ANNALES
UNIVERSITATIS MARIAE CURIE - SKLODOWSKA
LUBLIN - POLONIA
VOL. L, 15 SECTIO A 1996

ISCANDER R. NEZHMETDINOV (Kazan)

Stability of Geometric Properties of Convolutions of Univalent Harmonic Functions

ABSTRACT. Let $\mathcal{H}^0$ be the class of complex-valued harmonic functions f given by the formula (1).

If $\mathcal{M}$ is a subclass of $\mathcal{H}^0$ then a neighbourhood of $f \in \mathcal{M}$ may be defined by following an idea of Ruscheweyh. For given subclasses $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ of $\mathcal{H}^0$ the stability of the convolution $\mathcal{M} * \mathcal{N}$ with respect to $\mathcal{P}$ means that $f * g \in \mathcal{P}$ whenever $f \in \mathcal{M}, g \in \mathcal{N}$, rangle over sufficiently small neighbouhoods. Stability conditions in some special cases ($\mathcal{N} = \{\mathrm{id}\}$, $\mathcal{M}, \mathcal{P}$ starlike, or convex) are established.

It is a well-known fact that any function $f(z)$, harmonic in the unit disk $D = \{z : |z| < 1\}$ can be written as $f(z) = f_1(z) + \overline{f_2(z)}$, both $f_1(z)$ and $f_2(z)$ being regular functions in D. Consider the class $\mathcal{H}^0$ of normalized harmonic functions

$$f(z) = z + \sum_{|k|=2}^{\infty} a_k(f)\phi_k(z), \quad z \in D, \tag{1}$$

where $\phi_k(z) = z^k$ for $k \geq 2$ and $\phi_k(z) = \overline{z}^{|k|}$ for $k \leq -2$. We retain the notation introduced by Clunie and Sheil-Small [2], according to which the superscript "0" means that there is no term $a_1(f)\overline{z}$ in the expansion (1).

Denote by S_H^0, St_H^0 and K_H^0 subclasses of $\mathcal{H}^0$, consisting of univalent, starlike and convex univalent functions, respectively.

Given any $f, g \in \mathcal{H}^0$, define, as in [1], their Hadamard convolution

$$(2) \qquad (f * g)(z) = z + \sum_{|k|=2}^{\infty} a_k(f) a_k(g) \phi_k(z),$$

as well as their integral convolution

$$(3) \qquad (f \otimes g)(z) = z + \sum_{|k|=2}^{\infty} a_k(f) a_k(g) k^{-1} \phi_k(z).$$

We also introduce T–δ–neighborhoods of $f \in \mathcal{H}^0$,

$$(4) \qquad TN_\delta(f) = \left\{ g \in \mathcal{H}^0 : \sum_{|k|=2}^{\infty} T_k |a_k(g) - a_k(f)| \leq \delta \right\},$$

generalizing those studied both in [2] and [3]. Here $\{T_{\pm k}\}_{k=2}^{\infty}$ are sequences of positive real numbers.

By a $T - \delta$ –neighborhoods $TN_\delta(\mathcal{M})$ of a class $\mathcal{M} \subset \mathcal{H}^0$ we mean the union of all $TN_\delta(f)$, where f ranges over the whole class $\mathcal{M}$.

In accordance with [3] let us give the basic

Definition. Assume that $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are subclasses of $\mathcal{H}^0$ with

$$\mathcal{M} * \mathcal{N} = \{f * g : f \in \mathcal{M}, g \in \mathcal{N}\} \subset \mathcal{P}.$$

Then (2) is said to be $T - \mathcal{P}$ - stable on the pair $(\mathcal{M}, \mathcal{N})$ if there exists a $\delta > 0$ such that $TN_\delta(\mathcal{M}) * TN_\delta(\mathcal{N}) \subset \mathcal{P}$.

Avci and Złotkiewicz [1] used a quite elementary approach to prove some relations of the form $\mathcal{M} * TN_b(e) \subset \mathcal{P}$, where $e(z) = z$. Here we extend to the harmonic case the duality technique developed by Ruscheweyh [4] for analytic functions and apply it to deduce necessary and sufficient conditions for (2) and (3) to be stable on the pairs $(\mathcal{M}, \{e\})$, $\mathcal{M}$ being any of the classes St_H^0, K_H^0 or $\{e\}$. Furthermore, we shall present explicit expressions or equations for the stability constants

$$\delta_T(\mathcal{M} * \mathcal{N}, \mathcal{P}) = \sup\{\delta \geq 0 : TN_b(\mathcal{M}) * TN_b(\mathcal{N}) \subset \mathcal{P}\}.$$

In the sequel $\mathcal{P}$ will denote either of two classes S_H^0 and St_H^0.

Theorem 1. *The convolution* (2) *is:*

a) $T - K_H^0$*-stable on* $(\{e\},\{e\}) \iff T_k^{-1} = O(k^{-1})$, $k \to \pm\infty$; *moreover,*

$$\delta_T\left(\{e\} * \{e\}\,,\, K_H^0\right) = \inf\left\{T_k|k|^{-1} : |k| \geq 2\right\};$$

b) $T-\mathcal{P}$*-stable on* $(\{e\},\{e\}) \iff T_k^{-1} = O(|k|^{-1/2})$, $k \to \pm\infty$; *besides,*

$$\delta_T\left(\{e\} * \{e\}\,,\, \mathcal{P}\right) = \inf\left\{T_k|k|^{-1/2} : |k| \geq 2\right\};$$

c) $T - K_H^0$*-stable on* $(K_H^0,\{e\}) \iff T_k^{-1} = O(k^{-3})$, $k \to \pm\infty$, *where* $\delta = \delta_T(K_H^0 * \{e\}, K_H^0)$ *is the unique positive root of the equation*

$$\delta = \inf\left\{2T_k^2 k^{-2}(|k+1|T_k + 2\delta)^{-1} : |k| \geq 2\right\};$$

d) $T-\mathcal{P}$*-stable on* $(K_H^0,\{e\}) \iff T_k^{-1} = O(k^{-2})$, $k \to \pm\infty$, *in addition* $\delta = \delta_T(K_H^0 * \{e\},\mathcal{P})$ *can be found from*

$$\delta = \inf\left\{2T_k^2|k|^{-1}(|k+1|T_k + 2\delta)^{-1} : |k| \geq 2\right\};$$

e) $T - K_H^0$*-stable on* $(St_H^0,\{e\}) \iff T_k^{-1} = O(k^{-4})$, $k \to \pm\infty$, *furthermore,* $\delta = \delta_T(St_H^0 * \{e\}, K_H^0)$ *satisfies the equation*

$$\delta = \inf\left\{6T_k^2 k^{-2}[(k+1)(2k+1)T_k + 6\delta]^{-1} : |k| \geq 2\right\};$$

f) $T - \mathcal{P}$*-stable on* $(St_H^0,\{e\}) \iff T_k^{-1} = O(k^{-3})$, $k \to \pm\infty$, *where the constant* $\delta = \delta_T(St_H^0 * \{e\},\mathcal{P})$ *is the unique positive solution of the equation*

$$\delta = \inf\left\{6T_k^2|k|^{-1}\left[((k+1)(2k+1)T_k + 6\delta\right]^{-1} : |k| \geq 2\right\}.$$

First, let us state and prove sufficient conditions for $f \in \mathcal{H}^0$ to be univalent, starlike or convex in terms of the convolution (2). Set

$$X = \{h \in \mathcal{H}^0 : a_k(h) = k(k+i\alpha)/(1+i\alpha) \text{ with } \alpha \in \mathbb{R} \text{ for all } |k| \geq 2\},$$
$$Y' = \{h \in \mathcal{H}^0 : a_k(h) = (k+i\alpha)/(1+i\alpha) \text{ with } \alpha \in \mathbb{R} \text{ for all } |k| \geq 2\},$$
$$Z = \{h \in \mathcal{H}^0 : a_k(h) = (\phi_k(x) - \phi_k(y))/(x-y) \text{ with } |x|,|y| \leq 1\,,\, x \neq y\,, \text{ for all } |k| \geq 2\},$$

and $Y = Y' \cup Z$.

Observe that for any $|k| \geq 2$ we have $|a_k(h)| \leq k^2$ if $h \in X$ and $|a_k(h)| \leq |k|$ if $h \in Y$ (or Z).

Lemma 1. *If* $f \in \mathcal{H}^0$ *and for all* $h \in X$ *(*Y *or* Z*) there holds*

$$(6) \qquad (f * h)(z) \neq 0\,, \quad z \in D' = D \setminus \{0\}\,,$$

then $f \in K_H^0$ *(*St_H^0 *or* S_H^0*), respectively.*

Proof. Given $h \in Z$, we have

$$(f * h)(z) = z + \sum_{|k|=2}^{\infty} \frac{\phi_k(x) - \phi_k(y)}{x - y} a_k(f)\phi_k(z) = \frac{f(xz) - f(yz)}{x - y}\,.$$

Show that the condition (6) (holding for all $h \in Z$) implies the univalence of f. Assume the contrary, i. e., there are $z_1, z_2 \in D$ such that $z_1 \neq z_2$, whereas $f(z_1) = f(z_2)$. Without loss of generality, let $|z_1| \leq |z_2|$. Put $z' = z_2$ and $z_1 = xz'$, where $|x| \leq 1, y = 1, x \neq y$. Hence for certain $h \in Z$ we obtain $(f * h)(z') = (f(z_1) - f(z_2))/(x - y)$, and, by the contradiction, f is univalent in D.

Now, if (6) holds for any $h \in Y$, the above reasoning yields at once $f \in S_H^0$. For $h \in Y'$ we have

$$(f * h)(z) = z + \sum_{|k|=2}^{\infty} \frac{k + i\alpha}{1 + i\alpha} a_k(f)\phi_k(z) \neq 0\,,$$

or, equivalently, $zf'_z - \overline{z}f'_{\overline{z}} + i\alpha f \neq 0$, for $z \in D', a \in \mathbb{R}$. Therefore, by the normalization, $d\arg f(re^{i\theta})/d\theta = \mathrm{Re}[(zf'_z - \overline{z}f'_{\overline{z}})/f] > 0$, where $z = re^{i\theta}, 0 < r < 1$. Hence for $D_r = \{z : |z| < r\}, 0 < r < 1$, its image $f(D_r)$ is a domain starlike with respect to the origin, and so is $f(D)$. Thus, $f \in St_H^0$. The case when $h \in X$ is considered in a similar way, however, by the Choquet Theorem [1], we may discard an additional condition for univalence.

The following assertion appears to be useful in constructing examples which prove the sharpness of constants.

Lemma 2 (see [2]). *Let* m *be an integer,* $|m| \geq 2$. *Then we have*

$$z + c\phi_m(z) \in K_H^0 \iff |c| \leq m^{-2}\,,$$
$$z + c\phi_m(z) \in \mathcal{P} \iff |c| \leq |m|^{-1}\,.$$

Proof of Theorem 1.
Case (a). Let (2) be $T - K_H^0$-stable on $(\{e\}, \{e\})$, that is, for some $\delta > 0$ the inclusion $TN_\delta(e) * TN_\delta(e) \subset K_H^0$ holds. Put

$$f(z) = g(z) = z + \delta T_k^{-1}\phi_k(z) \in TN_\delta(e)\,.$$

Then

$$(f*g)(z) = z + \delta^2 T_k^{-2}\phi_k(z) \in K_H^0\,,$$

and, by Lemma 2, $\delta^2 T_k^{-2}$ for any $k, |k| \geq 2$. Thus, $T_k^{-1} = O(k^{-1})$ as $k \to \pm\infty$.

Conversely, from the latter relation it follows that a $M > 0$ exists such that $T_k^{-1} \leq M|k|^{-1}$ (or $T_k|k|_{-1} \geq M_{-1}$) for all $|k| \geq 2$. Therefore,

$$\delta' = \inf\left\{T_k|k|^{-1} : |k| > 2\right\} > 0\,.$$

Choose any $f, g \in TN_\delta(e), 0 < \delta < \delta'$, and $h \in X$. Then we have

$$|k| \leq \delta^{-1}T_k\,,\ |a_k(h)| \leq k^2 \quad \text{and } |z^{-1}\phi_k(z)| \leq |z| \quad \text{for any } ||k| \geq 2, z \in D'.$$

Since $g \in TN_\delta(e)$, there also holds $|a_k(g)| \leq \delta T_k^{-1}, |k| \geq 2$. By using all the inequalities, we can estimate

$$\left|\frac{f*g*h}{z}\right| = \left|1 + \sum_{|k|=2}^{\infty} \frac{a_k(f)a_k(g)a_k(h)}{z}\phi_k(z)\right|$$

$$\geq 1 - |z|\sum_{|k|=2}^{\infty} \delta k^2 T_k^{-1}|a_k(f)| \geq 1 - |z|\delta^{-1}\sum_{|k|=2}^{\infty} T_k|a_k(f)| \geq 1 - |z| > 0\,.$$

Thus, the condition (6) is valid for all $h \in X$, so $f*g \in K_H^0$. On the other hand, if $\delta > \delta'$, then an integer $m, |m| \geq 2$, can be found such that $T_m|m|^{-1} < \delta$. The above example shows that for $f(z) = g(z) = z + \delta T_m^{-1}\phi_m(z)$ their convolution $f*g \notin K_H^0$, whence $\delta_T(\{e\}*\{e\}, K_H^0)\delta'$.

Case (c). Let (2) be $T-K_H^0$–stable on $(K_H^0, \{e\})$. Then there exists a positive δ such that $TN_\delta(K_H^0) * TN_\delta(e) \subset K_H^0$. Choose the functions

$$f(z) = L_0(z) + \delta\eta_k T_k^{-1}\phi_k(z) \in TN_\delta(K_H^0\,,$$

where

$$L_0(z) = z + \sum_{|j|=2}^{\infty}(j+1)\phi_j(z)/2 \in K_H^0\,, \quad \eta_k = \text{sgn } k\,,$$

$$\text{and} \quad g(z) = z + \delta\eta_k T_k^{-1}\phi_k(z) \in TN_\delta(e)\,.$$

By the above assumption we have

$$(f*g)(z) = z + \left(\frac{|k+1|}{2} + \delta T_k^{-1}\right)\delta T_k^{-1} = \phi_k(z) \in K_H^0$$

and from Lemma 2 it follows that

$$\delta\frac{|k+1|}{2}T_k^{-1} \leq \left(\frac{|k+1|}{2}+\delta T_k^{-1}\right)\delta T_k^{-1} \leq k^{-2},$$

hence $T_k^{-1} = O(k^{-3})$ as $k \to \pm\infty$. Now assume that $\delta > \delta', \delta'$ being the unique positive root of $\delta = \inf\{2T_k^2 k^{-2}(|k+1|T_k + 2\delta)^{-1} : |k| \geq 2\}$. Since both parts of the equation are monotonous relative to δ, we have

$$\delta > \inf\left\{2T_k^2 k^{-2}(|k+1|T_k + 2\delta)^{-1} : |k| \geq 2\right\}$$

and, by the above, functions $f \in TN_\delta(K_H^0), g \in TN_\delta(e)$ can be chosen such that $f * g \notin K_H^0$.

On the other hand, let $T_k^{-1} = O(k^{-3})$ as $k \to \pm\infty$, i.e., there is a $M > 0$ such that $T_k^{-1} \leq M|k|^{-3}$, or, equivalenty, $T_k \geq M^{-1}|k|^3$ for all $|k| \geq 2$. Obviously, $T_k \geq (2M)^{-1}k^2|k+1|$ and hence

$$2T_k^2k^{-2}(|k+1|T_k + 2\delta)^{-1} \geq M^{-1}|k+1|T_k(|k+1|T_k+2\delta)^{-1}.$$

Since

$$|k+1|T_k \geq (2M)^{-1}k^2|k+1|^2 \geq 2M^{-1} \quad \text{for all} \quad |k| \geq 2,$$

we have

$$2T_k^2k^{-2}(|k+1|T_k + 2\delta)^{-1} \geq M^{-1}(1+\delta M)^{-1}.$$

Therefore, the equation (5) yields

$$\delta = \inf\{2T_k^2k^{-2}(|k+1|T_k+2\delta)^{-1} : |k| \geq 2\} \geq M^{-1}(1+\delta M)^{-1},$$

so that its root $\delta' \geq (\sqrt{5}-1)/2M$ and is positive.

Suppose now that $0 < \delta \leq \delta'$. Both parts of (5) being monotonous, with respect to δ, it follows that

$$\delta \leq 2T_k^2k^{-2}(|k+1|T_k+2\delta)^{-1} \quad \text{for all} \quad k, |k| \leq 2. \tag{7}$$

Assuming that $f \in TN_\delta(f_0)$ with $f_0 \in K_H^0, g \in TN_\delta(e)$ and $h \in X$, we get

$$\begin{aligned}\left|\frac{(f*g*h)(z)}{z}\right| &\geq \left|\frac{(f_0*e*h)(z)}{z}\right| - \left|\frac{(f_0*(g-e)*h)(z)}{z}\right| \\ &\quad - \left|\frac{((f-f_0)*(g-e)*f)(z)}{z}\right| \\ &\geq 1 - \sum_{|k|=2}^{\infty}\Big[|a_k(f_0)| + |a_k(f)-a_k(f_0)|\Big]\left|a_k(g)a_k(h)z^{-1}\phi_k(z)\right|.\end{aligned}$$

By the sharp estimate $|a_k(f_0)| \le |k+1|/2$ (found in [2] for $f_0 \in K_H^0$), we use the inequalities $|a_k(f) - a_k(f_0)| \le \delta T_k^{-1}$ and $a_k(h) \le k^2, |k| \ge 2$, to obtain

$$(8) \qquad \left|\frac{(f*g*h)(z)}{z}\right| \ge 1 - |z| \sum_{|k|=2}^{\infty} \left(\frac{|k+1|}{2} + \delta T_k^{-1}\right) k^2 |a_k(g)|,$$

where $z \in D'$. Hence (7) and (8) yield

$$\left|\frac{(f*g*h)(z)}{z}\right| \ge 1 - \delta^{-1}|z| \sum_{|k|=2}^{\infty} T_k\, |a_k(g)| \ge 1 - |z| > 0,$$

and, by Lemma 1, $f * g \in K_H^0$.

Case (e) is studied in a similar way by applying the sharp estimate $|a_k(f_0))| \le (k+1)(2k+1)/6$, valid for all $f_0 \in St_H^0$ (see [6]). To prove the remaining cases it suffices to replace the class X by Y (or by Z) and to repeat the previous reasoning.

The conclude with, let us state a counterpart of Theorem 1 for the integral convolution (3).

Theorem 2. *The convolution* (3) *is:*

a) $T - K_H^0$*-stable on* $(\{e\},\{e\}) \iff T_k^{-1} = O(|k|^{-1/2}),\ k \to \pm\infty$; *moreover,*

$$\delta_T(\{e\} \otimes \{e\},\, K_H^0) = \inf\left\{T_k |k|^{-1/2} : |k| \ge 2\right\};$$

b) $T - \mathcal{P}$*-stable on* $(\{e\},\{e\}) \iff T_k^{-1} = O(1),\ k \to \pm\infty$; *besides,*

$$\delta_T(\{e\} \otimes \{e\},\, \mathcal{P}) = \inf\{T_k : |k| \ge 2\};$$

c) $T - K_H^0$*-stable on* $(K_H^0, \{e\}) \iff T_k^{-1} = O(k^{-2}),\ k \to \pm\infty$, *where*

$$\delta_T(K_H^0 \otimes \{e\}, K_H^0) \iff \delta_T(K_H^0 * \{e\}, \mathcal{P});$$

d) $T - \mathcal{P}$*-stable on* $(K_H^0, \{e\}) \iff T_k^{-1} = O(k^{-1}),\ k \to \pm\infty$, *in addition* $\delta = \delta_T(K_H^0 \otimes \{e\}, \mathcal{P})$ *can be found from*

$$\delta = \inf\left\{2T_k^2(|k+1|T_k + 2\delta)^{-1} : |k| \ge 2\right\};$$

e) $T - K_H^0$*-stable on* $(St_H^0, \{e\}) \iff T_k^{-1} = O(k^{-3}),\ k \to \pm\infty$, *furthermore,*

$$\delta_T(St_H^0 \otimes \{e\}, K_H^0) = \delta_T(St_H^0 *, \mathcal{P});$$

f) $T - \mathcal{P}$*-stable on* $(St_H^0, \{e\}) \iff T_k^{-1} = O(k^{-2}),\ k \to \pm\infty$, *where* $\delta = \delta_T(St_H^0 \otimes \{e\}, \mathcal{P})$ *is the unique positive solution of the equation*

$$\delta = \inf\left\{6T_k^2\left[((k+1)(2k+1)T_k + 6\delta\right]^{-1} : |k| \ge 2\right\}.$$

References

[1] Avci, Y. and E. Złotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae Curie–Skłodowska Sect. A **44** (1990), 1–7.

[2] Clunie, J. and T. Sheil–Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.

[3] Nezhmetdinov, I. R., *Stability of geometric properties of convolutions of univalent functions in certain neighborhoods*, Izv. Vyssh. Uchebn. Zaved. Mat. **No 8** (1994), 46–54.

[4] Ruscheweyh, St., *Über die Faltung schlichter Funktionen*, Math. Z. **128** (1972), 85–92.

[5] Sheil–Small, T. and E. M. Silvia, *Neighborhoods of analytic functions*, J. Analyse Math. **52** (1989), 210–240.

[6] ———, *Constants for planar harmonic mappings*, J. London Math. Soc. **42(2)** (1990), 237–248.

Faculty of Mechanics and Mathematics
Kazan State University
Kazan, Russian Federation

received September 18, 1996