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Integral Means of Univalent Harmonic Maps

ABSTRACT. The main results obtained in this paper are the following. If $f = h + \bar{g}$ is a univalent harmonic map, then $g, h \in H^p$ and $f \in h^p$ for $p \in (0, A^{-2})$, where A is given in (3). This is an improvement of a result presented in [AL]. Moreover, a further improvement of the range: $p \in (0, 1/3)$ is established for close-to-convex harmonic maps.

1. Introduction. Statement of results. Let Δ denote the open unit disc in the complex plane and S_H denote the class of all complex valued, harmonic, sense-preserving univalent functions f in Δ normalized by

$$f(0) = 0, \quad f_z(0) = 1. \tag{1}$$

Each $f \in S_H$ can be expressed as

$$f = h + \bar{g}, \tag{2}$$

where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in Δ. It is known ([BH]) that

$$3 \leq A = \sup_{f \in S_H} |a_2| < 50. \tag{3}$$

Let $H^p\,(h^p)$, $0 < p < \infty$, denote the standard Hardy space of analytic (harmonic) functions on Δ. It is well-known that, if f is analytic and univalent in Δ, then $f \in H^p$ for $0 < p < \frac{1}{2}$ (see e.g. [D1, p. 50]).

In 1990 Y. Abu-Muhanna and A. Lyzzaik [AL] proved the following

Theorem A. *If* $f = h + \bar{g} \in S_H$, *then* $h, g \in H^p$ *and* $f \in h^p$ *for every* p, $p \in (0, (2A+2)^{-2})$, *where* A *is given by* (3).

In [BH] the authors proposed to find the exact set of all $p > 0$ such that $f \in h^p$, if $f \in S_H$. Here we extend the above cited range for p, namely we prove

Theorem 1. *Under the assumptions of* Theorem A, $h, g \in H^p$ *and* $f \in h^p$ *for* $0 < p < A^{-2}$.

Let K_H, C_H denote the subclasses of S_H consisting of harmonic mappings onto convex and close-to-convex regions, respectively. It has been shown in [CS] that if $f = h + \bar{g} \in K_H$, then h is close-to-convex and $|g(z)| < |h(z)|$ for $z \in \Delta \setminus \{0\}$. These facts imply

Theorem 2. *If* $f = h + \bar{g} \in K_H$, *then* $g, h \in H^p$ *and* $f \in h^p$ *for* $0 < p < \frac{1}{2}$.

In section 4 we show that the convex harmonic function [CS]

$$(4) \qquad \begin{aligned} f(z) &= (z - \frac{1}{2}z^2)(1-z)^{-2} - \frac{1}{2}\overline{z^2(1-z)^{-2}} \\ &= \operatorname{Re}\left(\frac{z}{1-z}\right) + i \operatorname{Im}\left(\frac{z}{(1-z)^2}\right), \quad z \in \Delta, \end{aligned}$$

is in $h^{\frac{1}{2}}$ (although $g, h \notin H^{\frac{1}{2}}$) but it is not in h^p for $p > \frac{1}{2}$. Therefore the exact range of $p > 0$ such that, $f \in h^p$ if f is a convex harmonic function, can be at most the interval $(0, \frac{1}{2}]$.

For close-to-convex harmonic mappings we get

Theorem 3. *If* $f = h + \bar{g} \in C_H$, *then* $h, g \in H^p$ *and* $f \in h^p$ *for* $0 < p < \frac{1}{3}$.

Because

$$\sup_{f \in K_H} |a_2| = 2, \qquad \sup_{f \in C_H} |a_2| = 3,$$

it seems natural to conjecture that, if $f = \bar{g} + h \in S_H$, then $g, h \in H^p$ and $f \in h^p$ for $0 < p < 1/A$, where A is given by (3).

2. Proof of Theorem 1. The proof of Theorem 1 is based on the following, below stated results.

For $p \in \mathbb{R}$ and f harmonic on Δ let us set

$$M_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\,, \quad 0 \le r < 1\,. \tag{5}$$

Now, let $\mathcal{A}$ denote the class of analytic, locally univalent functions h on Δ, normalized by

$$h(0) = 0\,, \quad h'(0) = 1\,,$$

and satisfying the condition

$$\left|\frac{zh''(z)}{h'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{2A|z|}{1-|z|^2} \tag{6}$$

for some constant $A \ge 1$. As in [P3, p.176; P2] we define

$$\beta_h(p) = \limsup_{r\to 1} \frac{\log M_p(r, h')}{-\log(1-r)}\,. \tag{7}$$

Theorem B. *If $h \in \mathcal{A}$, then for $p \in \mathbb{R}$*

$$\beta_h(p) \le -\frac{1}{2} + p + \sqrt{\frac{1}{4} - p + A^2p^2}\,. \tag{8}$$

To prove this theorem it is enough to proceed analogously as in the proof of Theorem 1 of [P1] (see also [P3, pp.176-182])

The next result we will need is due to T.Flett [F1], [F2] (see also [MP] for its simple proof).

Theorem C. *Let $0 < p < 1$ and h be an analytic function on Δ. If*

$$\int_0^1 (1-r)^{p-1} M_p(r, h')dr < \infty\,, \tag{9}$$

then $h \in H^p$.

Proof of Theorem 1. Let $f = h + \overline{g} \in S_H$. For fixed $\zeta \in \Delta$ consider the function

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)}\,.$$

Then $F \in S_H$. This fact implies that the analytic and locally univalent function h satisfies (6) with A given by (3). Now it follows from Theorem B that for each $\epsilon > 0$ there exists $C(\epsilon)$ such that

$$M_p(r,g') < M_p(r,h') \leq \frac{C(\epsilon)}{(1-r)^{-\frac{1}{2}+p+\sqrt{\frac{1}{4}-p+A^2p^2+\epsilon}}}\,.$$

Hence for $0 < p < 1$ and for arbitrarily fixed $\epsilon > 0$

$$\int_0^1 (1-r)^{p-1} M_p(r,g')dr < \int_0^1 (1-r)^{p-1} M_p(r,h')dr$$
$$\leq \int_0^1 \frac{C(\epsilon)}{(1-r)^{\frac{1}{2}+\sqrt{\frac{1}{4}-p+A^2p^2+\epsilon}}}dr\,.$$

The last integral is finite if

$$\frac{1}{2} + \sqrt{\frac{1}{4} - p + A^2p^2} < 1 - \epsilon < 1\,,$$

and this inequality holds if

$$p < A^{-2}\,.$$

So, in view of Theorem C, h and $g \in H^p$ for $p < A^{-2}$.

3. Proof of Theorem 3. Let k be the function defined by the formula

$$k(z) = \frac{(1+z)^2}{(1-z)^4}\,, \quad z \in \Delta\,. \tag{10}$$

We start with the following

Lemma. *If $f = \bar{g} + h \in C_H$ is a close-to-convex harmonic map, then for $0 < p < \infty$*

$$M_p(r,h') \leq M_p(r,k)\,. \tag{11}$$

Proof. It was shown in [CS] that $zh'(z) = F(z)G(z)$ if $f = \bar{g} + h \in C_H$ and $G(z) = e^{-i\alpha}z + a_2z^2 + \ldots$, $-\pi < \alpha < \pi$, is a starlike function and $F(z) = e^{i\alpha} + b_1 z + \ldots$ satisfies $|\arg F(z)| < \pi$. Hence

$$\log|h'(z)| = \log|F(z)| + \log\left|\frac{G(z)}{z}\right|\,.$$

Now notice that to prove our lemma it is enough to apply the reasoning similar to that in the proof of Theorem 7.2 of [D2, p. 229].

Lemma 1 implies immediately

Corollary. *If* $f = h + \bar{g} \in C_H$, *then* $g', h' \in H^p$ *for* $0 < p < \frac{1}{4}$.

Proof of Theorem 3. Assume that $f = h + \bar{g} \in C_H$ and $\frac{1}{4} < p < 1$. Then Lemma 1 and the Lemma in [D1, p.65] imply that there is a positive constant C such that

$$M_p(r, h') \leq \frac{C}{(1-r)^{4p-1}}.$$

From this

$$\int_0^1 (1-r)^{p-1} M_p(r, h') dr \leq \int_0^1 (1-r)^{-3p} dr.$$

The last integral is finite if $p < \frac{1}{3}$ and the assertion follows from Theorem C.

Remark. Notice that Theorem 3 implies the result of J.A. Cima and J.E. Livingston [CL]: If $f = h + \bar{g} \in S_H$ and $f(\Delta)$ is a starlike domain (with respect to zero), then $h, g \in H^p$ and $f \in h^p$ for $0 < p < 1/3$.

4. Examples.

1. Let $f = h + \bar{g}$ be given by formula (4). We claim that $f \notin h^p$ if $p > \frac{1}{2}$, whereas $f \in h^{1/2}$.

First assume that $\frac{1}{2} < p < 1$. We have

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \geq \int_{-\pi}^{\pi} |\mathrm{Im} f(re^{i\theta})|^p d\theta = r^p(1-r^2)^p \int_{-\pi}^{\pi} \frac{|\sin\theta|^p}{|1-re^{i\theta}|^{4p}} d\theta$$
$$= 2r^p(1-r^2)^p \int_0^{\pi} \frac{(\sin\theta)^p}{(1+r^2-2r\cos\theta)^{2p}} d\theta.$$

Making the substitution $t = \cos\theta$ gives

$$I_r = \int_0^{\pi} \frac{(\sin\theta)^p}{(1+r^2-2r\cos\theta)^{2p}} d\theta > \frac{1}{(1+r^2)^{2p}} \int_0^1 \frac{dt}{(1-t^2)^{(1-p)/2}(1-ct)^{2p}}$$
$$> 2^{-(3p+1)/2} \int_0^1 \frac{dt}{(1-ct)^{(3p+1)/2}} = \frac{1}{c2^{(3p-1)/2}(3p-1)} \left[(1-c)^{(-3p+1)/2} - 1\right],$$

where $c = 2r/(1+r^2)$.

Thus

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta = 2r^p(1-r^2)^p I_r \to \infty \quad \text{as} \quad r \to 1^-$$

and the first assertion made about the function f is proved.

Now we show that $f \in h^{1/2}$. Because

$$\sup_{0\le r<1}\int_0^{2\pi}\left|\operatorname{Re}\left(\frac{re^{i\theta}}{1-re^{i\theta}}\right)\right|^{1/2}d\theta \le \sup_{0\le r<1}\int_0^{2\pi}\left|\frac{re^{i\theta}}{1-re^{i\theta}}\right|^{1/2}d\theta<\infty$$

it is enough to show that $\operatorname{Im} f \in h^{1/2}$. Similarly as above we get

$$\begin{aligned}\int_0^{2\pi}|\operatorname{Im} f(re^{i\theta})|^{\frac12}d\theta &= \frac{2\sqrt{r(1-r^2)}}{1+r^2}\int_{-1}^{1}\frac{dt}{\sqrt[4]{1-t^2}(1-ct)}\\ &=\frac{4\sqrt{r(1-r^2)}}{1+r^2}\int_0^1\frac{dt}{\sqrt[4]{1-t^2}(1-(ct)^2)} \qquad (12)\\ &<\frac{4\sqrt{r(1-r^2)}}{1+r^2}\int_0^1\frac{dt}{\sqrt[4]{1-t}(1-ct)}.\end{aligned}$$

Expanding the function $t\to 1/(1-ct)$ into a power series and integrating term by term we obtain

$$\int_0^1\frac{dt}{\sqrt[4]{1-t}(1-ct)}=\sum_{n=0}^{\infty}\frac{n!}{\frac34\left(\frac34+1\right)\cdots\left(\frac34+n\right)}c^n=S(c). \qquad (13)$$

Using the fact that the gamma function can be expressed as

$$\Gamma(a)=\lim_{n\to\infty}\frac{n!n^a}{a(a+1)\cdots(a+n)}$$

one can easily check that the coefficients in the series in formula (13) are of order $n^{-3/4}=n^{(1/4-1)}$ as $n\to\infty$. This means that the function $S(c)$ "behaves" like the function $F(c)=(1-c)^{-1/4}$, i.e. the ratio $S(c)/F(c)$ has a positive limit as $c\to 1^-$. Hence there is a constant $C>0$ such that

$$S(c)<C(1-c)^{-1/4}=C\frac{\sqrt[4]{1+r^2}}{\sqrt{1-r}}.$$

Hence

$$\int_0^{2\pi}|\operatorname{Im} f(re^{i\theta})|^{1/2}d\theta<8C.$$

2. Consider the close-to-convex function $l=h+\bar g$ where

$$h(z)=\frac{z-z^2/2+z^3/6}{(1-z)^3},\quad g(z)=\frac{z^2/2+z^3/6}{(1-z)^3}.$$

It can be easily checked that neither g, nor h is in $H^{1/3}$. However, l is in $h^{\frac{1}{3}}$. It has been shown in [CS] that l can be expressed as

$$l(z) = \frac{1}{6}\operatorname{Re}\left(\left(\frac{1+z}{1-z}\right)^3 - 1\right) + \frac{1}{4} i \operatorname{Im}\left(\frac{1+z}{1-z}\right)^2 .$$

Because $\operatorname{Im}(l) \in h^{\frac{1}{3}}$ it is enough to prove that the integral

$$\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3\right|^{1/3} d\theta$$

is bounded as $r \to 1^-$. We have

$$\begin{aligned}\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3\right|^{1/3} d\theta &\le (1-r^2)\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^2} d\theta \\ &+ \sqrt[3]{12}(1-r^2)^{1/3}\int_0^{2\pi} \frac{\sin^{\frac{2}{3}}\theta}{1+r^2-2r\cos\theta} d\theta \\ &\le 2\pi + 4\sqrt[3]{12}(1-r^2)^{1/3}\int_0^1 \frac{dt}{(1-t)^{1/6}(1-ct)},\end{aligned} \tag{14}$$

where $c = 2r/(1+r^2)$ as above. Integrating term by term gives

$$\int_0^1 \frac{dt}{(1-t)^{1/6}(1-ct)} = \sum_{n=0}^{\infty} \frac{n!}{\frac{5}{6}\left(\frac{5}{6}+1\right)\cdots\left(\frac{5}{6}+n\right)} c^n .$$

Now it is enough to notice that the coefficients in the last series are of order $n^{1/6-1}$ as $n \to \infty$. Thus there is a constant $C > 0$ such that

$$\int_0^1 \frac{dt}{(1-t)^{1/6}(1-ct)} \le C(1-c)^{-1/6} = C(1-r)^{-1/3}(1+r^2)^{1/6}$$

This together with (14) proves that $\operatorname{Re}(l)$ is in $h^{1/3}$.

Open problem. Is f in $h^{1/2}$ ($h^{1/3}$) if f is a convex (close-to-convex) harmonic mapping ?

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References

[AL] Abu-Muhanna, Y. and A. Lyzzaik, *The boundary behaviour of harmonic univalent maps*, Pacific J. Math. **141** (1990), 1-20.

[BH] Bshouty, D. and W. Hengartner, *Univalent harmonic mappings in the plane*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **48** (1994), 1-42.

[CL] Cima, J.A. and A.E. Livingston, Integral smoothness properties of some harmonic mappings, Complex Variables **11** (1989), 95-110.

[CS] Clunie, J. and T.Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. AI **9** (1984), 3-25.

[D1] Duren, P.L., *Theory of H^p spaces*, Academic Press, New York - London 1970.

[D2] Duren, P.L., *Univalent functions*, Springer-Verlag, New York -Tokyo 1983.

[F1] Flett, T. M., *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. Appl. **39** (1972), 125-158.

[F2] Flett, T. M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, ibid., **38** (1972), 746-765.

[MP] Mateljevic, M. and M. Pavlovic, *Multipliers of H^p and BMOA*, Pacific J. Math. **146** (1990), 71-84.

[P1] Pommerenke, Ch., *On the integral means of the derivative of a univalent function*, J. London Math. Soc. **(2) 32** (1985), 254-258.

[P2] Pommerenke, Ch., *On the integral means of the derivative of a univalent function II*, Bull. London Math. Soc. **17** (1985), 565-570.

[P3] Pommerenke , Ch., *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin-Heidelberg- New York 1991.

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