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# On Types of Convergence of a Sequence of Defective Random Elements

ABSTRACT. We introduce concepts of vague essential convergence, vague convergence in probability and vague almost sure convergence of a sequence of defective random elements. Relations between these types of convergence and the classical ones are also investigated.

**1. Introduction and preliminaries.** Let $(\Omega, \mathcal{A}, P)$ be a generalized probability space, i.e. $\Omega$ is the set of elementary events, $\mathcal{A}$ is a $\sigma$-field of subsets of $\Omega$ and $P$ is a measure defined on $\mathcal{A}$ such that $P(\Omega) \leq 1$. If $P(\Omega) = 1$, then $P$ is said to be a proper probability measure, while $P$ with $P(\Omega) < 1$ is called a defective (imperfect) probability measure. Moreover, $(S, \varrho)$ stands for a metric space and $\mathcal{B} := \mathcal{B}(S)$ denotes the Borel $\sigma$-field of subsets of $S$. By a random element $X$ we mean the mapping $X : \Omega \to S$ such that $X^{-1}(B) \in \mathcal{A}$, $B \in \mathcal{B}$. In the case $S = \mathbb{R}$, $X$ is called a random variable. By $\overline{S}$ we denote the union of the space $S$ and some points $x_\infty, y_\infty, \ldots$ not belonging to $S$. In $\overline{S}$ we consider a topology generated by the following families of neighbourhoods of points:

$$\mathcal{B}(x) := \{U \subset \overline{S} : x \in U,\ U \subset S,\ U \text{ is open}\}, \quad x \in S.$$

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$$\mathcal{B}(x_\infty) := \Big\{ U \subset \overline{S} : \text{ there exist sets } A_1, A_2, \dots, A_n \text{ closed in } S \text{ and such that } U = \Big( S \setminus \bigcup_{i=1}^{n} A_i \Big) \cup \{x_\infty\} \Big\}, x_\infty \in \overline{S} \setminus S.$$

The extension $\overline{S}$ of $S$ is similar to that in [8].

By a defective random element $X$ we mean a mapping $X : \Omega \to \overline{S}$ such that $X^{-1}(B) \in \mathcal{A}$, $B \in \mathcal{B}(\overline{S})$, and $P[X^{-1}(S)] = a < P(\Omega)$. A defective random variable $X : \Omega \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, is characterized by the property $P[\omega : |X(\omega)| < \infty] = a < P(\Omega)$, or symbolically, by $0 < P[\omega : |X(\omega)| = \infty]$. The generalized probability distribution $P_X$ of a random element $X$ is defined by:

$$P_X(B) = P[X^{-1}(B)] = P[\omega : X(\omega) \in B], \quad B \in \overline{\mathcal{B}} = \mathcal{B}(\overline{S}).$$

The set of all random elements (defective and non-defective) defined on $(\Omega, \mathcal{A})$ is denoted by $\mathfrak{X}$ and the subset of non-defective random elements by $\mathfrak{X}_0$.

The defective random variables appear in a natural way in the renewal theory ([4]), the theory of physical measurement ([11]), or in the theory of probabilistic metric spaces ([10]). Here we quote a simple example from the theory of games.

**Example 1.** *Gambler's ruin* (cf. [3]). Let $X_1, X_2, \dots$ be independent, identically distributed random variables:

$$P\{X_k = -1\} = q, \quad P\{X_k = 1\} = p, \; p + q = 1 .$$

Write $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, and let, for any $a, b \in \mathrm{N}$, $t$ be the first $n$ such that $S_n = -a$ or $S_n = b$. Here $p$ is the probability of winning in a single game, $q$ of losing, $a$ is a capital of the gambler, $b$ denotes the intentional winning, while $t$ is the final moment of the game. If the capital of the gambler is unlimited then

$$t = \begin{cases} \text{first } n \text{ such that } S_n \geq b, \\ \infty \quad \text{if no such } n \text{ exists.} \end{cases}$$

If $p < q$, then $P\{t < \infty\} = (p/q)^b < 1$. Thus $t$ is a defective random variable which takes finite values with probability less than 1.

Let $\mathcal{P} = \mathcal{P}(S)$ be the class of all Borel measures $P$ defined on $(S, \mathcal{B}(S))$ such that $P(S) \leq 1$ and $\mathcal{P}_0 \subset \mathcal{P}$ is the subclass containing the proper probability measures ($P \in \mathcal{P}_0 \iff P(S) = 1$). Denote by $\mathcal{C}_b$ the set of all bounded continuous functions on $S$ and by $\mathcal{C}_M$ the subset of $\mathcal{C}_b$ containing the functions with bounded support, i.e. $f \in \mathcal{C}_M$ if $\exists_{\text{bounded set } D(f)} \forall_{x \notin D(f)} f(x) = 0$.

Now we need to recall the notions of weak and vague convergence of a sequence $\{P_n, n \geq 1\}$ of generalized probability measures.

We say that a sequence $\{P_n, n \geq 1\}$ of measures $P_n \in \mathcal{P}$ *weakly converges* to a measure $P \in \mathcal{P}$ ($P_n \xrightarrow{D} P$, $n \to \infty$) if for every function $f \in \mathcal{C}_b$

$$\lim_{n \to \infty} \int_S f dP_n = \int_S f dP.$$

A sequence $\{P_n, n \geq 1\}$ of measures $P_n \in \mathcal{P}$ *vaguely converges* to a measure $P \in \mathcal{P}$ ($P_n \xrightarrow{V} P$, $n \to \infty$) if for every function $f \in \mathcal{C}_M$

$$\lim_{n \to \infty} \int_S f dP_n = \int_S f dP.$$

It is known that, if $S$ is a separable metric space, then the following statements are true.

Weak convergence of a sequence $\{P_n, n \geq 1\}$ of proper probability measures, is characterized by the following equivalent conditions (cf. [1], [7]):

(i) $P_n \xrightarrow{D} P$, $n \to \infty$,
(ii) $\limsup_{n \to \infty} P_n(F) \leq P(F)$ for every closed set $F$,
(iii) $\liminf_{n \to \infty} P_n(G) \geq P(G)$ for every open set $G$,
(iv) $\lim_{n \to \infty} P_n(A) = P(A)$ for every set $A \in \mathcal{B}(S)$ such that $P(\partial A) = 0$,
(v) $\lim_{n \to \infty} \int_S f dP_n = \int_S f dP$ for every uniformly continuous function $f \in \mathcal{C}_b$.

For vague convergence of a sequence $\{P_n, n \geq 1\}$ with $P_n \in \mathcal{P}$, $n \geq 1$, the following conditions are equivalent (cf. [6]):

(i) $P_n \xrightarrow{V} P$, $n \to \infty$,
(ii) $\limsup_{n \to \infty} P_n(F) \leq P(F)$ and $\liminf_{n \to \infty} P_n(G) \geq P(G)$ for every bounded closed set $F$ and every bounded open set $G$, respectively,
(iii) $\lim_{n \to \infty} P_n(A) = P(A)$ for every bounded set $A \in \mathcal{B}(S)$ such that $P(\partial A) = 0$.

We say that a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ *weakly converges* to a random element $X \in \mathfrak{X}$ ($X_n \xrightarrow{D} X$, $n \to \infty$) if the sequence $\{P_{X_n}, n \geq 1\}$ of generalized probability distributions of $X_n$ weakly converges to $P_X$.

A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ *vaguely converges* to a random element $X \in \mathfrak{X}$, ($X_n \xrightarrow{V} X$, $n \to \infty$) if the sequence $\{P_{X_n}, n \geq 1\}$ of generalized probability distributions of $X_n$ vaguely converges to $P_X$.

By $\mathcal{C}_P$ we denote the family of continuity sets of a measure $P$, i.e. $A \in \mathcal{C}_P$ if $P(\partial A) = 0$, where $\partial A$ denotes the boundary of $A$. The family $\mathcal{C}_{P_X}$ will be denoted shortly by $\mathcal{C}_X$. The following concept of essential convergence in law was given in [12] (cf. [9], [2]).

A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}_0$ is said to be *essentially convergent in law* to a random element $X \in \mathfrak{X}_0$ ($X_n \xrightarrow{ED} X, n \to \infty$) if for every set $A \in \mathcal{C}_X$

$$P\{\limsup_{n\to\infty}[X_n \in A]\} = P\{\liminf_{n\to\infty}[X_n \in A]\}$$
$$= P[X \in A] \text{ with respect to } P \in \mathcal{P}_0(\Omega).$$

**Remark.** The essential convergence in law can be considered in the set $\mathfrak{X}$ of defective and non-defective random elements.

For the sake of completeness we recall the following notions.

We say that a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}_0$ *converges in probability* to a random element $X \in \mathfrak{X}_0$ if for any $\varepsilon > 0$ $\lim_{n\to\infty} P[\omega : \varrho(X_n, X) > \varepsilon] = 0$ ($P \in \mathcal{P}_0(\Omega)$) and we write $X_n \xrightarrow{P} X$, $n \to \infty$.

A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}_0$ is said to be *convergent almost surely* to a random element $X \in \mathfrak{X}_0$ (notation: $X_n \xrightarrow{a.s.} X$, $n \to \infty$) if $P[\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)] = 1$ ($P \in \mathcal{P}_0(\Omega)$).

The measure $Q$ is said to be *absolutely continuous* with respect to the measure $P$ (notation: $Q \prec P$), if for every sequence $\{A_n, n \geq 1\}$ of random events $A_n \in \mathcal{A}$ the following condition is fulfilled:

$$\lim_{n\to\infty} P(A_n) = 0 \Rightarrow \lim_{n\to\infty} Q(A_n) = 0\,.$$

The measures $P$ and $Q$ are equivalent ($P \equiv Q$) if $P \prec Q$ and $Q \prec P$.

The following results (cf. [5], [12]) will be useful in further considerations. For $X, X_n \in \mathfrak{X}_0$, $n \in \mathrm{N}$, $P, Q \in \mathcal{P}_0(\Omega)$:

$$X_n \xrightarrow{P} X \iff \forall_{Q\equiv P}\, Q_{X_n} \xrightarrow{D} Q_X,\ n \to \infty. \tag{1}$$

$$X_n \xrightarrow{a.s.} X \iff \forall_{Q\equiv P}\, X_n \xrightarrow{ED} X,\ n \to \infty. \tag{2}$$

**2. Vague essential convergence, vague convergence in probability and vague almost sure convergence.** We introduce the following concept of the vague essential convergence.

**Definition 1.** We say that a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ is *vaguely essentially convergent* to a random element $X \in \mathfrak{X}$ ($X_n \xrightarrow{VED} X$, $n \to \infty$) if for every bounded set $A \in \mathcal{C}_X$

$$P\{\limsup_{n\to\infty}[X_n \in A]\} = P\{\liminf_{n\to\infty}[X_n \in A]\} = P[X \in A],$$

where $P \in \mathcal{P}(\Omega)$.

**Theorem 1.** *Let $X, X_n \in \mathfrak{X}$, $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) $X_n \xrightarrow{VED} X$, $n \to \infty$,

(ii$_1$) $P\{\limsup_{n\to\infty}[X_n \in F]\} \leq P[X \in F]$ *for every bounded closed set* $F$,

*and*

(ii$_2$) $P\{\liminf_{n\to\infty}[X_n \in G]\} \geq P[X \in G]$ *for every bounded open set* $G$.

**Proof.** $(i) \Rightarrow (ii_1)$. Let $F$ be any given bounded and closed set contained in $S$. There exists a sequence of sets $F^{\delta_n} = \{x \in S : \varrho(x, F) < \delta_n\}$ with $\delta_n \to 0$, $n \to \infty$, such that $F^{\delta_n} \in \mathcal{C}_X$, $n \in \mathbb{N}$, and $F = \bigcap_{n=1}^{\infty} F^{\delta_n}$. Of course, the sets $F^{\delta_n}$ are bounded. Let $\varepsilon$ be an arbitrary positive number. There exists $n_0$ such that for $n \geq n_0$ we have

$$P[X \in F^{\delta_n}] < P[X \in F] + \varepsilon.$$

Hence we get

$$P\{\limsup_{n\to\infty}[X_n \in F]\} \leq P\{\limsup_{n\to\infty}[X_n \in F^{\delta_k}]\}$$
$$= P[X \in F^{\delta_k}] < P[X \in F] + \varepsilon,$$

for every $k \geq n_0$. Since $\varepsilon$ is arbitrary, we see that

$$P\{\limsup_{n\to\infty}[X_n \in F\} \leq P[X \in F].$$

$(i) \Rightarrow (ii_2)$. Assume that $G$ is any given open and bounded set contained in $S$. There exists a sequence of open sets $G_n$, $n \geq 1$, such that $G_n \subset G$, $G_n \in \mathcal{C}_X$ and $G = \bigcup_{n=1}^{\infty} G_n$. Let $H_n = \bigcup_{i=1}^{n} G_i$. Hence $H_n \in \mathcal{C}_X$, $G = \lim_{n\to\infty} H_n$, $H_n \subset H_{n+1}$ and $H_n$ is bounded for every $n$. Also, for any given $\varepsilon > 0$, there exists $n_0$ such that for $n \geq n_0$ the following inequality is true

$$P[X \in H_n] > P[X \in G] - \varepsilon.$$

Hence we have

$$P\{\liminf_{n\to\infty}[X_n \in G]\} \geq P\{\liminf_{n\to\infty}[X_n \in H_k]\} = P[X \in H_k] > P[X \in G] - \varepsilon$$

for any $\varepsilon > 0$ and $k \geq n_0$. Thus

$$P\{\liminf_{n\to\infty}[X_n \in G]\} \geq P[X \in G].$$

(ii$_1$) and (ii$_2$)$\Rightarrow$(i). Let $A$ be any given bounded set which is a continuity set of the measure $P_X$ ($A \in \mathcal{C}_X$). For any given $\varepsilon > 0$ there exist a closed set $F$ and an open set $G$ such that $G \subset A \subset F$ and

$$P[X \in F \setminus A] \leq \varepsilon \text{ and } P[X \in A \setminus G] \leq \varepsilon.$$

Hence, for any $\varepsilon > 0$,

$$\begin{aligned} P[X \in A] - \varepsilon \leq P[X \in G] &\leq P\{\liminf_{n\to\infty}[X_n \in G]\} \leq P\{\liminf_{n\to\infty}[X_n \in A]\} \\ &\leq P\{\limsup_{n\to\infty}[X_n \in A]\} \leq P\{\limsup_{n\to\infty}[X_n \in F]\} \leq P[X \in F] \\ &\leq P[X \in A] + \varepsilon. \end{aligned}$$

Thus

$$P\{\liminf_{n\to\infty}[X_n \in A]\} = P\{\limsup_{n\to\infty}[X_n \in A]\} = P[X \in A],$$

which completes the proof.

**Theorem 2.** *Let $X, X_n \in \mathfrak{X}$, $n \in \mathrm{N}$. If $X_n \xrightarrow{VED} X$, $n \to \infty$, then $X_n \xrightarrow{V} X$, $n \to \infty$.*

**Proof.** For every bounded set $A \in \mathcal{C}_X$ we have

$$\begin{aligned} P\{\liminf_{n\to\infty}[X_n \in A]\} &= P\left\{\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}[X_k \in A]\right\} = \lim_{n\to\infty} P\left\{\bigcap_{k\geq n}[X_k \in A]\right\} \\ &\leq \liminf_{n\to\infty} P[X_n \in A] \leq \limsup_{n\to\infty} P[X_n \in A] \\ &\leq \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A]\right\} = P\left\{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}[X_k \in A]\right\} \\ &= P\{\limsup_{n\to\infty}[X_n \in A]\} \end{aligned}$$

and

$$P\{\liminf_{n\to\infty}[X_n \in A]\} = P\{\limsup_{n\to\infty}[X_n \in A]\} = P[X \in A].$$

Hence

$$\liminf_{n\to\infty} P[X_n \in A] = \limsup_{n\to\infty} P[X_n \in A] = P[X \in A],$$

and so, for every bounded set $A \in \mathcal{C}_X$: $\lim_{n\to\infty} P[X_n \in A] = P[X \in A]$.

**Example 2.** *Vague convergence does not imply VED convergence.* Assume that $\{X_n, n \geq 1\}$ is a sequence of independent, identically distributed, non-degenerate random variables. Since all $X_n$ are identically distributed, the sequence $\{X_n, n \geq 1\}$ vaguely converges to a random variable $X$ which is identically distributed as $X_1$. Let now $A \in \mathcal{C}_X$ be a bounded set such that $0 < P[X_1 \in A] = a < 1$. Then

$$P\{\liminf_{n\to\infty}[X_n \in A]\} = \lim_{n\to\infty} P\left\{\bigcap_{k\geq n}[X_k \in A]\right\} = 0 \neq a = P[X \in A].$$

Thus $X_n \overset{VED}{\nrightarrow} X$, $n \to \infty$.

By the definitions of convergence ED and VED we get the following

**Corollary 1.** *Let* $X, X_n \in \mathfrak{X}$, $n \in \mathbb{N}$. *If* $X_n \xrightarrow{ED} X$, $n \to \infty$, *then* $X_n \xrightarrow{VED} X$, $n \to \infty$.

Starting with the equivalence formulas (1) and (2) we are able to introduce the concept of vague convergence in probability and vague almost sure convergence.

**Definition 2.** We say that a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ *vaguely converges in probability* to a random element $X \in \mathfrak{X}$ ($X_n \xrightarrow{VP} X$, $n \to \infty$) if it vaguely converges to $X$ with respect to every measure $Q \equiv P$, i.e.

$$X_n \xrightarrow{VP} X \iff \forall_{Q\equiv P}\, X_n \xrightarrow{V} X, \ n \to \infty.$$

**Definition 3.** We say that a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ *vaguely almost surely converges* to a random element $X \in \mathfrak{X}$ ($X_n \xrightarrow{Va.s.} X$, $n \to \infty$) if it vaguely essentially converges to $X$ with respect to every measure $Q \equiv P$, i.e.

$$X_n \xrightarrow{Va.s.} X \iff \forall_{Q\equiv P}\, X_n \xrightarrow{VED} X, \ n \to \infty.$$

By the Definitions 2, 3, Theorem 2 and (1), (2) we get the following statements.

**Corollary 2.** *Let $X$, $X_n \in \mathfrak{X}$, $n \in \mathbb{N}$. Then the following implications hold:*

(i) $X_n \xrightarrow{VP} X \Rightarrow X_n \xrightarrow{V} X$, $n \to \infty$,

(ii) $X_n \xrightarrow{Va.s.} X \Rightarrow X_n \xrightarrow{VED} X$, $n \to \infty$,

(iii) $X_n \xrightarrow{Va.s.} X \Rightarrow X_n \xrightarrow{VP} X$, $n \to \infty$,

(iv) $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{VP} X$, $n \to \infty$,

(v) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{Va.s.} X$, $n \to \infty$.

We shall see that without additional assumptions, none of the above mentioned implications is revertible.

**Example 3.** *Vague convergence does not imply VP convergence.*
Let $\Omega = [0,1]$ and let $P$ be a measure on $(\Omega, \mathcal{A})$ such that $P(\{0\}) = P(\{1\}) = 1/2$. Define the random variables $X$, $X_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, as follows:

$$X_n(\omega) = \begin{cases} 0, & \omega \in [0,1/2], \\ 1, & \omega \in (1/2,1], \end{cases} \qquad X(\omega) = \begin{cases} 1, & \omega \in [0,1/2], \\ 0, & \omega \in (1/2,1]. \end{cases}$$

Let $A$ be a continuity set of measure $P_X$. There are three possibilities:
(i) $0 \notin A$ and $1 \notin A$. Then $P[X_n \in A] = P[X \in A] = 0$, $n \in \mathbb{N}$.
(ii) $0 \in A$ and $1 \in A$. Then $P[X_n \in A] = P[X \in A] = 1$, $n \in \mathbb{N}$.
(iii) Exactly one of the numbers 0, 1 belongs to $A$. Then $P[X_n \in A] = P[X \in A] = \frac{1}{2}$, $n \in \mathbb{N}$.

In all cases we have $\lim_{n\to\infty} P[X_n \in A] = P[X \in A]$. Thus $X_n \xrightarrow{V} X$, $n \to \infty$, with respect to the measure $P$.

Now let $Q$ be a measure on $(\Omega, \mathcal{A})$ such that $Q(\{0\}) = \frac{1}{4}$, $Q(\{1\}) = \frac{3}{4}$ and let $B = (\frac{1}{2}, \frac{3}{2})$. Of course, $Q \equiv P$ and $B$ is the continuity set of measure $Q_X$. Moreover, $Q[X_n \in B] = \frac{3}{4}$, $n \in \mathbb{N}$, and $Q[X \in B] = \frac{1}{4}$. This implies $X_n \overset{V}{\nrightarrow} X$, $n \to \infty$, with respect to the measure $Q$. Thus, by Definition 2, $X_n \overset{VP}{\nrightarrow} X$, $n \to \infty$.

**Example 4.** *VED convergence does not imply Va.s. convergence.*
Let $(\Omega, \mathcal{A}, P)$ be such that $\Omega = [0,a]$, $a > 0$, and $P$ be defined as follows: $P(\{ra/4\}) = 1/8$, $r = 0,1,2,3,4$, and let $X, X_n : \Omega \to [1/a, \infty] = S$, $n = 1,2,\ldots$, be such that

$$X_n(\omega) = \begin{cases} 1/\omega, & \omega \neq 0 \\ \infty, & \omega = 0, \end{cases} \qquad X(\omega) = \begin{cases} \frac{1}{\omega + a/4}, & 0 < \omega \leq 3a/4, \\ \frac{1}{\omega - 3a/4}, & 3a/4 < \omega \leq a, \\ \infty, & \omega = 0. \end{cases}$$

Then $[X_n \in A] = [X_1 \in A]$, $n = 1, 2, \ldots$, $A \in \mathcal{B}$. Moreover, we have $P[X_n \in A] = P[X \in A]$, $n = 1, 2, \ldots$, $A \in \mathcal{B}$. Therefore, by the Definition 1, we conclude that $X_n \xrightarrow{VED} X$, $n \to \infty$.

Now define the measure $Q$ on $(\Omega, \mathcal{A})$ as follows: $Q(\{0\}) = Q(\{a/2\}) = Q(\{a\}) = 1/8$, $Q(\{a/4\}) = Q(\{3a/4\}) = 1/4$. We see that $P$ and $Q$ are concentrated on the set $K = \{0, a/4, a/2, 3a/4, a\}$. Obviously that $Q \equiv P$. Write now $B = (1/a, 2/a)$. Then we have

$$Q[X_n \in B] = Q[1/\omega \in B] = Q(\{3a/4\}) = 1/4, \; n = 1, 2, \ldots,$$

$$Q[X \in B] = Q\left[\frac{1}{\omega + a/4} \in B\right] = Q(\{a/2\}) = 1/8,$$

and so, $X_n \overset{VED}{\not\to} X$, $n \to \infty$, with respect to the measure $Q$. Hence $X_n \overset{Va.s.}{\not\to} X$, $n \to \infty$.

**Example 5.** *VP convergence does not imply Va.s. convergence.*
Let $\Omega = [0, a]$, and let $P$ be the Lebesgue measure on $(\Omega, \mathcal{A})$. Define the following family of random elements:

$$X_{2^k+r}(\omega) = \begin{cases} 2, & \omega \in \left[\frac{r}{2^k}a, \frac{r+1}{2^k}a\right], \\ 1, & \omega \in [0, a] \setminus \left[\frac{r}{2^k}a, \frac{r+1}{2^k}a\right], \end{cases} \tag{3}$$

$k = 0, 1, \ldots$, $r = 0, 1, \ldots, 2^k - 1$. Since for every $n \in \mathbb{N}$ there is exactly one pair of numbers $k, r \in \mathbb{N}$ such that $n = 2^k + r$, $0 \leq r < 2^k$, the sequence $\{X_n, n \geq 1\}$ of random elements $X : \Omega \to \mathbb{R}$ is defined correctly by (3). Moreover, $n \to \infty \iff k \to \infty$. Define also the random element $X : \Omega \to \mathbb{R}$ by $X(\omega) \equiv 1$. Let $A$ be any bounded continuity set of measure $P_X$.

(i) $1 \in A$. Then $P[X_n \notin A] = P[X_{2^k+r} \notin A] \leq P\left[\frac{r}{2^k}a, \frac{r+1}{2^k}a\right] = \frac{a}{2^k}$, where $n = 2^k + r$, $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} P[X_n \notin A] = 0, \tag{4}$$

and thus

$$\lim_{n\to\infty} P[X_n \in A] = P(\Omega) = P[X \in A]. \tag{5}$$

(ii) $1 \notin A$. We have $P[X_n \in A] = P[X_{2^k+r} \in A] \le P\left[\frac{r}{2^k}a, \frac{r+1}{2^k}a\right] = \frac{a}{2^k}$, $n = 2^k + r$, $n \in \mathbb{N}$. Then

$$(6) \qquad \lim_{n\to\infty} P[X_n \in A] = 0 = P[X \in A].$$

It follows from (5) and (6) that $X_n \xrightarrow{V} X$, $n \to \infty$, with respect to the measure $P$. Now, let $Q$ be a measure on $(\Omega, \mathcal{A})$ such that $Q \equiv P$ and let $A$ be a bounded continuity set of measure $Q_X$. Then it follows from (4) for $1 \in A$ that $\lim_{n\to\infty} Q[X_n \notin A] = 0$. Hence

$$(7) \qquad \lim_{n\to\infty} Q[X_n \in A] = Q(\Omega) = Q[X \in A].$$

However, if $1 \notin A$, then it follows from (6) that

$$(8) \qquad \lim_{n\to\infty} Q[X_n \in A] = 0 = Q[X \in A].$$

Therefore, by (7) and (8), we have $X_n \xrightarrow{V} X$, $n \to \infty$, with respect to the measure $Q$. Thus $X_n \xrightarrow{VP} X$, $n \to \infty$.

Let $A = \left[\frac{1}{2}, \frac{3}{4}\right]$. Of course, $A \in \mathcal{C}_{P_X}$. Moreover, for any $n \in \mathbb{N}$ we have

$$\bigcap_{k\ge n} [X_k \in A] = \emptyset \quad \text{and} \quad \bigcup_{k\ge n} [X_k \in A] = \Omega.$$

Hence

$$P\{\liminf_{n\to\infty} [X_n \in A]\} = \lim_{n\to\infty} P\left\{\bigcap_{k\ge n} [X_k \in A]\right\} = 0$$

and

$$P\{\limsup_{n\to\infty} [X_n \in A]\} = \lim_{n\to\infty} P\left\{\bigcup_{k\ge n} [X_k \in A]\right\} = P(\Omega).$$

It follows from the last equations and the definition of VED convergence that $X_n \overset{VED}{\nrightarrow} X$, $n \to \infty$, with respect to the measure $P$, and hence $X_n \overset{V a.s.}{\nrightarrow} X$, $n \to \infty$.

We will denote by $B \div C$ the symmetric difference of the sets $B$ and $C$.

**Lemma 1.** *Let* $X, X_n \in \mathfrak{X}$ $n \in \mathbb{N}$. *If* $X_n \xrightarrow{VP} X$, $n \to \infty$, *then* $P([X_n \in A] \div [X \in A]) \to 0$, $n \to \infty$, *for every bounded set* $A \in \mathcal{C}_X$.

**Proof.** Let $A$ be any given bounded set such that $P[X \in \partial A] = 0$. If $X_n \xrightarrow{VP} X$, $n \to \infty$, then $X_n \xrightarrow{V} X$, $n \to \infty$, $\forall_{Q \equiv P}$. Assume that $P[X \in A] > 0$. Define the measure $Q$ as follows:

$$Q(B) = \big(P(B|[X \in A]) + P(B)\big)/2.$$

Of course, $Q \equiv P$, and so, by the assumption, $Q[X_n \in A] \to Q[X \in A]$, $n \to \infty$. Therefore

$$\Big(P([X_n \in A]|[X \in A]) + P[X_n \in A]\Big) \xrightarrow[n\to\infty]{} \Big(P([X \in A]|[X \in A]) + P[X \in A]\Big).$$

By our assumption,

$$(9) \qquad P[X_n \in A] \to P[X \in A], \ n \to \infty,$$

and hence

$$(10) \qquad P([X_n \in A] \cap [X \in A]) \to P([X \in A]), \ n \to \infty.$$

From the equality

$$[X_n \in A] \div [X \in A] = \Big([X_n \in A] \setminus ([X_n \in A] \cap [X \in A])\Big) \cup \Big([X \in A] \setminus ([X_n \in A] \cap [X \in A])\Big),$$

$n = 1, 2, \ldots$, using (9) and (10) we get

$$P([X_n \in A] \div [X \in A]) \to 0, \ n \to \infty.$$

If $P[X \in A] = 0$, then

$$P([X_n \in A] \div [X \in A]) = P[X_n \in A] \to P[X \in A] = 0,$$

which completes the proof.

**Lemma 2.** *If a sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely almost surely converges to a random element $X \in \mathfrak{X}$, then for every bounded set $A \in \mathcal{C}_X$*

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}([X_k \in A] \div [X \in A])\Big\} = 0.$$

**Proof.** If $X_n \xrightarrow{Va.s.} X$, $n \to \infty$, then $X_n \xrightarrow{VED} X$, $n \to \infty$, $\forall_{Q\equiv P}$. Let $A$ be any given bounded $P_X$-continuity set. For $A$ such that $P_X(A) > 0$ we define the measure $Q$ as follows:

$$Q(B) = \big(P(B|[X \in A]) + P(B)\big)/2.$$

Obviously, $Q \equiv P$. Thus $X_n \xrightarrow{VED} X$, $n \to \infty$, for the measure $Q$. Hence it follows that

$$\lim_{n\to\infty} Q\Big\{\bigcup_{k\geq n}[X_k \in A]\Big\} = Q[X \in A]$$

and

$$\lim_{n\to\infty} Q\Big\{\bigcap_{k\geq n}[X_k \in A]\Big\} = Q[X \in A].$$

Thus, by the definition of the measure $Q$ and from

$$\text{(11)}\qquad \begin{aligned} &\lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \in A]\Big\} = P[X \in A] \quad \text{and} \\ &\lim_{n\to\infty} P\Big\{\bigcap_{k\geq n}[X_k \in A]\Big\} = P[X \in A], \end{aligned}$$

we get

$$\begin{aligned} \lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \in A] \cap [X \in A]\Big\} &= P[X \in A] \\ &= \lim_{n\to\infty} P\Big\{\bigcap_{k\geq n}[X_k \in A] \cap [X \in A]\Big\}. \end{aligned}$$

Therefore from (11) we have

$$\text{(12)}\qquad \lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \in A] \cap [X \notin A]\Big\} = 0,$$

$$\text{(13)}\qquad \lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \notin A] \cap [X \in A]\Big\} = 0.$$

Moreover,

$$\bigcup_{k\geq n}([X_k \in A] \div [X \in A])$$
$$= \bigcup_{k\geq n}\Big(([X_k \in A] \cap [X \notin A]) \cup ([X_k \notin A] \cap [X \in A])\Big)$$
$$= \bigcup_{k\geq n}([X_k \in A] \cap [X \notin A]) \cup \bigcup_{k\geq n}([X_k \notin A] \cap [X \in A]).$$

Hence by (12) and (13)

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \in A] \div [X \in A]\Big\} = 0$$

for every bounded set $A \in \mathfrak{C}_X$ such that $P_X(A) \neq 0$.

Now assume that $P_X(A) = 0$. Since $X_n \xrightarrow{Va.s.} X, n \to \infty, X_n \xrightarrow{V} X$, $n \to \infty$, by Corollary 2 . Consequently,

$$\lim_{n\to\infty} P[X_n \in A] = P[X \in A].$$

Hence by the equality $P_X(A) = 0$ we get

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\geq n}[X_k \in A] \div [X \in A]\Big\} = 0,$$

which completes the proof.

**Lemma 3.** *If* $X, X_n \in \mathfrak{X}, n \in \mathrm{N}$ *and*

$$\lim_{n\to\infty} P\{[X_n \in A] \div [X \in A]\} = 0$$

*for every bounded set* $A \in \mathfrak{C}_X$*, then* $X_n \xrightarrow{VP} X, n \to \infty$.

**Proof.** Assume that

$$\lim_{n\to\infty} P\{[X_n \in A] \div [X \in A]\} = 0$$

for every bounded set $A \in \mathfrak{C}_X$. Then, we have

$$\lim_{n\to\infty} P\{[X_n \in A] \cap [X \notin A]\} = 0$$

and

$$\lim_{n\to\infty} P\{[X \in A] \cap [X_n \notin A]\} = 0.$$

Moreover,

$$\lim_{n\to\infty} \Big(P[X_n \in A] - P[X \in A]\Big) \le \lim_{n\to\infty} P\Big\{[X_n \in A] \cap [X \notin A]\Big\} = 0$$

and

$$\lim_{n\to\infty} \Big(P[X \in A] - P[X_n \in A]\Big) \le \lim_{n\to\infty} P\Big\{[X \in A] \cap [X_n \notin A]\Big\} = 0.$$

Thus we get

$$P[X \in A] \le \lim_{n\to\infty} P[X_n \in A] \le P[X \in A],$$

for every bounded set $A \in \mathcal{C}_X$, which proves that $X_n \xrightarrow{V} X, n \to \infty$, with respect to the measure $P$.

Now let $Q$ be any measure such that $Q \equiv P$. It follows from our assumption that

$$\lim_{n\to\infty} Q\{[X_n \in A] \div [X \in A]\} = 0.$$

By a reasoning as above we get $X_n \xrightarrow{V} X$, $n \to \infty$, with respect to the measure $Q$. Hence by the Definition 2 we have $X_n \xrightarrow{VP} X$, $n \to \infty$, which completes the proof.

**Lemma 4.** *If* $X, X_n \in \mathfrak{X}, n \in \mathrm{N}$ *and*

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\ge n}([X_k \in A] \div [X \in A])\Big\} = 0$$

*for every bounded set* $A \in \mathcal{C}_X$, *then* $X_n \xrightarrow{Va.s.} X, n \to \infty$.

**Proof.** Let

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\ge n}([X_k \in A] \div [X \in A])\Big\} = 0$$

for every bounded set $A \in \mathcal{C}_X$. Then

$$\lim_{n\to\infty} P\Big\{\bigcup_{k\ge n}([X_k \in A] \cap [X \notin A])\Big\} = 0$$

and

$$\lim_{n\to\infty} P\left\{[X \in A] \cap \bigcup_{k\geq n} [X_k \notin A]\right\} = 0.$$

Moreover,

$$\lim_{n\to\infty} \left(P\left\{\bigcup_{k\geq n} [X_k \in A]\right\} - P[X \in A]\right)$$
$$\leq \lim_{n\to\infty} P\left\{\bigcup_{k\geq n} ([X_k \in A] \cap [X \notin A])\right\} = 0$$

and

$$\lim_{n\to\infty} \left(P[X \in A] - P\left\{\bigcap_{k\geq n} [X_k \in A]\right\}\right)$$
$$\leq \lim_{n\to\infty} P\left\{[X \in A] \cap \bigcup_{k\geq n} [X_k \notin A]\right\} = 0.$$

Hence we get

$$P[X \in A] \leq \lim_{n\to\infty} P\left\{\bigcap_{k\geq n} [X_k \in A]\right\} \leq \lim_{n\to\infty} P\left\{\bigcup_{k\geq n} [X_k \in A]\right\} \leq P[X \in A],$$

for every bounded set $A \in \mathcal{C}_X$, proving $X_n \xrightarrow{VED} X$, $n \to \infty$, with respect to the measure $P$.

From our assumption we get

$$\lim_{n\to\infty} Q\left\{\bigcup_{k\geq n} ([X_k \in A] \div [X \in A])\right\} = 0$$

for every measure $Q \equiv P$ and every bounded set $A \in \mathcal{C}_X$. Consequently, $X_n \xrightarrow{VED} X$, $n \to \infty$, with respect to the measure $Q \equiv P$. Thus, by the Definition 3, we get $X_n \xrightarrow{V a.s.} X$, $n \to \infty$, which completes the proof.

**Theorem 3.** *A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely converges in probability to a random element $X \in \mathfrak{X}$ if and only if for every bounded set $A \in \mathcal{C}_X$*

$$\lim_{n\to\infty} P\{[X_n \in A] \div [X \in A]\} = 0.$$

**Proof.** This is an immediate consequence of Lemmas 1 and 3.

**Theorem 4.** *A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely almost surely converges to a random element $X \in \mathfrak{X}$ if and only if for every bounded set $A \in \mathcal{C}_X$*

$$\lim_{n\to\infty} P\left\{ \bigcup_{k\geq n} [X_k \in A] \div [X \in A] \right\} = 0.$$

**Proof.** This is an immediate consequence of Lemmas 2 and 4.

**Theorem 5.** *A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely almost surely converges to a random element $X \in \mathfrak{X}$ if and only if it is vaguely convergent in probability and vaguely essentially convergent to $X$.*

**Proof.** If $X_n \xrightarrow{Va.s.} X$, $n \to \infty$, then, ewidently, $X_n \xrightarrow{VED} X$, $n \to \infty$, by Corollary 2 (ii). Moreover, $X_n \xrightarrow{VP} X$, $n \to \infty$, by Corollary 2 (iii).

Now, assume that $\{X_n, n \geq 1\}$ vaguely converges in probability and is vaguely essentially convergent to $X$. It is sufficient to prove that $X_n \xrightarrow{VED} X$, $n \to \infty$, with respect to every measure $Q \equiv P$. Since $X_n \xrightarrow{VED} X$, $n \to \infty$, with respect to the measure $P$, we get

$$P\left\{ \lim_{n\to\infty} \bigcup_{k\geq n} [X_k \in A] \right\} = P\left\{ \lim_{n\to\infty} \bigcap_{k\geq n} [X_k \in A] \right\},$$

for every bounded set $A \in \mathcal{C}_X$. Hence we have

$$\lim_{n\to\infty} P\left\{ \left( \bigcup_{k\geq n} [X_k \in A] \right) \setminus \left( \bigcap_{k\geq n} [X_k \in A] \right) \right\} = 0,$$

and so, for every measure $Q \equiv P$,

$$\lim_{n\to\infty} Q\left\{ \left( \bigcup_{k\geq n} [X_k \in A] \right) \setminus \left( \bigcap_{k\geq n} [X_k \in A] \right) \right\} = 0,$$

or

$$\lim_{n\to\infty} Q\left\{ \bigcup_{k\geq n} [X_k \in A] \right\} = \lim_{n\to\infty} Q\left\{ \bigcap_{k\geq n} [X_k \in A] \right\}.$$

Hence by the inequalities

$$Q\left\{ \bigcap_{k\geq n} [X_k \in A] \right\} \leq Q\{[X_n \in A]\} \leq Q\left\{ \bigcup_{k\geq n} [X_k \in A] \right\}$$

we get

$$\lim_{n\to\infty} Q\Big\{\bigcap_{k\geq n}[X_k \in A]\Big\} = \lim_{n\to\infty} Q\Big\{\bigcup_{k\geq n}[X_k \in A]\Big\} \tag{14}$$
$$= \lim_{n\to\infty} Q[X_n \in A].$$

The assumption $X_n \xrightarrow{VP} X$, $n \to \infty$, implies $X_n \xrightarrow{V} X$, $n \to \infty$, with respect to every measure $Q \equiv P$. Thus, for the measure $Q \equiv P$, $\lim_{n\to\infty} Q[X_n \in A] = Q[X \in A]$ for every bounded set $A \in \mathcal{C}_X$. Hence by (14) we get

$$\lim_{n\to\infty} Q\Big\{\bigcap_{k\geq n}[X_k \in A]\Big\} = \lim_{n\to\infty} Q\Big\{\bigcup_{k\geq n}[X_k \in A]\Big\} = Q[X \in A],$$

for every measure $Q \equiv P$ and every bounded set $A \in \mathcal{C}_X$. Therefore $X_n \xrightarrow{VED} X$, $n \to \infty$, with respect to every measure $Q \equiv P$, which completes the proof.

**Example 6.** *VED convergence does not imply VP convergence.* Let $\Omega = [0,1]$ and let $P$ be the Lebesgue measure on $[0,1]$. Moreover, let $S = [0,1]$. We define the random variables $X$, $X_n$, $n = 1, 2, \ldots$, as follows:

$$X_n(\omega) = \omega, \quad X(\omega) = \begin{cases} 1, & \omega = 0, \\ \omega + 1/2, & 0 < \omega < 1/2, \\ \omega - 1/2 = \omega + 1/2 - 1, & 1/2 \leq \omega \leq 1. \end{cases}$$

$X$, $X_n$, $n \in \mathbf{N}$ are uniformly distributed on $[0,1]$.
$P\{\liminf_{n\to\infty}[X_n \in A]\} = P\{\limsup_{n\to\infty}[X_n \in A]\} = P[X \in A]$ for every $A \in \mathcal{B}$. Therefore, $X_n \xrightarrow{VED} X$, $n \to \infty$. Now, let $Q$ be the measure on $[0,1]$ with density $f(x) = 2x$. Moreover, let $A = [0, 1/2]$. Of course, $Q \equiv P$.
$\lim_{n\to\infty} Q([X_n \in A]) = Q([X_1 \in A]) = Q([0,1/2]) = \int_0^{1/2} 2x dx = 1/4$. On the other hand, $Q([X \in A]) = Q([1/2, 1]) = \int_{1/2}^1 2x dx = 3/4$ and so $X_n \not\xrightarrow{V} X$, $n \to \infty$, with respect to the measure $Q \equiv P$. Thus $X_n \not\xrightarrow{VP} X$, $n \to \infty$.

**Theorem 6.** *A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely converges in probability to a constant $c$ if and only if it converges vaguely to $c$.*

**Proof.** The necessity of the condition follows immediately from the Corollary 2 (*i*), and so we must only prove its sufficiency. Assume therefore that

$X_n \xrightarrow{V} c$, $n \to \infty$, and let $A$ be any bounded $P_c$-continuity set. It follows from the conditions equivalent to the vague convergence, that

$$\lim_{n\to\infty} P[X_n \in A] = P[c \in A]. \tag{15}$$

Moreover, we have

$$[X_n \in A] \div [c \in A] = ([X_n \in A] \cap [c \notin A]) \cup ([X_n \notin A] \cap [c \in A]), \tag{16}$$

$n = 1, 2, \ldots$.

Suppose that $c \notin A$. Then

$$\lim_{n\to\infty} P\{[X_n \notin A] \cap [c \in A]\} = \lim_{n\to\infty} P\{[X_n \notin A] \cap \emptyset\} = 0,$$

and

$$\lim_{n\to\infty} P\{[X_n \in A] \cap [c \notin A]\} = \lim_{n\to\infty} P\{[X_n \in A] \cap \Omega\}$$
$$= \lim_{n\to\infty} P[X_n \in A] = P[c \in A] = 0$$

by (15). Hence by (16) we get

$$\lim_{n\to\infty} P\{[X_n \in A] \div [c \in A]\} = 0.$$

Now, let $c \in A$. Then

$$\lim_{n\to\infty} P\{[X_n \in A] \cap [c \notin A]\} = \lim_{n\to\infty} P\{[X_n \in A] \cap \emptyset\} = 0,$$

and, by (15),

$$\lim_{n\to\infty} P\{[X_n \notin A] \cap [c \in A]\} = \lim_{n\to\infty} P[X_n \notin A] = \lim_{n\to\infty} P\{\Omega \setminus [X_n \in A]\}$$
$$= P(\Omega) - \lim_{n\to\infty} P[X_n \in A] = P(\Omega) - P[c \in A] = P(\Omega) - P(\Omega) = 0.$$

Thus, by (16) we have

$$\lim_{n\to\infty} P\{[X_n \in A] \div [c \in A]\} = 0,$$

using Theorem 3 we are done.

**Theorem 7.** *A sequence $\{X_n, n \geq 1\}$ of random elements $X_n \in \mathfrak{X}$ vaguely almost surely converges to a constant $c$ if and only if it vaguely essentialy converges to $c$.*

**Proof.** If $X_n \xrightarrow{Va.s.} c$, then $X_n \xrightarrow{VED} c$ by Corollary 2 (*ii*).

Assume therefore that $X_n \xrightarrow{VED} c$, $n \to \infty$. Let $A$ be any bounded $P_c$-continuity set. By definition of VED convergence we have

$$(17) \quad \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A]\right\} = \lim_{n\to\infty} P\left\{\bigcap_{k\geq n}[X_k \in A]\right\} = P[c \in A].$$

Moreover, the following equation holds:

$$(18) \quad \begin{aligned}\bigcup_{k\geq n}([X_k \in A] \div [c \in A]) = &\left(\bigcup_{k\geq n}[X_k \in A] \cap [c \notin A]\right) \\ &\cup \left(\bigcup_{k\geq n}[X_k \notin A] \cap [c \in A]\right).\end{aligned}$$

We consider two cases:

(a) $c \notin A$. We have

$$\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \notin A] \cap [c \in A]\right\} = \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \notin A] \cap \emptyset\right\} = 0$$

and, by (17),

$$\begin{aligned}\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A] \cap [c \notin A]\right\} &= \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A] \cap \Omega\right\} \\ &= \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A]\right\} = P[c \in A] = 0.\end{aligned}$$

Thus, by (18),

$$\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}([X_k \in A] \div [c \in A])\right\} = 0.$$

(b) $c \in A$. Then

$$\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A] \cap [c \notin A]\right\} = \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \in A] \cap \emptyset\right\} = 0$$

and

$$\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \notin A]\cap[c\in A]\right\} = \lim_{n\to\infty} P\left\{\bigcup_{k\geq n}[X_k \notin A]\right\}$$

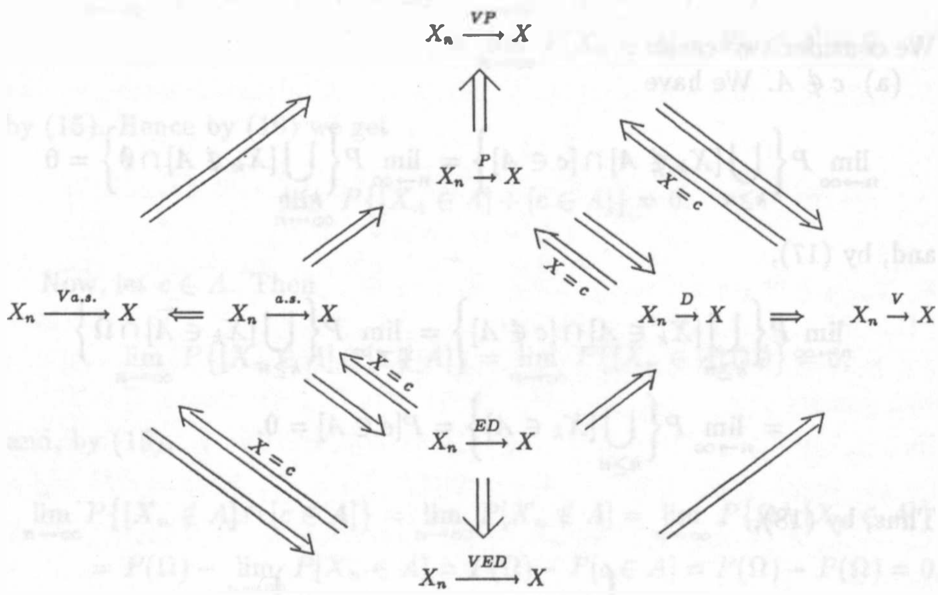
$$= \lim_{n\to\infty} P\left\{\Omega\setminus\bigcap_{k\geq n}[X_k\in A]\right\} = P(\Omega) - \lim_{n\to\infty} P\left\{\bigcap_{k\geq n}[X_k\in A]\right\}$$

$$= P(\Omega) - P[c\in A] = P(\Omega) - P(\Omega) = 0$$

by (17). Hence and from (18) it follows that

$$\lim_{n\to\infty} P\left\{\bigcup_{k\geq n}\left([X_k\in A]\div[c\in A]\right)\right\} = 0\,,$$

and using Theorem 4 we are done.




The diagram of the relations between various types of convergences. Without additional assumptions, none of the above implications is revertible.

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