 5–6, 8] in studying the class $\mathcal{S}_H$ of all functions f which are complex valued, harmonic, orientation preserving, univalent mappings of Δ, with the normalization

$$f(0) = 0 < f_z(0). \tag{1}$$

If we let F and G be in $\mathcal{H}(\Delta)$ and satisfy $\operatorname{Re} F = \operatorname{Re} f$, $\operatorname{Re} G = \operatorname{Im} f$, then

$$f = h + \overline{g} \text{ with } h = \frac{F + iG}{2} \text{ and } g = \frac{F - iG}{2}, \tag{2}$$

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and the Jacobian of f, given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2, \tag{3}$$

is positive on Δ. For uniqueness of the presentation (2) we usually assume $h(0) = g(0) = 0$. From (1)–(3) it follows that for $f = h + \overline{g} \in \mathcal{S}_H$ we have $|g'(0)| < |h'(0)| = h'(0)$, and hence the sets

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\overline{z}}(0) = 0\} \tag{4}$$

and

$$\left\{ f - \left(\overline{g'(0)} / h'(0) \right) \overline{f} : f = h + \overline{g} \in \mathcal{S}_H \right\}$$

are the same. Assuming that $f_z(0)$ varies within a bounded (resp. compact) set W of positive numbers, we obtain a normal (resp. compact) family

$$\{ f \in \mathcal{S}_H : f_z(0) \in W \} \quad \left(\text{resp. } \{ f \in \mathcal{S}_H^0 : f_z(0) \in W \} \right),$$

for the proof see [3].

In contrast to conformal mappings, harmonic univalent functions f are not at all determined (up to normalization (1)) by their image domains. Given a general simply connected domain $\Omega \subsetneqq \mathbb{C}$ with any prescribed point $w_0 \in \Omega$, it is natural to study harmonic orientation preserving univalent mappings f of Δ onto Ω with $f(0) = w_0$. If f is such a mapping,

$$f(z) = w_0 + \sum_{j=1}^{\infty} a_j z^j + \overline{\sum_{j=1}^{\infty} b_j z^j},$$

then $J_f(0) = |a_1|^2 - |b_1|^2 > 0$, all the affine transformations

$$w \mapsto T(w) = w_0 + te^{-i\alpha} \left[\overline{a}_1(w - w_0) - \overline{b}_1(\overline{w} - \overline{w}_0) \right], \ t > 0, \ \alpha \in \mathbb{R}, \tag{5}$$

map $\mathbb{C}$ onto itself univalently, $T(w_0) = w_0$, and the function

$$z \mapsto T\left(f\left(e^{i\alpha} z\right)\right) - w_0$$

is in the class (4). Since the set $T(\Omega)$ is affinely similar Ω, and as equality $T(\Omega) = \Omega$ may occur for a suitable choice of $t > 0$ and $\alpha \in \mathbb{R}$, we may restrict ourselves to the class

$$\mathcal{S}_H(\Delta, \Omega) = \{ f : f - w_0 \in \mathcal{S}_H^0,\ f(\Delta) = \Omega \} . \tag{6}$$

Let us consider the following examples.

Example 1. Choose $\alpha \in \mathbb{R}$ and $t > 0$ so that

$$\operatorname{Im} \{ e^{i\alpha}(a_1 - b_1) \} = 0 < t = 1 / \operatorname{Re} \{ e^{i\alpha}(a_1 + b_1) \} .$$

Then (5) maps every strip

$$\{ w : a < \operatorname{Im} w < b \},\ a < \operatorname{Im} w_0 < b,$$

onto itself. Indeed, under the above assumptions, $\operatorname{Im} T(w) \equiv \operatorname{Im} w$, and if

$$\operatorname{Re} \{ e^{ix}(a_1 + b_1) \} = \operatorname{Im} \{ e^{ix}(a_1 - b_1) \} = 0$$

for a real x, then $|a_1| = |b_1|$, which contradicts the positivity of (3). Hence the desired choice of α and t is possible.

Example 2. Let $\operatorname{Re} w_0$ be positive, and take $\alpha \in \mathbb{R}$ and $t > 0$ so that

$$\operatorname{Im} \{ e^{i\alpha}(a_1 + b_1) \} = 0 < t = 1 / \operatorname{Re} \{ e^{i\alpha}(a_1 - b_1) \} .$$

Like before, the choice of parameters α, t is possible, and (5) maps the right half-plane onto itself. Here $\operatorname{Re} T(w) \equiv \operatorname{Re} w$.

Example 3. If $-\infty \leqslant a < w_0 < b \leqslant +\infty$, $t = 1 / |a_1 - b_1|$ and $e^{-i\alpha} = (a_1 - b_1) / |a_1 - b_1|$, then (5) maps every set

$$\Omega_{a,b} = \mathbb{C} \setminus \{ (-\infty, a] \cup [b, +\infty) \} \tag{7}$$

onto itself. Observe here that $T(x) = x$ for all real x and

$$\operatorname{Im} T(w) \equiv \left[\left(|a_1|^2 - |b_1|^2 \right) / |a_1 - b_1|^2 \right] \operatorname{Im} w.$$

Hengartner and Schober [6] and later Cima and Livingston [2] considered the case of Ω being a strip, Abu-Muhanna and Schober [1] considered the case of Ω being a wedge or half-plane, and Livingston [8] considered the case of $\Omega = \mathbb{C} \setminus (-\infty, a]$, $a < 0$.

Our purpose is to study the closure of the class $\mathcal{S}_H(\Delta, \Omega_{a,b})$ for arbitrary $a < 0 < b$, including the limit cases $a = -\infty$ or $b = +\infty$, see (6–7) with $w_0 = 0$. It appears that $\overline{\mathcal{S}_H(\Delta, \Omega_{a,b})}$ is the union of a disjoint uncountable collection of compact convex sets with integral representations of Choquet's type. This paper contains results presented by the second author on the international conference "Planar harmonic mappings", Technion (Haifa, Israel), May 8-15, 1995.

2. Auxiliary results. This section presents some preliminaries and provides a detailed exposition of the class $\mathcal{S}_0$, defined by (23). Observe first that the set

$$\mathcal{H}_H(\Delta) = \{h + \overline{g} : \; h, g \in \mathcal{H}(\Delta), \, g(0) = 0\},$$

with the topology of locally uniform convergence, is a locally convex topological vector space that contains $\mathcal{H}(\Delta)$ and $\overline{\mathcal{H}(\Delta)} = \{\overline{g} : g \in \mathcal{H}(\Delta)\}$ as its subspaces. The topological dual space $\mathcal{H}'_H(\Delta)$ can be represented by complex measures with compact supports in Δ [6]. Let $h, g \in \mathcal{H}(\Delta)$ with $g(0) = 0$ and $L \in \mathcal{H}'_H(\Delta)$. Then $L(h + \overline{g}) = L_1(h) + \overline{L_2(g)}$, where both the functionals $L_1 = L|\mathcal{H}(\Delta)$ and $g \mapsto L_2(g) = \overline{\left(L|\overline{\mathcal{H}(\Delta)}\right)(\overline{g})}$ are in $\mathcal{H}'(\Delta)$. Thus, for continuous complex-linear functionals on $\mathcal{H}_H(\Delta)$, the sequence form of Toeplitz type is possible.

Let $\mathcal{A}$ be a subset of a locally convex topological vector space. We shall use the notation $\mathrm{E}\,\mathcal{A}$, $\sigma\mathcal{A}$, $\mathrm{co}\,\mathcal{A}$ and $\overline{\mathrm{co}}\mathcal{A}$ to denote the *set of extreme points* of $\mathcal{A}$, the *set of support points* of $\mathcal{A}$, the *convex hull* of $\mathcal{A}$ and the *closed convex hull* of $\mathcal{A}$, respectively. The *set of all probability measures* on K we denote by $\mathbb{P}_K$.

Let $\mathcal{P}$ be the class of functions $p \in \mathcal{H}(\Delta)$ for which $p(0) = 1$ and $\mathrm{Re}\, p > 0$ on Δ. From the Riesz-Herglotz representation formula we get

$$\mathcal{P} = \overline{\mathrm{co}}\{p_\eta : |\eta| = 1\} = \{p_\mu : \mu \in \mathbb{P}_{\partial\Delta}\}, \tag{8}$$

where

(9)
$$p_\eta(z) \equiv \frac{1+\eta z}{1-\eta z} \quad \text{for} \quad |\eta| = 1 \quad \text{and}$$
$$p_\mu(z) \equiv \int_{\partial\Delta} p_\eta(z) d\mu(\eta) \quad \text{for} \quad \mu \in \mathbb{P}_{\partial\Delta}.$$

Also

(10)
$$\mathrm{E}\,\mathcal{P} = \{p_\eta : |\eta| = 1\} \quad \text{and} \quad \sigma\mathcal{P} = \mathrm{co}(\mathrm{E}\,\mathcal{P}),$$

see e. g. [4].

Remark 1. Applying the Lebesgue dominated convergence theorem, we may deduce from (9) that for any $\mu \in \mathbb{P}_{\partial\Delta}$ and $\eta \in \partial\Delta$, the function $z \mapsto (1-\overline{\eta}z)\,p_\mu(z)$ has a nontangential limit at the point η:

(11)
$$(1-\overline{\eta}z)\,p_\mu(z) \to 2\mu(\{\overline{\eta}\}) \quad \text{as} \quad \Delta \ni z \to \eta \quad \text{with} \quad z-\eta = O(1-|z|).$$

In particular, (11) implies that for $\eta \in \partial\Delta$, $\delta > 0$ and all $p_\mu \in \mathcal{P}$ having analytic extensions to $\Delta \cup \{z : 0 < |z-\eta| < \delta\}$, the following equivalences hold:

(i) $\mu(\{\overline{\eta}\}) = 0$ iff p_μ is analytic at η;

(ii) $\mu(\{\overline{\eta}\}) > 0$ iff p_μ has a simple pole at η.

The class $\mathcal{P}$ will play an important role in the considered sets of harmonic functions. For any $-1 \leqslant c \leqslant 1$, $z \in \Delta$ and $p \in \mathcal{P}$, denote

(12)
$$q_c(z) \equiv q(z,c) \equiv z \,/\, (1-2cz+z^2)$$

and

(13)
$$k(z,c,p) \equiv \mathrm{Re} \int_0^z q_c'(\xi)p(\xi)d\xi + i\,\mathrm{Im}\, q_c(z).$$

By means of (8–9) and (12–13), we may define some classes of harmonic typically real functions:

(14)
$$\mathcal{F}(c) = \{k(\cdot,c,p) : p \in \mathcal{P}\}, \; -1 \leqslant c \leqslant 1,$$

and

$$\mathcal{F} = \bigcup_{-1\leqslant c\leqslant 1} \mathcal{F}(c). \tag{15}$$

Clearly, $\mathcal{F} \subset \mathcal{H}_H(\Delta)$, and from (8–10) we deduce easily that

$$\mathcal{F}(c) = \left\{ \int_{\partial\Delta} k(\cdot, c, p_\eta) d\mu(\eta) : \mu \in \mathbb{P}_{\partial\Delta} \right\}, \ -1 \leqslant c \leqslant 1, \tag{16}$$

$$\mathrm{E}\,\mathcal{F}(c) = \{k(\cdot, c, p_\eta) : |\eta| = 1\}, \ \sigma\mathcal{F}(c) = \mathrm{co}(\mathrm{E}\,\mathcal{F}(c)) \tag{17}$$

and

$$\mathrm{E}\,\mathcal{F} = \bigcup_{-1\leqslant c\leqslant 1} \mathrm{E}\,\mathcal{F}(c). \tag{18}$$

Each class $\mathcal{F}(c)$ is compact convex and invariant under the mapping $f \mapsto \hat{f}$, where

$$\hat{f}(z) \equiv \overline{f(\overline{z})}.$$

In fact,

$$f = k(\cdot, c, p) \ \text{implies} \ \hat{f} = k(\cdot, c, \hat{p}). \tag{19}$$

Moreover, if we denote $\tilde{p}(\xi) \equiv p(-\xi)$, then $p \in \mathcal{P}$ iff $\tilde{p} \in \mathcal{P}$,

$$\begin{aligned} &-k(-z, c, p) \equiv k(z, -c, \tilde{p}) \quad \text{and} \\ &\mathcal{F}(-c) = \{z \mapsto -f(-z) : f \in \mathcal{F}(c)\}, \ -1 \leqslant c \leqslant 1. \end{aligned} \tag{20}$$

Also, $\mathcal{F}(1)$ is homeomorphic to the class discussed by A. E. Livingston [8]:

$$\overline{\mathcal{S}_H(\Delta, \Omega_{a,+\infty})} = \left\{ \frac{af}{f(-1^+)} : f \in \mathcal{F}(1) \right\}, \tag{21}$$

who observed that

$$\mathrm{E}\,\overline{\mathcal{S}_H(\Delta, \Omega_{a,+\infty})} = \left\{ \frac{af}{f(-1^+)} : f \in \mathrm{E}\,\mathcal{F}(1) \right\}. \tag{22}$$

Obviously, the union $\mathcal{F}$ is also compact but not convex. Let us consider the class

$$(23)\qquad \mathcal{S}_0 = \left\{ f \in \mathcal{S}_H^0 : \ f_z(0) = 1 \text{ and } f(\Delta \setminus \mathbb{R}) = \mathbb{C} \setminus \mathbb{R} \right\},$$

and let

$$\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}, \qquad \mathbb{C}_+ = \{z : \operatorname{Im} z > 0\},$$
$$\Delta_- = \Delta \cap \mathbb{C}_-, \qquad \Delta_+ = \Delta \cap \mathbb{C}_+.$$

Using standard methods [1–3, 5–6, 8] we shall examine the class $\mathcal{F}$ (which is interesting in itself). Its properties and some convexity techniques allow us to extend the results of Livingston [8] to the classes

$$\mathcal{S}_H(\Delta, \Omega_{a,b}) \ \text{ with } \ a < 0 < b \ (a \neq -\infty \text{ or } b \neq +\infty).$$

Theorem 1. $\mathcal{S}_0 \subset \mathcal{F} \subset \mathcal{S}_H^0$. *Moreover, for each $f \in \mathcal{F}$, every horizontal line has a non-empty connected intersection with $f(\Delta)$.*

Proof. Let $f = h + \overline{g} \in \mathcal{S}_0$ with $h, g \in \mathcal{H}(\Delta)$ and $g(0) = 0$. Then $a = g'/h'$ satisfies the hypothesis of Schwarz's lemma, and, like in [3, 6, 8], we first observe that

$$(h - g) \circ f^{-1}(w) \equiv (f - 2\operatorname{Re} g) \circ f^{-1}(w) \equiv w - 2\operatorname{Re}\left[g \circ f^{-1}(w)\right].$$

Since $|g'| < |h'|$ everywhere, we conclude that

$$\frac{\partial}{\partial t} \operatorname{Re}\left[(h - g) \circ f^{-1}(t + i\alpha)\right] = \frac{\partial}{\partial t}\left[(h - g) \circ f^{-1}(t + i\alpha)\right] \in \mathbb{R} \setminus \{0\}$$

for all $t \in \mathbb{R}$, if $\alpha \in \mathbb{R} \setminus \{0\}$, and all $t \in f^{-1}(\mathbb{R})$, if $\alpha = 0$ (by definition, $f^{-1}(\mathbb{R}) \subset \mathbb{R}$). In fact, $h' - g'$ does not vanish on Δ, and, for any fixed α and $z(t) \equiv f^{-1}(t + i\alpha)$ we have

$$1 = [f(z(t))]' = f_z(z(t))z'(t) + f_{\overline{z}}(z(t))\overline{z'(t)},$$

i. e. $z'(t) \neq 0$ for all $t \in \mathbb{R}$. Consequently, the function $(h - g) \circ f^{-1}$ maps univalently each horizontal line into itself. Hence $h - g$ is a classically normalized univalent function on Δ with $\operatorname{Im}(h-g) = \operatorname{Im} f = 0$ a. e. on $\partial\Delta$. By uniqueness in the Riemann mapping theorem, $h - g = q_c$ for a suitable $c \in [-1,1]$, see (12). Thus

$$f = \operatorname{Re}(h+g) + i\operatorname{Im}(h-g) = k\left(\cdot, c, \frac{1+a}{1-a}\right) \in \mathcal{F}, \text{ see (13).}$$

Let now $f = k(\cdot, c, p) \in \mathcal{F}(c)$ for some $c \in [-1,1]$ and $p \in \mathcal{P}$. Analogously to [3, 6, 8] we conclude that the function $f \circ q_c^{-1}$ maps horizontal lines into themselves, and

$$\frac{\partial}{\partial t}\left[f \circ q_c^{-1}(t+i\alpha)\right] = \operatorname{Re}\left[p \circ q_c^{-1}(t+i\alpha)\right] > 0$$

for every $t \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, and every $t \in q_c^{-1}(\mathbb{R})$ and $\alpha = 0$. This means that the functions $t \mapsto \operatorname{Re}\left[f \circ q_c^{-1}(t+i\alpha)\right]$, $\alpha \in \mathbb{R}$, are strictly increasing, i. e. $f \in \mathcal{S}_H^0$ and every horizontal line has a nonempty intersection with $f(\Delta)$.

The following Lemma will be useful for our next results.

Lemma 1. *Suppose η to be v or $\overline{v}$, where $v = e^{i\gamma}$ with $0 \leqslant \gamma \leqslant \pi$, let $c = \cos\gamma$, and let $p \in \mathcal{P}$ be analytic at η. Consider the function*

$$F(z) = \int_0^z q_c'(\xi) p(\xi) d\xi, \quad z \in \Delta. \tag{24}$$

Then the function w_η defined by

(25)
$$w_\eta(z) \equiv \begin{cases} F(z) - p(\eta)q_c(z) + \dfrac{\eta p'(\eta)}{\eta - \overline{\eta}} \log(1 - \overline{\eta} z), & \text{if } \eta^2 \neq 1, \\ F(z) - p(\eta)q_c(z) - [p'(\eta) + \eta p''(\eta)] \log \dfrac{1}{1-\eta z} + \dfrac{2p'(\eta)}{1-\eta z}, & \\ & \text{if } \eta^2 = 1, \end{cases}$$

is analytic at η. Furthermore, if $\operatorname{Re} p(\eta) = 0$, then

$$\eta p'(\eta) < 0 \tag{26}$$

and

$$\eta p'(\eta) + \operatorname{Re}\left[\eta^2 p''(\eta)\right] \leqslant 0. \tag{27}$$

Proof. The functions

$$z \mapsto F'(z) - p(\eta)q_c'(z) - \eta^2 p'(\eta)/[(1-\eta^2)(z-\eta)]\ ,\ \text{if } \eta^2 \neq 1,$$

and

$$z \mapsto F'(z) - p(\eta)q_\eta'(z) + 2\eta p'(\eta)/(z-\eta)^2$$
$$+ [p'(\eta) + \eta p''(\eta)]/(z-\eta)\ ,\ \text{if } \eta^2 = 1,$$

are analytic at $z = \eta$. The property (26) follows from the obvious facts:

$$\beta(x) \stackrel{\text{def}}{=} \operatorname{Re} p\left(\eta e^{ix}\right) \geqslant \operatorname{Re} p(\eta) = \beta(0) = 0 \ \text{ for all real } x,$$

$p'(\eta) \neq 0$ (examine the net change in the argument of $p - p(\eta)$) and

$$\operatorname{Re} p(x\eta) \geqslant \operatorname{Re} p(\eta) = 0 \ \text{ for } \ 0 < x \leqslant 1.$$

Finally, (27) is a consequence of the evident inequality $\beta''(0) \geqslant 0$.

Theorem 2. *Fix any $\gamma \in [0, \pi]$ and suppose that $p \in \mathcal{P}$ is analytic at $v = e^{i\gamma}$ (resp. at $\overline{v} = e^{-i\gamma}$). Consider the function $f = k(\cdot, c, p)$ with $c = \cos\gamma$. If $\operatorname{Re} p(v) > 0$ (resp. $\operatorname{Re} p(\overline{v}) > 0$), then $f(\Delta_+) = \mathbb{C}_+$ (resp. $f(\Delta_-) = \mathbb{C}_-$).*

Proof. Let $\eta = v$ or $\eta = \overline{v}$, and consider the function (24). By Lemma 1, the function (25) is analytic at η. Since $|\arg(1 - \overline{\eta}z)| < \pi/2$ for all $z \in \Delta$, the unrestricted limit

$$\lim_{\Delta \ni z \to \eta} [(1 - \overline{\eta}z)\log(1 - \overline{\eta}z)]$$

exists and equals 0. Thus

$$F(z) = q_c(z)[p(\eta) + o(1)] + w_\eta(z) \ \text{ as } \ \Delta \ni z \to \eta.$$

Finally, on the preimages of horizontal lines, i. e. on the sets

$$\{z \in \Delta : \operatorname{Im} f(z) = \operatorname{Im} q_c(z) = \alpha\} \ \text{ with } \ \alpha \neq 0 \ \text{ and } \ \alpha \operatorname{Im}\eta \geqslant 0, \tag{28}$$

we have

$$\operatorname{Re} f(z) = \operatorname{Re} F(z) \to \pm\infty \ \text{ as } \ \operatorname{Re} q_c(z) \to \pm\infty.$$

The proof is complete.

As a corollary we get

Theorem 3. *For* $-1 \leqslant c \leqslant 1$, *let* $\mathcal{S}_0(c) = \mathcal{S}_0 \cap \mathcal{F}(c)$. *Then*

(i) $$\overline{\mathcal{S}_0(c)} = \mathcal{F}(c) \quad \text{for} \quad -1 \leqslant c \leqslant 1,$$

(ii) $$\overline{\mathcal{S}_0} = \mathcal{F},$$

(iii)
$$(1-\lambda)\mathcal{S}_0(c) + \lambda\mathcal{F}(c) = \mathcal{S}_0(c) \quad \text{for all} \quad 0 \leqslant \lambda < 1 \quad \text{and} \quad -1 \leqslant c \leqslant 1.$$

Proof. (i). It is sufficient to observe that each $f = k(\cdot, c, p) \in \mathcal{F}(c)$ is the locally uniform limit of the sequence $f_n = k(\cdot, c, p_n)$, where $p_n(z) \equiv p((1-1/n)z)$. Clearly, all the $f_n \in \mathcal{F}(c)$, and all the numbers $\operatorname{Re} p_n(v)$, $\operatorname{Re} p_n(\overline{v})$ are positive. So the conclusion follows from the previous theorem.

(ii). The class $\mathcal{F}$ is compact as the image of the compact set $[-1,1] \times \mathcal{P}$ under the continuous mapping $(c,p) \mapsto k(\cdot, c, p)$. By Theorems 1 and 2,

$$\overline{\mathcal{S}_0} = \overline{\bigcup_{-1 \leqslant c \leqslant 1} \mathcal{S}_0(c)} \supset \bigcup_{-1 \leqslant c \leqslant 1} \overline{\mathcal{S}_0(c)} = \mathcal{F} \supset \overline{\mathcal{S}_0}.$$

(iii). If $f = (1-\lambda)f_1 + \lambda f_2$ with $0 \leqslant \lambda < 1$, $f_1 \in \mathcal{S}_0(c)$ and $f_2 \in \mathcal{F}(c)$, then on the level set (28) we have

$$\operatorname{Re} f_1(z) \to +\infty, \quad \liminf \operatorname{Re} f_2(z) > -\infty \quad \text{as} \quad \operatorname{Re} q_c(z) \to +\infty,$$

and

$$\operatorname{Re} f_1(z) \to -\infty, \quad \limsup \operatorname{Re} f_2(z) < +\infty \quad \text{as} \quad \operatorname{Re} q_c(z) \to -\infty.$$

This means that $f \in \mathcal{S}_0(c)$.

The next two theorems complete Theorem 2.

Theorem 4. *Under the assumptions and notation of* Lemma 1, *the function* $f = k(\cdot, c, p)$ *with* $\operatorname{Re} p(\eta) = 0$ *has the following properties:*

(i) *if* $\eta^2 \neq 1$, *then*

$$f(\Delta_+) = D(v) \cap \mathbb{C}_+ \quad \text{for the case } \eta = v,$$

and

$$f(\Delta_-) = D(\overline{v}) \cap \mathbb{C}_- \quad \textit{for the case } \eta = \overline{v},$$

where $D(\eta)$ *is a strip of the form*

(29) $\{(x,y) : |x + y\,\mathrm{Im}\,p(\eta) - \mathrm{Re}\,w_\eta(\eta)| < -\pi\eta\, p'(\eta)/\,[4\sin\gamma]\}$;

(ii) *if* $\eta^2 = 1$ *and* $\eta p'(\eta) + \mathrm{Re}\,p''(\eta) < 0$, *then* $f(\Delta \setminus \mathbb{R}) = \mathbb{C} \setminus \mathbb{R}$, *i. e.* $f \in \mathcal{S}_0$;

(iii) *if* $\eta^2 = 1$ *and* $\eta p'(\eta) + \mathrm{Re}\,p''(\eta) = 0$, *then*

$$f(\Delta \setminus \mathbb{R}) = \{(x,y) : y \neq 0, (x - \phi(y))\eta > 0\},$$

where

$$\phi(y) = -y\,\mathrm{Im}\,p(\eta) - p'(\eta) - \pi\,\mathrm{Im}\,p''(\eta)\,\mathrm{sign}(y)/2 + \mathrm{Re}\,w_\eta(\eta). \tag{30}$$

The proof of the parts (ii)–(iii) needs a simple lemma.

Lemma 2.

(i) *If* $\eta^2 = 1$, $z = q_\eta^{-1}\left(\eta/t^2 + i\alpha\right)$, $t > 0$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$, *then*

$$\begin{aligned} \frac{t(1+\eta z)}{1-\eta z} &= 2\sqrt{1 + \frac{t^2(1+4i\alpha\eta)}{4}} \\ &= 2\left[1 + \frac{t^2(1+4i\alpha\eta)}{8}\right] + O(t^4) \text{ as } t \to 0^+. \end{aligned} \tag{31}$$

In particular,

$$\frac{t}{1-\eta z} = 1 + \frac{t}{2} + O(t^2) \text{ and } \log\frac{t}{1-\eta z} = \frac{t}{2} + O(t^2) \text{ as } t \to 0^+.$$

(ii) *If* $\eta^2 = 1$, $z = q_\eta^{-1}\left(-\eta/t^2 + i\alpha\right)$, $t > 0$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$, *then*

$$\begin{aligned} \frac{t(1+\eta z)}{1-\eta z} &= 2i\sqrt{1 - \frac{t^2(1+4i\alpha\eta)}{4}} \\ &= 2i\eta\,\mathrm{sign}(\alpha)\left[1 - \frac{t^2(1+4i\alpha\eta)}{8}\right] + O(t^4) \end{aligned} \tag{32}$$

as $t \to 0^+$. *In particular,*

$$\frac{t}{1-\eta z} = \eta i \operatorname{sign}(\alpha) + \frac{t}{2} + O(t^2) \text{ and}$$
$$\log \frac{t}{1-\eta z} = \frac{(\pi - t)\eta i \operatorname{sign}(\alpha)}{2} + O(t^2) \text{ as } t \to 0^+.$$

Proof. If $z/(1-\eta z)^2 = \eta/t^2 + i\alpha$, $t > 0$, then

$$t^2(1+\eta z)^2/(1-\eta z)^2 = 4\left[1 + t^2(1+4i\alpha\eta)/4\right],$$

i. e. (31) holds because of $\operatorname{Re}[(1+\eta z)/(1-\eta z)] > 0$. Similarly, if $z/(1-\eta z)^2 = -\eta/t^2 + i\alpha$, $t > 0$, then

$$t^2(1+\eta z)^2/(1-\eta z)^2 = -4\left[1 - t^2(1+4i\alpha\eta)/4\right],$$

i. e. (32) holds because of $\operatorname{Re}[(1+\eta z)/(1-\eta z)] > 0$.

Proof of Theorem 4. (i). Fix α so that $\alpha \operatorname{Im}\eta > 0$. If $z \in \Delta$, $\operatorname{Im} q_c(z) = \alpha$ and $\operatorname{Re} q_c(z) \to \pm\infty$, then, according to (25),

$$\operatorname{Re} f(z) = \operatorname{Re} F(z) \to -\alpha \operatorname{Im} p(\eta) \mp \frac{\pi\eta p'(\eta)\operatorname{sign}(\alpha)}{4\operatorname{Im}\eta} + \operatorname{Re} w_\eta(\eta),$$

and hence the form (29).

(ii)–(iii). It is sufficient to apply (25) and Lemma 2. If $\eta^2 = 1$, $z = q_\eta^{-1}\left(\eta/t^2 + i\alpha\right)$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$\begin{aligned}\operatorname{Re} F(z) &= -\alpha \operatorname{Im} p(\eta) - p'(\eta) + \operatorname{Re} w_\eta(\eta) \\ &\quad - \frac{2p'(\eta) + [p'(\eta) + \eta \operatorname{Re} p''(\eta)]\, t \log t}{t} + O(t)\end{aligned}$$

as $t \to 0^+$, i. e. for $t \to 0^+$ we get

$$\operatorname{Re} F(z) \to \begin{cases} +\infty & \text{if } \eta = 1, \\ -\infty & \text{if } \eta = -1. \end{cases}$$

If now $\eta^2 = 1$, $z = q_\eta^{-1}\left(-\eta/t^2 + i\alpha\right)$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$\begin{aligned}\operatorname{Re} F(z) = -\alpha \operatorname{Im} p(\eta) - p'(\eta) - \frac{\pi \operatorname{sign}(\alpha) \operatorname{Im} p''(\eta)}{2} + \operatorname{Re} w_\eta(\eta) \\ - \eta\left[\eta p'(\eta) + \operatorname{Re} p''(\eta)\right] \log t + O(t) \text{ as } t \to 0^+,\end{aligned}$$

i. e. for $t \to 0^+$ we obtain

$$\operatorname{Re} F(z) \to \begin{cases} (-\eta)\infty, & \text{if } \eta p'(\eta) + \operatorname{Re} p''(\eta) < 0, \\ -\alpha \operatorname{Im} p(\eta) - p'(\eta) - \frac{\pi \operatorname{sign}(\alpha) \operatorname{Im} p''(\eta)}{2} + \operatorname{Re} w(\eta), \\ & \text{if } \eta p'(\eta) + \operatorname{Re} p''(\eta) = 0, \end{cases}$$

and hence the shape of (30). This finishes the proof.

Theorem 5. *Let η be either v or $\overline{v}$, where $v = e^{i\gamma}$ with $0 \leqslant \gamma \leqslant \pi$, and let $c = \cos\gamma$. Consider any function $f = k(\cdot, c, p_\mu)$ with $\mu \in \mathbb{P}_{\partial\Delta}$ and $\mu(\{\overline{\eta}\}) > 0$, see (9), (12–13) and Remark 1. Then*

(i) $$f(\Delta_+) = \mathbb{C}_+ \quad \textit{for} \quad \operatorname{Im}\eta > 0,$$

(ii) $$f(\Delta_-) = \mathbb{C}_- \quad \textit{for} \quad \operatorname{Im}\eta < 0,$$

and

(iii) $$f \in \mathcal{S}_0, \quad \textit{if} \quad \eta^2 = 1.$$

To prove it we need

Lemma 3. *Let η be either v or $\overline{v}$, where $v = e^{i\gamma} = c + is$ with $0 < \gamma < \pi$, and consider the function $f = k(\cdot, c, p_{\overline{\eta}})$. If we denote*

$$D = \left\{(x, y) :\ 0 < x + |y|c/s + (\gamma - sc)\big/(4s^3) < \pi\big/(4s^3)\right\},$$

then

(i) $$f(\Delta) = \mathbb{C}_+ \cup \{(x, y) \in D : y \leqslant 0\} \quad \textit{for } \eta = v,$$

and

(ii) $$f(\Delta) = \mathbb{C}_- \cup \{(x, y) \in D : y \geqslant 0\} \quad \textit{for } \eta = \overline{v}.$$

Proof. Since the total cluster sets of f at all points of $\partial\Delta\setminus\{v,\overline{v}\}$ consist of real numbers, it is sufficient to examine the cluster sets of f at η and $\overline{\eta}$, and next to calculate the limits $f(-1^+)$, $f(1^-)$. To this end, denote

$$F(z)\equiv\int_0^z q_c(\xi)p_{\overline{\eta}}(\xi)d\xi.$$

A simple calculation shows that

$$\operatorname{Re}f(z)\equiv\operatorname{Re}F(z)\equiv -A\operatorname{Re}\frac{\eta z}{1-\eta z}-B\operatorname{Im}\frac{\overline{\eta}z}{(1-\overline{\eta}z)^2}-C\arg\frac{1-\eta z}{1-\overline{\eta}z},$$

where $A=c/(2s^2)$, $B=\operatorname{sign}(\operatorname{Im}\eta)/(2s)$, $C=\operatorname{sign}(\operatorname{Im}\eta)/(4s^3)$ and $\arg 1=0$. On every set (28) we have

(33)
$$z=q_c^{-1}\left(\frac{1}{t}+i\alpha\right)=\eta-\frac{i\eta}{2\operatorname{Im}\eta}t-\left(\frac{\alpha\eta}{2\operatorname{Im}\eta}+\frac{i}{8\operatorname{Im}^3\eta}\right)t^2+O(t^3),$$

whenever $\alpha\operatorname{Im}\eta>0$ and $t\to 0$.

Similarly,

(34)
$$z=q_c^{-1}\left(\frac{1}{t}+i\alpha\right)=\overline{\eta}+\frac{i\overline{\eta}}{2\operatorname{Im}\eta}t+\left(\frac{\alpha\overline{\eta}}{2\operatorname{Im}\eta}+\frac{i}{8\operatorname{Im}^3\eta}\right)t^2+O(t^3),$$

if $\alpha\operatorname{Im}\overline{\eta}=-\alpha\operatorname{Im}\eta>0$ and $t\to 0$.

(I) <u>The cluster set at</u> η. Since $|\arg[(1-\eta z)/(1-\overline{\eta}z)]|<\pi$ for all $z\in\Delta$, and since for z defined by (33),

$$\operatorname{Im}\left[\overline{\eta}z/(1-\overline{\eta}z)^2\right]=-8\alpha\operatorname{Im}^2\eta/t+O(1)\text{ as }t\to 0,$$

we get

$$\operatorname{Re}F(z)=4\alpha\operatorname{Im}\eta/t+O(1)\text{ as }t\to 0.$$

Thus $\operatorname{Re}F(z)\to\pm\infty$ as $t\to 0^\pm$.

(II) <u>The cluster set at</u> $\overline{\eta}$. On the curve (34) we have

$$\frac{\eta z}{1-\eta z}=-1+\frac{2i\operatorname{Im}\eta}{t}\left[1+\left(i\alpha-\frac{\eta}{4\operatorname{Im}^2\eta}\right)t+O(t^2)\right],$$

i. e.

$$\operatorname{Re}\frac{\eta z}{1-\eta z} = -\frac{1}{2} - 2\alpha \operatorname{Im}\eta + O(t) \text{ as } t \to 0.$$

Moreover, on the curves (34) the following limit passages

$$\arg\frac{1-\eta z}{1-\bar{\eta} z} \to \begin{cases} \arg\eta + \pi, & \text{if } -\pi < \arg\eta < 0 \text{ and } t \to 0^+, \\ \arg\eta - \pi, & \text{if } 0 < \arg\eta < \pi \text{ and } t \to 0^+, \\ \arg\eta, & \text{if } t \to 0^-, \end{cases}$$

hold. In fact, the Möbius transformation $z \mapsto t = (1-\eta z)/(1-\bar{\eta} z)$ is conformal and the points $t = \eta, 1, -\eta, 0$ correspond to $z = -1, 0, 1, \bar{\eta}$, respectively. Finally observe that

$$\operatorname{Re} F(-1^+) = \operatorname{Re}\eta/(4\operatorname{Im}^2\eta) - (\arg\eta)/(4\operatorname{Im}^3\eta) = (sc-\gamma)/(4s^3)$$

and

$$\operatorname{Re} F(1^-) = (\pi + sc - \gamma)/(4s^3)\,.$$

Lemma 4. *If* $\eta^2 = 1$ *and* $f = k(\cdot, \eta, p_\eta)$, *then*

(i) $\qquad f(\Delta) = \mathbb{C} \setminus (-\infty, -1/6]$ *for* $\eta = 1$,

and

(ii) $\qquad f(\Delta) = \mathbb{C} \setminus [1/6, +\infty)$ *for* $\eta = -1$.

Proof. Observe first that

$$f(z) \equiv \operatorname{Re}\left\{\frac{\eta}{6}\left[\left(\frac{1+\eta z}{1-\eta z}\right)^3 - 1\right]\right\} + i \operatorname{Im} q_\eta(z),$$

and hence the radial limits: $f(-\eta) = -\eta/6$ and $\eta f(\eta) = +\infty$. From Lemma 2 we deduce that

1° on the curve $z = q_\eta^{-1}(\eta/t^2 + i\alpha)$, $t > 0$, with $\alpha \neq 0$, we have

$$\left(\frac{1+\eta z}{1-\eta z}\right)^3 = \frac{8}{t^3}\left[1 + \frac{3}{8}t^2(1 + 4i\alpha\eta)\right] + O(t) \text{ as } t \to 0^+,$$

and

2° on the curve $z = q_\eta^{-1}\left(-\eta/t^2 + i\alpha\right)$, $t > 0$, with $\alpha \neq 0$, we have

$$\left(\frac{1+\eta z}{1-\eta z}\right)^3 = \frac{-8i\eta\,\mathrm{sign}(\alpha)}{t^3}\left[1 - \frac{3}{8}t^2(1+4i\alpha\eta)\right] + O(t) \text{ as } t \to 0^+,$$

i. e.

$$\mathrm{Re}\left(\frac{1+\eta z}{1-\eta z}\right)^3 = \frac{-12|\alpha|}{t} + O(t) \text{ as } t \to 0^+.$$

Thus, for $\eta = 1$ or $\eta = -1$, $\mathrm{Re}\,F(z) \to \pm\infty$ as $\mathrm{Re}\,q_\eta(z) \to \pm\infty$.

Proof of Theorem 5. By assumption, $f = k(\cdot, c, p_{\overline{\eta}})$ for $\mu(\{\overline{\eta}\}) = 1$, and

$$f = \mu(\{\overline{\eta}\})\,k(\cdot, c, p_{\overline{\eta}}) + [1 - \mu(\{\overline{\eta}\})]\,k(\cdot, c, p_\nu),$$

if $\mu(\{\overline{\eta}\}) < 1$ and $\nu = [\mu - \mu(\{\overline{\eta}\})\,\delta_{\overline{\eta}}]/(1 - \mu(\{\overline{\eta}\}))$, where δ_ξ means the Dirac measure concentrated at ξ. If now $\mu(\{\overline{\eta}\}) < 1$ and $\mathrm{Re}\,q_\eta(z) \to +\infty$ (resp. $\mathrm{Re}\,q_\eta(z) \to -\infty$), then $\liminf \mathrm{Re}\,k(\cdot, c, p_\nu) > -\infty$ (resp. $\limsup \mathrm{Re}\,k(\cdot, c, p_\nu) < +\infty$). According to Lemmas 3–4, the suitable passage to the limit is possible and the proof is complete.

Remark 2. Consider $f = k(\cdot, c, p_\xi)$ with $-1 \leqslant c \leqslant 1$ and $|\xi| = 1$. It follows from Lemmas 1–3 and Theorems 4–5 that

(i) $f \in \mathcal{S}_0$ iff $\xi = c$, i. e. iff $c^2 = 1$ and $\xi = c$;

(ii) in the case $c^2 = 1$ and $\xi \neq c$, the set $f(\Delta)$ is the union of three disjoint sets: two convex wedges $V_1 \subset \mathbb{C}_+$ and $V_2 \subset \mathbb{C}_-$ whose the common boundary $\overline{V}_1 \cap \overline{V}_2$ is a real half-plane starting at some α, and $\overline{V}_1 \cap \overline{V}_2 \setminus \{\alpha\}$;

(iii) in the case $-1 < \mathrm{Re}\,\xi = c < 1$, the set $f(\Delta)$ is the union of a non-horizontal strip and one from the two half-planes $\mathbb{C}_+$ or $\mathbb{C}_-$ (strictly speaking, the case implies that $f(\Delta)$ is the union of: a half-strip, a half-plane and a real segment);

(iv) in the case $c^2 \neq 1$ and $\mathrm{Re}\,\xi \neq c$, the set $f(\Delta)$ is the union of three disjoint sets: two non-horizontal half-strips $V_1 \subset \mathbb{C}_+$ and $V_2 \subset \mathbb{C}_-$ whose the common boundary $\overline{V}_1 \cap \overline{V}_2$ is a closed real segment joining some α_1 and α_2, and $\overline{V}_1 \cap \overline{V}_2 \setminus \{\alpha_1, \alpha_2\}$.

Remark 3. In cases (ii)–(iv) of the previous remark the real components are as large as possible. For any function $f = k(\cdot, c, p_\mu)$

with a finitely discrete measure μ, its range $f(\Delta)$ may have some real slits. However, the only possibilities for $f(\Delta_+)$ and $f(\Delta_-)$ are: half-planes (upper or lower), half-strips or wedges. For instance, $(1-\lambda)k(\cdot, c, p_\xi) + \lambda k(\cdot, c, p_{\overline{\xi}}) \in \mathcal{S}_0$ whenever $0 < \lambda < 1$ and $\operatorname{Re}\xi = c$.

Remark 4. A property of the Poisson integral concerning cluster sets at boundary points is well known, see [6; proof of Th. 2.9] or [14; Th. IV.3]. Consider any $f \in \mathcal{F}$ and let $\alpha < \beta$ with $0 \notin (\alpha, \beta)$. If the set $f(\Delta) \cap \{w : \alpha < \operatorname{Im} w < \beta\}$ is bounded to the left (resp. to the right), then it is bounded to the left (resp. to the right) by a straight line segment. This observation leads us to open questions:

1° Is there an $f \in \mathcal{F}$ whose range differs from that described in Remark 3?

2° Find the range of $k(\cdot, c, p_\mu)$ for a singular and non-atomic measure $\mu \in \mathbb{P}_{\partial\Delta}$.

Remark 5. The best bounds for $|a_n|$, $|b_n|$, $||a_n| - |b_n||$ and $|a_n - b_n|$ among all functions f in $\mathcal{F}$, where $f(z) \equiv \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=2}^{\infty} b_k z^k}$, are assumed in the classes $\mathcal{F}(1)$ and $\mathcal{F}(-1)$ so that Th. 5 from [8] with $a = 1/6$ extends to the whole class $\mathcal{F}$. In fact, if $f = k(\cdot, \cos\gamma, p_\mu)$, $0 \leqslant \gamma \leqslant \pi$, then a simple calculation shows that

$$na_n = \int_{\partial\Delta} \left[\sum_{j=1}^{n} \frac{j\sin(j\gamma)}{\sin\gamma} \xi^{n-j}\right] d\mu(\xi), \; n = 1, 2, \ldots,$$

and

$$nb_n = \int_{\partial\Delta} \left[\sum_{j=1}^{n-1} \frac{j\sin(j\gamma)}{\sin\gamma} \xi^{n-j}\right] d\mu(\xi), \; n = 2, 3, \ldots.$$

Thus

$$|a_n| \leqslant \sum_{j=1}^{n} j^2/n = (n+1)(2n+1)/6, \; n = 1, 2, \ldots,$$

$$|b_n| \leqslant \sum_{j=1}^{n-1} j^2/n = (n-1)(2n-1)/6, \; n = 2, 3, \ldots,$$

and

$$||a_n| - |b_n|| \leqslant |a_n - b_n| = \left|\frac{\sin(n\gamma)}{\sin\gamma}\right| \leqslant n, \ n = 2, 3, \ldots,$$

with equality only for $\gamma = 0$ with $\mu = \delta_1$ and for $\gamma = \pi$ with $\mu = \delta_{-1}$. Let us add that $\mathcal{F}(1)$ and $\mathcal{F}(-1)$ do not play an essential role in extremal problems over $\mathcal{F}$. For instance, $\operatorname{Re} a_3 \geqslant -4/3$ for all $f = k(\cdot, \cos\gamma, p_\mu) \in \mathcal{F}$ with equality only for $\gamma = \pi/2$ with $\mu = \delta_i$ or $\mu = \delta_{-i}$.

3. Some convexity tools. The main result concerns the existence of non-trivial variations preserving a system of constraints.

Theorem 6 [9, 13]. *Let $\mathcal{A}$ be a non-empty compact convex subset of a locally convex Hausdorff vector space with zero element θ. Suppose that $\Phi : \mathcal{A} \to \mathbb{R}^n$ is affine continuous. Then for every $a \in \mathcal{A}$ at least one of the following holds:*

(i) $a \in \operatorname{co}\{e_1, \ldots, e_{n+1}\}$ *for some* $e_1, \ldots, e_{n+1} \in \operatorname{E}\mathcal{A}$,

or

(ii) *there exists a variational formula:*

$$a + \varepsilon b \in \mathcal{A} \text{ for all } -1 \leqslant \varepsilon \leqslant 1 \text{ with } b \neq \theta,$$

which preserves the constraints:

$$\Phi(a + \varepsilon b) = \Phi(a) \text{ for all } -1 \leqslant \varepsilon \leqslant 1.$$

Application 1 [9, 13]. *Under the assumption of* Theorem 6, *for every non-empty compact convex set $W \subset \Phi(\mathcal{A})$, the preimage $\Phi^{-1}(W)$ is a compact convex subset of $\mathcal{A}$ with*

$$\operatorname{E}\Phi^{-1}(W) \subset \left\{ x = \sum_{j=1}^{n+1} \lambda_j e_j : \ \lambda_j \geqslant 0, \ e_j \in \operatorname{E}\mathcal{A}, \right.$$
$$\left. \sum_{j=1}^{n+1} \lambda_j = 1 \text{ and } \Phi(x) \in W \right\}.$$

For better supersets see [9, 13]. The proof for $n = 1$ and $\mathcal{A} = \mathbb{P}_{[a,b]}$ one can also find in [7]. As a corollary we get

Application 2 [11, 13]. *Let $\mathcal{A}$ be a non-empty compact convex subset of a locally convex Hausdorff vector space X and let*

$$\mathcal{Z} = \{\lambda x : \ \lambda \geqslant 0, \, x \in \mathcal{A}\}.$$

Suppose that $\Phi : X \to \mathbb{R}^n$ is linear continuous with $\theta \notin \Phi(\mathcal{A})$ (here θ means the zero element in $\mathbb{R}^n$). Then for every compact convex set $W \subset \Phi(\mathcal{Z})$, the preimage $(\Phi|\mathcal{Z})^{-1}(W)$ is compact convex with

$$\mathrm{E}(\Phi|\mathcal{Z})^{-1}(W) \subset \mathcal{B} \subset (\Phi|\mathcal{Z})^{-1}(\partial W),$$

where
(35)

$$\mathcal{B} = \Big\{ x = \sum_{j=1}^{n} \lambda_j e_j : \ \lambda_j \geqslant 0, \, e_j \in \mathrm{E}\,\mathcal{A}, \Phi(e_j) \neq \Phi(e_s) \text{ for } j \neq s \text{ and } \Phi(x) \in \partial W \Big\}.$$

In (35) we do not claim that $\lambda_1 + \cdots + \lambda_n = 1$.

Application 3 [8, 13]. *Suppose X is a locally convex Hausdorff vector space, $\phi : X \to \mathbb{C}$ is positively homogeneous (i. e. $\phi(\lambda x) = \lambda\phi(x)$ for all $\lambda \geqslant 0$ and $x \in X$), $\mathfrak{c} \in \mathbb{C} \setminus \{0\}$ and $\mathcal{A}$ is a compact convex subset of $\phi^{-1}(\mathfrak{c})$. Let $\psi : \mathcal{A} \to \mathbb{R}$ be affine continuous with $0 \notin \psi(\mathcal{A})$ and let $\mathcal{B} = \{a/\psi(a) : a \in \mathcal{A}\}$. Then*

1) $\mathcal{B}$ is compact convex,
2) the map $a \mapsto a/\psi(a)$ is a homeomorphism of $\mathcal{A}$ onto $\mathcal{B}$,
3) $\mathrm{E}\,\mathcal{B} = \{a/\psi(a) : \ a \in \mathrm{E}\,\mathcal{A}\}$.

In the proof there is no loss of generality in assuming that

$$\psi(\mathcal{A}) = [a_0, b_0] \subset (0, +\infty)$$

(if otherwise, instead of ψ consider $-\psi$). Then $\mathcal{B} \subset \mathcal{A}_M = \{\lambda x : 0 \leqslant \lambda \leqslant M, \, x \in \mathcal{A}\}$ for all $M \geqslant 1/a_0$, $\mathrm{E}\,\mathcal{A}_M \subset M \cdot (\mathrm{E}\,\mathcal{A}) \cup \{\theta\}$ and $\mathcal{B} = \psi^{-1}(1)$, where ψ means the affine extension of ψ to $\{\lambda x : \lambda \geqslant 0, \, x \in \mathcal{A}\}$. Thus we may apply Application 1 to $\mathcal{A}_M$, ψ and $W = \{1\}$. The conclusion $\mathrm{E}\,\mathcal{B} \subset \{a/\psi(a) : a \in \mathrm{E}\,\mathcal{A}\}$ follows from the fact that M may be arbitrarily large. A direct proof of Application 3 one can find in [8, 13].

4. The class $\mathcal{S}_H(\Delta, \Omega_{a,b})$. We follow the notation of (6–9), (12–15) and (23). Let us consider

$$\mathcal{G} = \{\lambda f :\ \lambda \geqslant 0 \ \text{ and } f \in \mathcal{F}\}, \tag{36}$$

the cone generated by $\mathcal{F}$. The main idea is to show that the set

$$\mathcal{G}_{a,b} = \{u \in \mathcal{G} :\ u(\Delta) \cap \mathbb{R} = (a,b)\} \tag{37}$$

is the closure of $\mathcal{S}_H(\Delta, \Omega_{a,b})$. Next, since (36) is the union of the convex sets

$$\mathcal{G}(c) = \{\lambda f :\ \lambda \geqslant 0 \ \text{ and } f \in \mathcal{F}(c)\},\ -1 \leqslant c \leqslant 1, \tag{38}$$

we are able to conclude that (37) is the union of some compact convex sets

$$\mathcal{G}_{a,b}(c) = \{u \in \mathcal{G}(c) :\ u(\Delta) \cap \mathbb{R} = (a,b)\} \tag{39}$$

with integral representations of Choquet's type. For short, *let $u(-1)$ and $u(1)$ stand for the radial limits $u(-1^+)$ and $u(1^-)$*, respectively. Clearly, the functionals

$$u \mapsto \Phi_A(u) = u(1) + Au(-1),\ A > 0, \tag{40}$$

are well-defined on (36), see a lemma below, and all of them are continuous on each $\mathcal{G}(c)$ with $-1 < c < 1$. Some properties of (40) one can find in

Lemma 5. *Assume $0 \leqslant \gamma \leqslant \pi$, $c = \cos\gamma$, $s = \sin\gamma$, and consider $f = k(\cdot, c, p)$ with $p \in \mathcal{P}$. Then*

$$k(x, c, p_{-1}) \leqslant f(x) \leqslant k(x, c, p_1) \ \text{for all } -1 \leqslant x \leqslant 1, \tag{41}$$

and, excluding $x_0 = c = \pm 1$, if $f(x_0) = k(x_0, c, p_{\pm 1})$ for an $x_0 \in [-1, 0) \cup (0, 1]$, then $f = k(\cdot, c, p_{\pm 1})$, respectively. In particular,

(i) $f(\pm 1) = \pm\infty$ *and* $\mp f(\mp 1) \in [1/6, 1/2]$ *iff* $c = \pm 1$,

(ii) $\{f(1) :\ f \in \mathcal{F}(c)\} = \left[\dfrac{\pi - \gamma - s}{2(1+c)s}, \dfrac{\pi - \gamma + s}{2(1-c)s}\right]$ *if* $-1 < c < 1$,

(iii)

$$\{f(-1) :\ f \in \mathcal{F}(c)\} = \left[\frac{-\gamma - s}{2(1+c)s}, \frac{-\gamma + s}{2(1-c)s}\right] \ \text{if } -1 < c < 1,$$

and, except for $c = \pm 1$,

(iv) $\Phi_A(\mathcal{F}(c)) = [\Phi_A(k(\cdot, c, p_{-1})), \Phi_A(k(\cdot, c, p_1))]$

$$= \left[\frac{\pi - (1+A)(\gamma+s)}{2(1+c)s}, \frac{\pi - (1+A)(\gamma-s)}{2(1-c)s}\right].$$

Moreover, $0 \in \Phi_A(\mathcal{F}(c))$ *if and only if*

$$\phi(\gamma) \stackrel{\text{def}}{=} \frac{\pi}{\gamma + \sin\gamma} - 1 \leqslant A \leqslant \psi(\gamma) \stackrel{\text{def}}{=} \frac{\pi}{\gamma - \sin\gamma} - 1, \tag{42}$$

i. e. iff $\phi^{-1}(A) \leqslant \gamma \leqslant \psi^{-1}(A)$.

Proof. For $-1 < x < 1$ we have $f'(x) > 0$, so the radial limits $f(-1)$, $f(1)$ exist. To see (41), consider first the case $0 < x < 1$, use (9) and next (20). Since

$$0 = k(x_0, c, p) - k(x_0, c, p_{-1})$$

$$= \begin{cases} \int_0^{x_0} q_c'(t)\,\mathrm{Re}[p(t) - p_{-1}(t)]dt & \text{if } 0 < x_0 \leqslant 1, \\ \int_0^{-x_0} q_c'(-t)\,\mathrm{Re}[p_1(t) - p(-t)]dt & \text{if } -1 \leqslant x_0 < 0, \end{cases}$$

then $p = p_{-1}$. The similar conclusion follows from equality $0 = k(x_0, c, p_1) - k(x_0, c, p)$.

(i). A simple calculation shows that

$$k(x, 1, p_{-1}) \equiv -k(-x, -1, p_1) \equiv x/(1-x)$$

and

$$k(x, 1, p_1) \equiv -k(-x, -1, p_{-1}) \equiv x(x^2+3) / [3(1-x)^3].$$

Thus, (i) follows from (41).

(ii–iii). Similarly,

$$k(x, c, p_1) \equiv -k(-x, -c, p_{-1}) \equiv \frac{\arctan[xs/(1-cx)]}{(1-c)s} + \frac{x-c}{1-c} q_c(x),$$

and the conclusion follows from (41), too.

(iv). Combining (ii) with (iii) we deduce (iv).

Remark 6. Both the functions ϕ and ψ, defined in (42), strictly decrease on $(0,\pi]$, $\phi(0^+)=\psi(0^+)=+\infty$, $\phi(\pi)=\psi(\pi)=0$, $\phi<\psi$ on $(0,\pi)$, and $\phi(\pi-\gamma)\psi(\gamma)\equiv 1$. Also, the condition

$$f\in\mathcal{F}(c) \text{ and } \Phi_A(f)=\Phi_A(k(\cdot,c,p_{\pm 1}))$$

is equivalent to $f=k(\cdot,c,p_{\pm 1})$, respectively.

Remark 7. From Lemma 5(i), Theorem 3 and Application 3 we conclude the above-mentioned results of Livingston: (21–22) so that

$$\overline{\mathcal{S}_H(\Delta,\Omega_{a,+\infty})}=\left\{a\int_{\partial\Delta}\frac{k(\cdot,1,p_\xi)}{k(-1,1,p_\xi)}d\mu(\xi):\ \mu\in\mathbb{P}_{\partial\Delta}\right\}.$$

Analogously,

$$\begin{aligned}\overline{\mathcal{S}_H(\Delta,\Omega_{-\infty,b})}&=\{bf/f(1):\ f\in\mathcal{F}(-1)\}\\&=\left\{b\int_{\partial\Delta}\frac{k(\cdot,-1,p_\xi)}{k(1,-1,p_\xi)}d\mu(\xi):\ \mu\in\mathbb{P}_{\partial\Delta}\right\}.\end{aligned}$$

In case $-\infty<a<0<b<+\infty$ the problem occurs more complicated. By definition, $\mathcal{S}_H(\Delta,\Omega_{a,b})=\mathcal{S}_0\cap\mathcal{G}_{a,b}=\{af/f(-1):\ f\in\mathcal{S}_0,\ \Phi_{-b/a}(f)=0\}$, see (36–37) and (40). Further on, we shall use Theorem 3, Lemma 6, and either Application 2 with $n=2$ or, simultaneously, Application 1 with $n=1$ and Application 3.

Theorem 7. *Let $-\infty<a<0<b<+\infty$ and $A=-b/a$. Then*

$$\overline{\mathcal{S}_H(\Delta,\Omega_{a,b})}=\bigcup_{\phi^{-1}(A)\leqslant\gamma\leqslant\psi^{-1}(A)}\mathcal{G}_{a,b}(\cos\gamma) \tag{43}$$

and each set (39) takes the form

$$\mathcal{G}_{a,b}(c)=\{af/f(-1):\ f\in\mathcal{F}(c)\ \text{and}\ \Phi_A(f)=0\}. \tag{44}$$

Remark 8. Exactly two members of the union (43) are singletons:

$$\mathcal{G}_{a,b}(c)=\left\{\frac{ak(\cdot,c,p_\eta)}{k(-1,c,p_\eta)}\right\} \text{ for either } c=\cos\phi^{-1}(A) \text{ and } \eta=-1$$

or $c = \cos\psi^{-1}(A)$ and $\eta = 1$.

Proof of Theorem 7. It is evident that (39) and (44) are equivalent. Because of Theorem 3 and Lemma 5, we only need to show that

$$\mathcal{G}_{a,b}(\cos\gamma) \subset \overline{\mathcal{S}_H(\Delta, \Omega_{a,b})} \text{ for all } \phi^{-1}(A) \leqslant \gamma \leqslant \psi^{-1}(A).$$

The proof will be divided into 2 steps.

STEP 1. We may restrict our attention to the case

$$\phi^{-1}(A) < \gamma < \psi^{-1}(A). \tag{45}$$

In fact, both classes $\mathcal{G}_{a,b}(\cos\phi^{-1}(A))$ and $\mathcal{G}_{a,b}(\cos\psi^{-1}(A))$ are singletons. If we consider a sequence (f_n) with

$$f_n = ak(\cdot, \cos\gamma_n, p_n)/k(-1, \cos\gamma_n, p_n) \in \mathcal{G}_{a,b}(\cos\gamma_n),$$

where $p_n \in \mathcal{P}$ and $\gamma_n \to \left[\phi^{-1}(A)\right]^+$, then for each convergent subsequence (p_{k_n}) of (p_n) we have $p = \lim p_{k_n} \in \mathcal{P}$ and

$$\begin{aligned} \Phi_A(k(\cdot\,, \cos\phi^{-1}(A), p_{-1})) &= 0 \\ = \Phi_A(k(\cdot, \cos\gamma_{n_k}, p_{n_k})) &\to \Phi_A(k(\cdot, \cos\phi^{-1}(A), p)). \end{aligned}$$

By Remark 6, $p = p_{-1}$ so that $\mathcal{G}_{a,b}(\phi^{-1}(A)) = \{\lim f_n\}$. Analogously, replacing $\phi^{-1}(A)$ by $\psi^{-1}(A)$ and $\gamma_n \to \left[\phi^{-1}(A)\right]^+$ by $\gamma_n \to \left[\psi^{-1}(A)\right]^-$, we show that the only element of $\mathcal{G}_{a,b}(\psi^{-1}(A))$ can be locally uniform approximated by arbitrary elements from $\mathcal{G}_{a,b}(\cos\gamma)$ as $\gamma \to \left[\psi^{-1}(A)\right]^-$.

STEP 2. Assume (45) so that

$$\Phi_A(k(\cdot, \cos\gamma, p_{-1})) < 0 < \Phi_A(k(\cdot, \cos\gamma, p_1)). \tag{46}$$

Choose any $f = ak(\cdot, \cos\gamma, p)/k(-1, \cos\gamma, p)) \in \mathcal{G}_{a,b}(\cos\gamma)$, where $p \in \mathcal{P}$, and define p_n by

$$p_n(z) \equiv p((1 - 1/n)z).$$

Consider the functions

$$(47)\qquad \check{p}_n = \begin{cases} (1-\lambda_n)p_{-1} + \lambda_n p_n & \text{if } \Phi_A(k(\cdot\cos\gamma, p_n)) \geqslant 0, \\ (1-t_n)p_1 + t_n p_n & \text{if otherwise,} \end{cases}$$

and

$$(48)\qquad f_n = ak(\cdot, \cos\gamma, \check{p}_n)/k(-1, \cos\gamma, \check{p}_n),$$

where λ_n, t_n are real numbers such that

$$(49)\qquad \Phi_A(k(\cdot, \cos\gamma, \check{p}_n)) = 0.$$

The function (48) is correctly defined since, according to (46), $0 < \lambda_n \leqslant 1$ and $0 < t_n < 1$, which means that $\check{p}_n \in \mathcal{P}$. Hence by Theorem 3, we deduce that $f_n \in \mathcal{S}_H(\Delta, \Omega_{a,b})$ for all n. Because

$$\Phi_A(k(\cdot, \cos\gamma, p_n)) \to \Phi_A(k(\cdot, \cos\gamma, p)) = 0,$$

(46–47) and (49) imply that $\check{p}_n \to p$, i. e. $\mathcal{S}_H(\Delta, \Omega_{a,b}) \ni f_n \to f$, and the proof is finished.

Remark 9. If $b = -a$, then $A = 1$ and $\phi^{-1}(1) = 0.8317\ldots$, $\psi^{-1}(1) = \pi - \phi^{-1}(1) = 2.3098\ldots$, $\cos\phi^{-1}(1) = -\cos\psi^{-1}(1) = 0.6736\ldots$

Theorem 8. *Let $-\infty < a < 0 < b < +\infty$, $A = -b/a$ and fix $c = \cos\gamma$ with $\phi^{-1}(A) < \gamma < \psi^{-1}(A)$.*

(i) *The sets*

$$\Gamma_1 = \{\xi \in \partial\Delta : \ \Phi_A(k(\cdot, c, p_\xi)) \leqslant 0\}$$

and

$$\Gamma_2 = \{\xi \in \partial\Delta : \ \Phi_A(k(\cdot, c, p_\xi)) > 0\}$$

are complementary arcs of the unit circle $\partial\Delta$ with $-1 \in \Gamma_1$ and $1 \in \Gamma_2$, both symmetric with respect to the real axis.

(ii) *For abbreviation, denote $\Gamma = \Gamma_1 \times \overline{\Gamma}_2$ and*

$$l(z, \xi) \equiv k(z, c, p_\xi)/k(-1, c, p_\xi).$$

Then for (44) we have the integral representation formula

$$\mathcal{G}_{a,b}(c) = \left\{ \int_{\Gamma_1 \times \Gamma_2} a \left[\frac{l(1,\eta) + A}{l(1,\eta) - l(1,\xi)} l(\cdot, \xi) \right.\right.$$
$$\left.\left. + \frac{l(1,\xi) + A}{l(1,\xi) - l(1,\eta)} l(\cdot, \eta) \right] d\mu(\xi,\eta) : \mu \in \mathbb{P}_\Gamma, \mu(\Gamma_1 \times \Gamma_2) = 1 \right\}. \tag{50}$$

Moreover, each integrand in (50) is an extreme point of $\mathcal{G}_{a,b}(c)$, and hence of the whole class $S_H(\Delta, \Omega_{a,b})$.

Proof. (i). Put $x = \operatorname{Re}\xi$. Then both the functions

$$x \mapsto k(1, c, p_\xi) = \int_0^1 q_c'(t) \frac{1 - t^2}{1 - 2xt + t^2} dt$$

and

$$x \mapsto k(-1, c, p_\xi) = -k(1, -c, p_{-\xi}) = -\int_0^1 q_c'(-t) \frac{1 - t^2}{1 + 2xt + t^2} dt$$

are continuous and strictly increasing on $[-1, 1]$, so are

$$x \mapsto \Phi_A(k(\cdot, c, p_\xi)) \quad \text{for all} \quad A > 0.$$

(ii). Use Application 2 to the class $\mathcal{A} = \mathcal{F}(c)$ with $\Phi(u) = (u(-1), u(1))$ and $W = \{(a, b)\}$, and next refer to the Krein-Milman theorem.

Remark 10. In the proof of Theorem 8(ii) we may also take advantage of Application 3 with $\mathcal{A} = \mathcal{F}(c)$, $\phi(f) = f_z(0)$, $c = 1$ and $\psi(f) = f(-1)$, to show that the set

$$\{af/f(-1) : f \in \mathcal{F}(c)\} \tag{51}$$

is compact convex with

$$\mathrm{E}\{af/f(-1) : f \in \mathcal{F}(c)\} = \{af/f(-1) : f \in \mathrm{E}\,\mathcal{F}(c)\},$$

and next use Application 1 to the class (51) with $\Phi(u) = u(-1)$ and $W = \{b\}$.

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