ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA
VOL. L, 13 SECTIO A 1996

PRZEMYSŁAW MATUŁA (Lublin)

A Remark on the Weak Convergence of Sums of Associated Random Variables

ABSTRACT. We study the central limit theorem and invariance principle for associated sequences. Under appropriate conditions the exact Berry-Essen bound $O(n^{-1/2})$ for the rate of convergence in the CLT is obtained. We also prove the CLT for sequences of random variables with infinite variance.

1. Introduction. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of associated random variables, i. e., for every finite subcollection $X_{n_1}, X_{n_2}, ..., X_{n_k}$ and coordinatewise nondecreasing functions $f, g : \mathbb{R}^k \to \mathbb{R}$ the inequality

$$Cov(f(X_{n_1}, X_{n_2}, ..., X_{n_k}), g(X_{n_1}, X_{n_2}, ..., X_{n_k})) \geq 0$$

holds, whenever this covariance is defined.

Associated processes play very important role in mathematical physics and statistics. Many recent papers deal with limit theorems for such processes (see for example [1], [2], [3], [4], [7], [9] and references therein).

To begin with let us give a brief exposition of some recent results on weak convergence of associated processes. We will restrict our attention to the central limit theorem, rate of convergence in the CLT and the invariance

1991 *Mathematics Subject Classification.* 60F05, 60F17, 60B10.
Key words and phrases. Central limit theorem, rate of convergence, invariance principle, associated random variables.

principle. In the following we assume $EX_n = 0$, $EX_n^2 < \infty$, $n \in \mathbb{N}$ and put

$$S_n = \sum_{k=1}^{n} X_k, \quad \sigma_n^2 = ES_n^2, \quad \tau_n^2 = \sum_{k=1}^{n} EX_k^2;$$
$$u(n) = \sup_{k\in\mathbb{N}} \sum_{j:|j-k|\geq n} Cov(X_j, X_k), \quad n \in \mathbb{N}\cup\{0\}.$$

Conditions for the convergence $S_n/\sigma_n \xrightarrow{d} N(0,1)$ have been established by several authors.

Newman (cf. [7]) assumed that $(X_n)_{n\in\mathbb{N}}$ is a strictly stationary sequence satisfying $0 < \sigma^2 = Var(X_1) + 2\sum_{k=2}^{\infty} Cov(X_1, X_k) < \infty$. The assumption of stationarity was relaxed by Cox and Grimmett (cf. [3]), who considered processes satisfying

(1) $$u(n) = o(1), \ u(0) < \infty,$$

(2) $$\inf_{n\in\mathbb{N}} EX_n^2 > 0,$$

(3) $$\sup_{n\in\mathbb{N}} E|X_n|^3 < \infty.$$

The result of Cox and Grimmett was generalized by Birkel (cf. [2]), who showed that (1) and

(4) $$\inf_{n\in\mathbb{N}} n^{-1}\sigma_n^2 > 0$$

(5) $$\lim_{n\to\infty} \sigma_n^{-2} \sum_{k=1}^{n} EX_k^2 I[|X_k| \geq \epsilon\sigma_n] = 0, \text{ for every } \epsilon > 0$$

imply the central limit theorem.

The rate of convergence in the CLT was investigated by Wood (cf. [11]) and Birkel (cf. [1]). Wood considered stationary processes and his result maximally leads to

$$\Delta_n := \sup_{x\in\mathbb{R}} |P[S_n/\sigma_n \leq x] - \Phi(x)| = O(n^{-1/5}),$$

where $\Phi(x)$ denotes here and in the sequel the standard normal distribution.

Birkel proved that if $u(n) = O(e^{-\lambda n})$, (3) and (4) are satisfied, then $\Delta_n = O(n^{-1/2} \log^2 n)$. He also pointed out that it is an open question whether the Berry-Essen rate $O(n^{-1/2})$ is available. We give an answer to this question.

Further problem of our interest is the invariance principle, that is convergence of $W_n(t) := S_{[nt]}/\sigma_n \xrightarrow{d} W(t)$, $t \in [0,1]$ in $D[0,1]$, where W denotes the Wiener process. Results of this kind obtained Newman and Wright (cf. [8]) and Birkel (cf. [2]). A more general situation was considered by Matuła and Rychlik (cf. [6]), who studied the convergence of

$$W_n^*(t) = S_{m_n(t)}/\sigma_n \tag{6}$$

where

$$m_n(t) = \max(i : \ k_i \le t k_n) \tag{7}$$

and $0 = k_0 < k_1 < k_2 < \dots$ is a sequence satisfying

$$\lim_{n\to\infty} \max_{1\le i\le n} (k_i - k_{i-1})/k_n = 0. \tag{8}$$

It is worth pointing out that the assumptions (1) and (4) play a very important role in the above mentioned results. But let us observe that the condition (4), ensuring that σ_n^2 grows at least as n is generally not appropriate for the nonstationary case. Moreover, as the example below demonstrates, (1) may be sometimes useless.

Example 1. Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent and identically distributed (abbr.: i.i.d.) random variables with $E\xi = 0$, $E\xi^2 = 1$. Write

$$X_n = \begin{cases} \xi_n, & n \ne 2^k, k \in \mathrm{N} \\ \xi_n + \xi, & n = 2^k, k \in \mathrm{N}. \end{cases}$$

$(X_n)_{n\in\mathrm{N}}$ is an associated sequence and

$$S_n/\sigma_n = (\xi_1 + \dots + \xi_n + [\log n]\xi)/\sqrt{n + [\log n]^2} \xrightarrow{d} N(0,1),$$

but $u(n) = \infty$.

Our ourpose is to study the CLT and the invariance principle without the assumptions (1) and (4). We present conditions under which the Berry-Essen rate $\Delta_n = O(n^{-1/2})$ is available. An attempt to prove the CLT for associated r.v.'s with infinite variance is also made.

2. Results.

Theorem 1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of associated random variables such that* $EX_n = 0$, $EX_n^2 < \infty$, $n \in \mathbb{N}$. *If*

$$\lim_{n\to\infty} \sigma_n^2/\tau_n^2 = 1 \tag{9}$$

and (5) *holds, then*

$$S_{m_n(t)}/\sigma_n \xrightarrow{d} W(t) \text{ in } D[0,1], \tag{10}$$

with $k_n = \tau_n^2$.

In order to prove Theorem 1 we need the following

Theorem 2. *Under the hypotheses of* Theorem 1

$$S_n/\sigma_n \xrightarrow{d} N(0,1), \text{ as } n \to \infty.$$

The following example shows that our result cannot be obtained from the CLT mentioned in the Introduction.

Example 2. Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables such that $P[\xi = \pm 1] = P[\xi = \pm n^{-1/2}] = 1/2$. Define

$$X_n = \begin{cases} \xi_n, & n \neq 2^{2^k}, k \in \mathbb{N} \\ \xi_n + \xi, & n = 2^{2^k}, k \in \mathbb{N}. \end{cases}$$

It is easy to see that

$$\sigma_n^{-3} \sum_{k=1}^{n} E|X_k|^3 \le \frac{C_1 + C_2 \log\log n}{(\sum_{k=1}^n VarX_k)^{3/2}} \le \frac{C_1 + C_2 \log\log n}{(\log n)^{3/2}} \to 0,$$

moreover,

$$1 \le \sigma_n^2/\tau_n^2 \le 1 + [\log\log n]/(1 + \ldots + 1/n) \to 1.$$

Thus (5) and (9) hold, but for the sequence $(X_n)_{n\in\mathbb{N}}$ neither (1), (2) nor (4) is satisfied.

In the next theorem we prove the CLT for associated random variables with infinite variance. As far as we know, this is the first result of this kind. Let us recall that the distribution function of a centered variable X belongs to the domain of attraction of the standard normal law if there exists a sequence $(B_n)_{n\in\mathbb{N}}$ such that $(X_1' + \ldots + X_n')/B_n \xrightarrow{d} N(0,1)$, where $X_1', \ldots, X_n'$ are independent copies of X.

Theorem 3. *Let $(X_n)_{n\in N}$ be a sequence of centered associated random variables with the same distribution belonging to the domain of attraction of the standard normal law with the normalizing sequence $(B_n)_{n\in N}$. If $E|X_k| < \infty$, $E|X_kX_m| < \infty$, $k \neq m$; $k, m \in N$ and*

$$\sum\sum_{1\leq k<m\leq n} Cov(X_k, X_m)/B_n^2 \to 0\,, \quad \text{as } n\to\infty\,,$$

then $S_n/B_n \xrightarrow{d} N(0,1)$, as $n\to\infty$.

In the following example we construct a sequence of associated random variables with infinite variance which satisfies the conditions of Theorem 3. For such a sequence the results from the Introduction cannot be applied.

Example 3. Let $(\xi_n)_{n\in N}$ be a sequence of i.i.d. random variables such that $P[\xi_n = \pm k] = c/k^3$, $k\in N$ and let $(\eta_n)_{n\in N}$ be a sequence of i.i.d. r.v.'s independent of $(\xi_n)_{n\in N}$ with $P[\eta_n = \pm 1] = 1/2$. Define

$$X_n = \begin{cases} \xi_n + \eta_n, & n \neq 2^k, k\in N \\ \xi_n + \eta_1, & n = 2^k, k \in N\,. \end{cases}$$

We see that $(X_n)_{n\in N}$ is a sequence of equidistributed associated random variables with $EX_n = 0$, $EX_n^2 = +\infty$, moreover, $h(x) = E|X_n|^2 I[|X_n| < x]$ is slowly varying, therefore the distribution of X_n belongs to the domain of attraction of $N(0,1)$, with normalizing sequence $(B_n)_{n\in N}$, say (cf. [5]). Moreover, for , $k\neq m$

$$Cov(X_k, X_m) = \begin{cases} 1, & k = 2^p \text{ and } m = 2^q; p, q \in N \\ 0, & \text{otherwise}\,. \end{cases}$$

Thus

$$\sum\sum_{1\leq k<m\leq n} Cov(X_k, X_m)/B_n^2 \leq \log^2 n/B_n^2 \to 0\,,$$

since $B_n = n^{1/2}l(n)$, where l is slowly varying.

The conditions which yield the Berry-Essen bound $\Delta_n = O(n^{-1/2})$ in the CLT for associated sequences are given in the following

Theorem 4. *Let $(X_n)_{n\in N}$ be a sequence of associated random variables such that:*

(11) $$EX_n = 0,\ n\in N;\ \sup_{n\in N} E|X_n|^3 < \infty,\ \inf_{n\in N} EX_n^2 > 0\,;$$

for every $n \in \mathbb{N}$ *there exists* $I_n \subset \{1, 2, ..., n\}$ *such that*

$$\sigma_n^2 - Var\left(\sum_{k\in I_n} X_k\right) - \sum_{k\in\{1,\ldots,n\}\setminus I_n} Var X_k \le C_1/\sqrt{n} \tag{12}$$

and $Card\, I_n \le C_2 n^{1/3}$, *where* C_1, C_2 *are constants not depending on* n. *Then there exists a constant* C_3 *independent of* n, *such that for all* $n \in \mathbb{N}$

$$\Delta_n \le C_3/\sqrt{n}. \tag{13}$$

It is easy to observe that a sequence $(X_n)_{n\in\mathbb{N}}$ defined in Example 1, provided additionally $E|\xi_n|^3 < \infty$, fulfills the assumptions of Theorem 4 with $I_n = \{k \le n,\ k = 2^p,\ p \in \mathbb{N}\}$. Let us consider another example, which demonstrates that our results cannot be obtained from that of Birkel [1].

Example 4. Let $\xi_0, \xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables such that $E\xi_n = 0$, $E\xi_n^2 = 1$, $E|\xi_n|^3 < \infty$,, $n \in \mathbb{N} \cup \{0\}$. Let us put $X_n = \xi_n + 2^{-n}\xi_0$. It is not hard to check that , $u(n) = O(e^{-\lambda n})$ for some $\lambda > 0$ and (3), (4) are satisfied. Thus Theorem 2.1 of [1] yields $\Delta_n = O(n^{-1/2}\log^2 n)$. But in this case also the assumptions of our Theorem 4 are satisfied with $I_n = \{1, 2, ..., [\log n]\}$ so that our theorem provides a better rate of convergence.

3. Proofs.

Lemma 1. *Suppose* $X_1, ..., X_n$ *are associated with joint and marginal characteristic functions* $\varphi(t_1, ..., t_n)$ *and* $\varphi_k(t)$, *respectively, and* $Cov(X_j, X_k)$ *is defined for* $j \ne k$. *Then*

$$\left|\varphi(t_1, ..., t_n) - \prod_{k=1}^{n} \varphi_k(t_k)\right| \le \sum\sum_{1\le k<m\le n} |t_k t_m| Cov(X_k, X_m).$$

This Lemma is a refined version of the Newman inequality (cf. [7, 8]), however, we do not require the variances to be finite. The proof is similar, so we omit details.

Proof of Theorem 2. It follows from (9) and (5) that

$$\lim_{n\to\infty} \tau_n^{-2} \sum_{k=1}^{n} EX_k^2 I[|X_k| \ge \epsilon\tau_n] = 0.$$

Therefore a sequence $(X_n')_{n\in\mathbb{N}}$ of independent random variables such that X_n' has the same distribution as X_n fulfills the CLT:

$$S_n'/\tau_n := (X_1' + \ldots + X_n')/\tau_n \xrightarrow{d} N(0,1).$$

From the Lemma we get

$$\left|\varphi_{S_n/\sigma_n}(t) - \varphi_{S_n'/\sigma_n}(t)\right| \le \frac{1}{2}t^2\frac{\sigma_n^2 - \tau_n^2}{\sigma_n^2},$$

where $\varphi_{S_n/\sigma_n}(t)$, $\varphi_{S_n'/\sigma_n}(t)$ denote the corresponding characteristic functions. Therefore $S_n/\sigma_n \xrightarrow{d} N(0,1)$.

Proof of Theorem 1. We apply Theorem 2 of [6]. By our Theorem 2 the CLT holds, therefore it remains to prove that for $p, q \in \mathbb{N}, p < q$,

$$\lim_{n\to\infty} \sigma_n^{-2} ES_{m_n(p)}S_{m_n(q)} = p. \tag{14}$$

The Lindeberg condition (5) implies the Feller condition

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{EX_k^2}{\sigma_n^2} = 0. \tag{15}$$

From the definition of $m_n(p)$ we get

$$\tau^2_{m_n(p)} \le p\tau_n^2 \text{ and } \tau^2_{m_n(p)+1} > p\tau_n^2,$$

thus

$$p - \frac{EX^2_{m_n(p)+1}}{\sigma^2_{m_n(p)+1}} \cdot \frac{\sigma^2_{m_n(p)+1}}{\tau^2_{m_n(p)+1}} \cdot \frac{\tau^2_{m_n(p)+1}}{\tau_n^2} < \frac{\tau^2_{m_n(p)}}{\tau_n^2} \le p.$$

As a consequence of (9) and (15)

$$\lim_{n\to\infty} \frac{\tau^2_{m_n(p)}}{\tau_n^2} = \lim_{n\to\infty} \frac{\sigma^2_{m_n(p)}}{\sigma_n^2} = p. \tag{16}$$

Furthermore

$$ES^2_{m_n(p)} \le ES_{m_n(p)}S_{m_n(q)} \le ES^2_{m_n(q)} - \left(\tau^2_{m_n(q)} - \tau^2_{m_n(p)}\right) \tag{17}$$

and (14) follows from (17) and (16).

Proof of Theorem 3. The proof is based on the Lemma and is similar to that of Theorem 2 so we omit details.

Proof of Theorem 4. Let us put $\bar{\sigma}_n^2 = Var\left(\sum_{k\in I_n} X_k\right) + \sum_{k\in I_n'} VarX_k$, $I_n' = \{1,2,...,n\} \setminus I_n$, $L_n = \bar{\sigma}_n^{-3}\left(E|\sum_{k\in I_n} X_k|^3 + \sum_{k\in I_n'} E|X_k|^3\right)$.

Let $f_n(t)$ denote the characteristic function of $S_n/\bar{\sigma}_n$ and $\bar{f}_n(t)$ the characteristic function of $\left(Y + \sum_{k\in I_n'} Y_k\right)/\bar{\sigma}_n$, where Y, Y_k, $k \in I_n'$ are independent and Y, Y_k have the same distribution as $\sum_{k\in I_n} X_k$ and X_k, respectively. Then applying estimates known for independent random variables (cf. [10], pp. 155-157 and 161), we have with $T = 1/4L_n$

$$\sup_{x\in\mathbb{R}} |P[S_n/\sigma_n \le x] - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P[S_n/\bar{\sigma}_n \le x] - \Phi(x)|$$

$$+ \sup_{x\in\mathbb{R}} |\Phi(x\sigma_n/\bar{\sigma}_n) - \Phi(x)| \le C_1 \int_{-T}^{T} \left|\frac{f_n(t) - \bar{f}_n(t)}{t}\right| dt$$

$$+C_2 \int_{-T}^{T} \left|\frac{\bar{f}_n(t) - e^{-t^2/2}}{t}\right| dt + C_3 T^{-1} + C_4\left(\frac{\sigma_n}{\bar{\sigma}_n} - 1\right)$$

$$\le C_5 T^2 \frac{\sigma_n^2 - \bar{\sigma}_n^2}{\bar{\sigma}_n^2} + C_6 L_n + C_4 \frac{\sigma_n^2 - \bar{\sigma}_n^2}{\bar{\sigma}_n(\sigma_n + \bar{\sigma}_n)}.$$

By our assumptions

$$L_n \le \frac{(CardI_n)^2 \sum_{k\in I_n} E|X_k|^3 + \sum_{k\in I_n'} E|X_k|^3}{(\sum_{k=1}^n VarX_k)^{3/2}}$$

$$\le C_7\left((CardI_n)^3 + n\right) n^{-3/2} \le C_8/\sqrt{n}$$

and

$$L_n \bar{\sigma}_n \ge C_9 > 0,$$

where $C_1, ..., C_9$ denote absolute constants. This ends the proof.

References

[1] Birkel, T., *On the convergence rate in the central limit theorem for associated random variables*, Ann. Probab. **16** (1988), 1685–1698.

[2] Birkel, T., *The invariance principle for associated processes*, Stoch. Proc. Appl. **27** (1988), 57–71.

[3] Cox, J. T. and G. Grimmett, *Central limit theorems for associated random variables and the percolation model*, Ann. Probab. **12** (1984), 514–528.

[4] Esary, J., F. Proschan and D. Walkup, *Association of random variables with applications*, Ann. Math. Statist. **38** (1967), 1466–1474.

[5] Feller, W., *An introduction to probability theory and its applications*, John Wiley, New York, 1966.

[6] Matula, P. and Z. Rychlik, *The invariance principle for nonstationary sequences of associated random variables*, Ann. Inst. H. Poincaré Probab. Statist. **26** (1990), 387–397.

[7] Newman, C. M., *Normal fluctuation and the FKG inequalities*, Comm. Math. Phys. **74** (1980), 119–128.

[8] Newman, C. M. and A. L. Wright, *An invariance principle for certain dependent sequences*, Ann. Probab. **9** (1981), 671–675.

[9] Newman, C. M., *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in Y.L. Tong, ed., Inequalities in Statistics and Probability (IMS, Hayword CA) (1984), 127–140.

[10] Petrov, V. V., *Limit theorems for sums of independent random variables (in Russian)*, Nauka, Moscow, 1987.

[11] Wood, T. E., *A Berry–Essen theorem for associated random variables*, Ann. Probab. **11** (1983), 1042-1047.

Instytut Matematyki UMCS
Plac Marii Curie–Skłodowskiej 1
20–031 Lublin, Poland
e-mail matula@golem.umcs.lublin.pl

received February 10, 1996