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Moment Generating Functions of First Hitting Times for the Bidimensional Geometric Brownian Motion

Abstract. Let $W(t)$ be a Brownian motion. Then $X(t) = \exp[W(t)]$ is known as a geometric Brownian motion. In this paper, the bidimensional geometric Brownian motion is considered. Explicit solution, in terms of moment generating functions, are obtained to all the first passage time problems in the Cartesian coordinate system that can be solved by using either the method of separation of variables or two special cases of the method of similarity solutions.

1. Introduction. Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions (or Wiener processes) with zero mean and infinitesimal variance $\sigma^2 = 2$. Then $X_i(t) \doteq \exp[W_i(t)]$, for $i = 1, 2$, are independent processes known as geometric Brownian motions (see Karlin and Taylor [5, p. 175]). As mentioned in [5], the geometric Brownian motion is frequently used in economics applications and to model population growth processes as well. Its infinitesimal parameters are given by

$$\mu_{X_i}(x_i) = x_i \tag{1.1}$$

and

$$\sigma^2_{X_i}(x_i) = 2x_i^2 . \tag{1.2}$$

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In addition, a diffusion process with the above infinitesimal parameters can also be considered as a "lognormal process", that is, a process with a lognormal transition density function, such as the one considered by Capocelli and Ricciardi [2] (see also Kannan [4, p. 281–283]).

Next, writing $X(t) \doteq X_1(t)$ and $Y(t) \doteq X_2(t)$ for simplicity, the processes $X(t)$ and $Y(t)$ satisfy the following stochastic differential equations:

$$\begin{cases} dX(t) = X(t)dt + \sqrt{2X^2(t)}\, dW_1(t)\,, \\ dY(t) = Y(t)dt + \sqrt{2Y^2(t)}\, dW_2(t)\,. \end{cases} \tag{1.3}$$

Let

$$T(x,y) = \inf\{t \geq 0 : (X(t), Y(t)) \in D | X(0) = x\,, Y(0) = y\}\,, \tag{1.4}$$

where D is a subset of $\mathbb{R}^2$. Then the moment generating function of the random variable T, that is

$$M(x,y;a) \doteq E\{\exp[-aT(x,y)]\}\,, \tag{1.5}$$

where we assume that a is a real, positive parameter, satisfies the Kolmogoroff backward differential equation

$$x^2 M_{xx} + y^2 M_{yy} + x M_x + y M_y = aM\,. \tag{1.6}$$

Furthermore, the function M is such that

$$0 < M(x,y;a) \leq 1 \tag{1.7}$$

for any values of x and y and any positive value of a (since we may assume that $T(x,y)$ is not identical to infinity). We also have the boundary condition:

$$M(x,y;a) = 1 \;\; \forall (x,y) \in \partial D\,, \tag{1.8}$$

where ∂D is the boundary of the stopping region D.

In this paper, we are interested in computing the function $M(x,y;a)$ in all the cases where equation (1.6), subject to (1.7) and (1.8), can be solved by using either the method of separation of variables or a technique called the method of similarity solutions. The method of similarity solutions (see [3], for instance) is a technique that enables to reduce a second order partial differential equation such as equation (1.6) to an ordinary differential equation by supposing, for example, that the function M is of the form

$$M(x,y;a) = N(x+y;a)\,. \tag{1.9}$$

Here, we consider two special cases of the method of similarity solutions: in Section 3, we assume that

$$M(x,y;a) = N(f(x)+g(y);a), \tag{1.10}$$

whereas in Section 4 we look for solutions of the form

$$M(x,y;a) = Q(p(x)g(y);a). \tag{1.11}$$

In both cases, we assume that the functions of x and y do not contain constant terms. Furthermore, since the method of similarity solutions is sometimes equivalent to the method of separation of variables, we consider this case first, in Section 2.

Finally, in order to obtain well-defined random variables T, we suppose that the boundary ∂D of the stopping region contains an uncountable number of points. We also assume that:
H1: the boundary ∂D of the stipping region depends on both $X(t)$ and $Y(t)$, so that the problems considered are really bidimensional;
H2: ∂D consists of a single curve (and the boundary is absorbing).

2. The method of separation of variables. We first assume that the function $M(x,y;a)$ can be written as

$$M(x,y;a) = F(x;a)\,G(y;a). \tag{2.1}$$

Then equation (1.6) is transformed into the system of ordinary differential equations (dropping the argument a):

$$x^2F''(x) + xF'(x) - (k + k_1a)F(x) = 0, \tag{2.2}$$

$$y^2G''(y) + yG'(y) + (k - k_2a)G(y) = 0, \tag{2.3}$$

where k, k_1 and k_2 are constants and

$$k_1 + k_2 = 1. \tag{2.4}$$

Let

$$\mu \doteq \sqrt{k + k_1 a}. \tag{2.5}$$

Then the general solution of equation (2.2) is

$$F(x) = \begin{cases} C_1x^\mu + C_2x^{-\mu} & \text{if } \mu \neq 0, \\ C_1 + C_2\ln x & \text{if } \mu = 0, \end{cases} \tag{2.6}$$

where C_1 and C_2 are constants. Similarly, we may write that

$$G(y) = \begin{cases} K_1 y^{\nu} + K_2 y^{-\nu} & \text{if } \nu \neq 0, \\ K_1 + K_2 \ln y & \text{if } \nu = 0, \end{cases} \tag{2.7}$$

where K_1 and K_2 are constants and

$$\nu \doteq \sqrt{k_2 a - k}. \tag{2.8}$$

Next, we have:

$$\lim_{a \to \infty} M(x, y; a) = 0, \tag{2.9}$$

so that either $F(x; a)$ or $G(y; a)$ (or both) must depend on a. Moreover, the boundary ∂D of the stopping region is defined by

$$\partial D = \{(x, y) \in \mathbb{R}^2 : F(x; a)\, G(y; a) = 1\}. \tag{2.10}$$

Since ∂D obviously cannot depend on the parameter a, we deduce from (2.9) and (2.10) that we must set

$$k = 0 \tag{2.11}$$

above and that the function

$$M(x, y; a) = \pm x^{\pm\sqrt{k_1 a}}\, y^{\pm\sqrt{k_2 a}}. \tag{2.12}$$

Using the relation (2.4) between the constants k_1 and k_2, we may write that

$$M(x, y; a) = (\pm x^{\pm\sqrt{k_1}}\, y^{\pm\sqrt{1-k_1}})^{\sqrt{a}}. \tag{2.13}$$

Finally, since, according to the hypothesis H1 in Section 1, ∂D must depend on both $X(t)$ and $Y(t)$ (and cannot be complex), we take

$$0 < k_1 < 1. \tag{2.14}$$

We can now state the proposition that follows.

Proposition 2.1. *Under the hypotheses* H1 *and* H2 *stated in* Section 1, *the only first passage time problems in the Cartesian coordinate system, for the bidimensional geometric Brownian motion defined by equations* (1.3), *for which the moment generating function* $M(x,y;a)$ *can be obtained by the method of separation of variables are the ones where*

$$(2.15) \qquad \partial D = \{(x,y) \in \mathbb{R}^2 : \pm x^{\pm\sqrt{k_1}}\, y^{\pm\sqrt{1-k_1}} = 1\},$$

where $0 < k_1 < 1$. *Then the function* M *is given by formula* (2.13). *Finally, in* (2.13) *and* (2.15) *the signs are chosen so that*

$$(2.16) \qquad 0 < \pm x^{\pm\sqrt{k_1}}\, y^{\pm\sqrt{1-k_1}} \le 1 .$$

Proof. All that remains to mention is the fact that the origin is a natural, hence inaccessible boundary for the geometric Brownian motion (see Kannan [4, p. 281]). Therefore, if, for instance, we take $k_1 = 1/2$ and if we suppose that $x > 0$ and $y > 0$, then all the signs in (2.13) and (2.15) will be chosen equal to $+$ if $xy < 1$, whereas we will choose the signs $-$ in the exponents if $xy > 1$. Moreover, the fact that the origin is a natural boundary for the geometric Brownian motion implies that there will be only one absorbing barrier in the Cartesian plane, as formulated in hypothesis H2.

Finally, using (2.16), it is obvious that the condition (1.7) is satisfied. □

3. The method of similarity solutions.

In this section, we suppose that the moment generating function $M(x,y;a)$ is of the form

$$(3.1) \qquad M(x,y;a) = N(z;a),$$

where

$$(3.2) \qquad z \doteq f(x) + g(y).$$

Furthermore, we assume, for simplicity, that the functions f and g do not contain constant terms. Then, equation (1.6) becomes (dropping the argument a, as in Section 2)

(3.3)
$$\left[x^2(f')^2 + y^2(g')^2\right] N''(z) + \left[x^2 f'' + y^2 g'' + xf' + yg'\right] N'(z) = aN(z).$$

In order to be able to solve equation (1.6) by this special case of the method of similarity solutions, we deduce from (3.3) that there are three possibilities: 1) the function $N(z)$ is linear in z:

$$(3.4) \qquad N(z) = \alpha + \beta z ;$$

2) the function $N(z)$ is of the form

$$N(z) = \alpha e^{\beta z}; \tag{3.5}$$

3) equation (3.3) may be rewritten as

$$\phi(z)N''(z) + \psi(z)N'(z) = aN(z). \tag{3.6}$$

In the first case, we must solve

$$\left[x^2 f''(x) + y^2 g''(y) + x f'(x) + y g'(y)\right]\beta = a\left[\alpha + \beta(f(x) + g(y))\right]. \tag{3.7}$$

It follows that

$$x^2 f''(x) + x f'(x) - a f(x) = e_1 \tag{3.8}$$

and

$$y^2 g''(y) + y g'(y) - a g(y) = e_2, \tag{3.9}$$

where e_1 and e_2 are constants such that $e_1 + e_2 = a(\alpha/\beta)$.

Now, as in Section 2, we obtain:

$$f(x) = c_{11} x^{\sqrt{a}} + c_{12} x^{-\sqrt{a}} - (e_1/a), \tag{3.10}$$

$$g(y) = c_{21} y^{\sqrt{a}} + c_{22} y^{-\sqrt{a}} - (e_2/a), \tag{3.11}$$

where c_{ij} is a constant for $i, j = 1, 2$. Hence, we have:

$$N(f(x) + g(y); a) = \beta\left[c_{11} x^{\sqrt{a}} + c_{12} x^{-\sqrt{a}} + c_{21} y^{\sqrt{a}} + c_{22} y^{-\sqrt{a}}\right]. \tag{3.12}$$

But, it is obvious that we cannot find constants c_{ij} such that the boundary ∂D of the stopping region satisfies the hypotheses H1 and H2 in Section 1 and does not depend on the parameter a. Therefore, we may conclude that there is no solution of the form $N(z) = \alpha + \beta z$.

Next, in the second case above, we may write that

$$M(x, y; a) = \alpha e^{\beta f(x)} e^{\beta g(y)}. \tag{3.13}$$

Consequently, this case is a special instance of the method of separation of variables. Thus, it has already been treated in Section 2. So, we can limit ourselves to the third possibility. We deduce from (3.6) that

$$x^2 (f'(x))^2 = K f(x) + f_0, \tag{3.14 a}$$

$$y^2 (g'(y))^2 = K g(y) + g_0, \tag{3.14 b}$$

$$x^2 f''(x) + x f'(x) = C f(x) + f_1, \tag{3.14 c}$$

$$y^2 g''(y) + y g'(y) = C g(y) + g_1, \tag{3.14 d}$$

where K, C, f_i and g_i are constants, for $i = 0, 1$. Next, since $f(x)$ cannot be a constant (in order to satisfy the hypothesis H1) and $x \neq 0$, differentiating equation (3.14 a) with respect to x, we obtain:

$$f''(x) + \frac{1}{x} f'(x) - \frac{K}{2x^2} = 0 . \tag{3.15}$$

The general solution of this last differential equation is

$$f(x) = \frac{K}{4} \ln^2 x + K_{10} \ln x + K_{11} , \tag{3.16}$$

where K_{10} and K_{11} are constants. Since we assumed that $f(x)$ does not contain a constants term, we set K_{11} equal to zero, so that

$$f(x) = \frac{K}{4} \ln^2 x + K_{10} \ln x . \tag{3.17}$$

Substituting equation (3.17) into (3.14 c) we get:

$$\frac{K}{2} = \frac{CK}{4} \ln^2 x + CK_{10} \ln x + f_1 . \tag{3.18}$$

Considering the fact that K and K_{10} cannot be both equal to zero at the same time, we may conclude that

$$C = K = f_1 = 0 \tag{3.19}$$

or, if $K \neq 0$

$$C = 0 \text{ and } f_1 = K/2 . \tag{3.20}$$

I) We shall treat the case when $C = K = f_1 = 0$ first. Then, we have:

$$f(x) = K_{10} \ln x . \tag{3.21}$$

Similarly, we find that $g_1 = 0$ and

$$g(y) = K_{20} \ln y . \tag{3.22}$$

In (3.21) and (3.22), K_{10} and K_{20} must be different from 0.

Next, let

$$h_0 = f_0 + g_0 . \tag{3.23}$$

Then, using (3.14 a)–(3.14 d) and (3.19), we may write that equation (3.6) is

$$h_0 N''(z) = aN(z)\,. \tag{3.24}$$

From (3.14 a), (3.14 b), (3.21) and (3.22) we obtain that both f_0 and g_0 are positive; thus, so is h_0.

The general solution of equation (3.24) is

$$N(z) = \gamma_1 e^{\delta z} + \gamma_2 e^{-\delta z}\,, \tag{3.25}$$

where γ_1 and γ_2 are constants and

$$\delta \doteq \sqrt{a/h_0}\,. \tag{3.26}$$

Now, as in Section 2, we use the fact that the boundary ∂D of the stopping region must be independent of the parameter a; it follows that we must set γ_1 or γ_2 equal to 0 in (3.25). Therefore, we may write that

$$N(z) = \gamma e^{\pm \delta z}\,, \tag{3.27}$$

where γ is a constant. Hence, we deduce that we have retrieved the solutions obtained in Section 2, by the method of separation of variables, which is confirmed when we substitute equations (3.21) and (3.22) into (3.27). Note also that (3.14 a) and (3.14 b) imply that

$$K_{10}^2 + K_{20}^2 = h_0\,, \tag{3.28}$$

so that the two methods really yield the same solutions.

II) Next, we treat the case when $C = 0$, but $K \neq 0$. We know that f_1 must then be equal to $K/2$. Similarly, we can show that g_1 is also equal to $K/2$. It implies that equation (3.3) can be written as

$$(Kz + h)N''(z) + KN'(z) = aN(z)\,, \tag{3.29}$$

where

$$h \doteq f_0 + g_0\,. \tag{3.30}$$

In fact, substituting equation (3.17) into (3.14 a), we find that $f_0 = K_{10}^2$ and, similarly, $g_0 = K_{20}^2$. Therefore, equation (3.28) is still verified and $h = h_0$.

Let

$$N(z) = L(w), \tag{3.31}$$

where

$$w \doteq z + h/K. \tag{3.32}$$

We find that the function L satisfies the following differential equation:

$$wL''(w) + L'(w) = \frac{a}{K} L(w). \tag{3.33}$$

The general solution of equation (3.33) is given by

$$L(w) = l_1 J_0(2d_1\sqrt{w}) + l_2 Y_0(2d_1\sqrt{w}), \tag{3.34}$$

where l_1 and l_2 are constants, J_0 and Y_0 are Bessel functions, and

$$d_1 \doteq \sqrt{-a/K}. \tag{3.35}$$

Note that

$$Kz + h \geq 0. \tag{3.36}$$

Indeed, from (3.14 a) and (3.14 b) we have:

$$Kz + h = x^2(f'(x))^2 + y^2(g'(y))^2. \tag{3.37}$$

It follows that

$$\operatorname{sgn}(w) = \operatorname{sgn}(K). \tag{3.38}$$

a) If K is negative, d_1 is real and we write:

$$L(w) = l_1 J_0(2id_1\sqrt{-w}) + l_2 Y_0(2id_1\sqrt{-w}). \tag{3.39}$$

The function L can be rewritten as (see [1, p. 375])

$$L(w) = l_3 I_0(2d_1\sqrt{-w}) + l_4 K_0(2d_1\sqrt{-w}), \tag{3.40}$$

where l_3 and l_4 are other constants, and I_0 and K_0 are modified Bessel functions.

There will be two cases: $-w \geq r$ or $0 \leq -w \leq r$. When $-w \geq r$, we must set l_3 equal to zero. Indeed, we have (see [1, p. 374]):

$$\lim_{x \to \infty} I_0(x) = \infty . \tag{3.41}$$

Using the fact that $N(z)$ must be equal to 1 on ∂D, we deduce that

$$N(z) = \frac{K_0(2d_1\sqrt{-z-(h/K)})}{K_0(2d_1\sqrt{r})} \tag{3.42}$$

if $-w \geq r$. Conversely, when $0 \leq -w \leq r$, we set l_4 equal to zero, because (see [1, p. 374])

$$\lim_{x \downarrow 0} K_0(x) = \infty . \tag{3.43}$$

Then, we have:

$$N(z) = \frac{I_0\left(2d_1\sqrt{-z-(h/K)}\right)}{I_0\left(2d_1\sqrt{r}\right)} \tag{3.44}$$

if $0 \leq -w \leq r$.

b) When the constant K is positive, we write:

$$L(w) = l_1 J_0\left(2id_2\sqrt{w}\right) + l_2 Y_0\left(2id_2\sqrt{w}\right) , \tag{3.45}$$

where

$$d_2 \doteq \sqrt{a/K} . \tag{3.46}$$

Again, we have:

$$L(w) = l_3 I_0\left(2d_2\sqrt{w}\right) + l_4 K_0\left(2d_2\sqrt{w}\right) . \tag{3.47}$$

We find that

$$N(z) = \frac{K_0\left(2d_2\sqrt{z+(h/K)}\right)}{K_0(2d_2\sqrt{r})}, \qquad \text{if } w \geq r \tag{3.48}$$

and

$$N(z) = \frac{I_0\left(2d_2\sqrt{z+(h/K)}\right)}{I_0(2d_2\sqrt{r})} \qquad \text{if } 0 \leq w \leq r . \tag{3.49}$$

Next, it is easy to see that formulas (3.42) and (3.48), and formula (3.44) and (3.49), correspond to the same first passage time problem, respectively. Therefore, we may assume that the constant K is positive, as, in fact, could be expected. Finally, it is also a simple matter to check (see [1, p. 374) that the functions $N(z)$ in formula (3.48) and (3.49) satisfy the condition

$$0 \leq N(z) \leq 1 . \tag{3.50}$$

We can now state the most important result of this paper.

Proposition 3.1. *Under the same hypotheses as in* Proposition 2.1, *the only first passage time problems in the Cartesian coordinate system, apart from those already obtained by the method of separation of variables, for the bidimensional geometric Brownian motion defined by equations* (1.3), *for which the moment generating function* $M(x,y;a)$ *can be obtained by supposing that* $M(x,y;a) = N(f(x)+g(y);a)$, *where* f *and* g *contain no constant terms, are the ones where*

(3.51)
$$\partial D = \left\{(x,y) \in \mathbb{R}^2 : \left(\frac{K}{2}\ln x + K_{10}\right)^2 + \left(\frac{K}{2}\ln y + K_{20}\right)^2 = r \geq 0\right\},$$

where K, K_{10} *and* K_{20} *are constants and we may assume that* $K > 0$. *The function* $M(x,y;a)$ *is given by*

$$(3.52)\qquad M(x,y;a) = \frac{K_0\left(\frac{2}{K}\sqrt{a}\left[\left(\frac{K}{2}\ln x + K_{10}\right)^2 + \left(\frac{K}{2}\ln y + K_{20}\right)^2\right]\right)}{K_0\left(2\sqrt{ar/K}\right)}$$

if

$$(3.53)\qquad \left(\frac{K}{2}\ln x + K_{10}\right)^2 + \left(\frac{K}{2}\ln y + K_{20}\right)^2 \geq r.$$

When

$$(3.54)\qquad 0 \leq \left(\frac{K}{2}\ln x + K_{10}\right)^2 + \left(\frac{K}{2}\ln y + K_{20}\right)^2 \leq r,$$

the function K_0 *is replaced by* I_0 *in formula* (3.52).

Proof. Formula (3.52) is obtained directly from (3.48), (3.37), (3.17) and the formula

$$(3.55)\qquad g(y) = \frac{K}{4}\ln^2 y + K_{20}\ln y,$$

which corresponds to (3.17). □

4. The case when $M(x,y;a) = Q(p(x)q(y);a)$**.** To conclude this paper, we consider another special case of the method of similarity solutions: we suppose that the moment generating function M can be written as

$$(4.1)\qquad M(x,y;a) = Q(u;a),$$

where

$$u \doteq p(x)q(x). \tag{4.2}$$

We assume, as in the preceding section, that the functions p and q do not contain constant terms. We also assume that p and q are not of the form $p(x) = e^{f(x)}$ and $q(y) = e^{g(y)}$; otherwise, since we could write that $\ln(u) = \ln[e^{f(x)}e^{g(y)}] = f(x) + g(y)$, the solutions obtained in Section 3 would reappear.

In terms of Q, the Kolmogoroff backward equation (1.6) becomes (dropping the argument a)

$$\begin{aligned}[x^2(p'q)^2 + y^2(pq')^2]Q''(u) + [x^2p''q + y^2pq'' + xp'q \\ + ypq']Q'(u) = aQ(u).\end{aligned} \tag{4.3}$$

As in Section 3, there are three possibilities:
1) the function $Q(u)$ is linear in u:

$$Q(u) = \lambda + \theta u; \tag{4.4}$$

2) $Q(u)$ is an exponential function of the form

$$Q(u) = \lambda e^{\theta u}; \tag{4.5}$$

3) equation (4.3) may be rewritten as

$$\Phi(u)Q''(u) + \Psi(u)Q'(u) = aQ(u). \tag{4.6}$$

In the first case above, equation (4.3) becomes

$$[x^2p''q + y^2pq'' + xp'q + ypq']\theta = a[\lambda + \theta(pq)]. \tag{4.7}$$

Using the fact that p and q do not contain constant terms, we deduce that λ must be equal to zero. But, Q then has the form

$$Q(p(x)q(y); a) = \theta p(x)q(y), \tag{4.8}$$

so that the variables x and y can be separated. Thus, this case has already been treated in Section 2.

Next, when $Q(u)$ is the exponential function defined in (4.5), we must solve the differential equation

$$[x^2(p'q)^2 + y^2(pq')^2]\theta^2 + [x^2p''q + y^2pq'' + xp'q + ypq']\theta = a. \tag{4.9}$$

Here, because of our assumption concerning the functions p and q, we must conclude that equation (4.9) has no solutions, unless the parameter a is equal to zero, which is not allowed.

Finally, if equation (4.6) is valid, then (since we have assumed that $p(x) \neq e^{f(x)}$ and $q(y) \neq e^{g(y)}$ we have:

(4.10 a) $$x^2(p'(x))^2 = \varepsilon_1 p^2(x),$$

(4.10 b) $$y^2(q'(y))^2 = \varepsilon_2 q^2(y),$$

(4.10 c) $$x^2 p''(x) + xp'(x) = \delta_1 p(x),$$

(4.10 d) $$y^2 q''(y) + yq'(y) = \delta_2 q(y),$$

where ε_i and δ_i are constants, for $i = 1, 2$. But, equations (4.10 c) and (4.10 d) yield (see Section 2) that

(4.11) $$p(x) = \alpha_1 x^{\sqrt{\delta_1}} + \alpha_2 x^{-\sqrt{\delta_1}}$$

and

(4.12) $$q(y) = \beta_1 y^{\sqrt{\delta_2}} + \beta_2 y^{-\sqrt{\delta_2}},$$

where α_i and β_i are constants, for $i = 1, 2$. Hence, it is easy to see that the solutions in this case will be the same as those already obtained, in Section 2, by the method of separation of variables.

Remark. Differentiating equations (4.10 a) and (4.10 b) with respect to x and y, respectively, we find that we must have:

(4.13) $$\varepsilon_i = \delta_i$$

for $i = 1, 2$. Then, the functions $p(x)$ and $q(y)$ defined in (4.11) and (4.12) satisfy equations (4.10 a) and (4.10 b), respectively. We also find that the solutions will be of the form

(4.14) $$Q(p(x)q(y); a) = \left(\pm x^{\pm\sqrt{\delta_1/(\delta_1+\delta_2)}}\, y^{\pm\sqrt{\delta_2/(\delta_1+\delta_2)}}\right)^{\sqrt{a}}.$$

We have thus proved the proposition that follows.

Proposition 4.1. *Under the same hypotheses as in* Proposition 2.1, *there are no other first passage time problems, in the Cartesian coordinate system, than those already obtained in* Section 2 *and* Section 3, *for the bidimensional geometric Brownian motion defined by* (1.3), *for which the moment* *generating function* $M(x, y; a)$ *is of the form* $M(x, y; a) = Q(p(x)q(y); a)$, *where we assume that* p *and* q *contain no constant terms.*

Remark. Note that the method of separation of variables is a particular instance of the special case of the method of similarity solutions considered in Section 4. Indeed the method of separation of variables corresponds to the case when $Q(u) = u$ (assuming that the functions $p(x)$ and $q(y)$ may depend on a). Consequently, we could have considered only the method of similarity solutions.

Similarly, using the complex logarithm function, we can show that if a solution of the form $Q(p(x)q(y); a)$ exists, then this solution can be written in the form $N(f(x) + g(y); a)$. Thus, all the cases considered in this paper could be reduced to the case when $M(x, y; a) = N(f(x) + g(y); a)$.

5. Conclusion. In this paper, we have obtained explicit expressions for the moment generating functions of first passage times for the bidimensional geometric Brownian motion in the Cartesian coordinate system. We solved, under two hypotheses, all the problems that are solvable by using either the method of separation of variables or two special cases of a technique known as the method of similarity solutions.

The aim of the paper was threefold. Firstly, first passage time distributions are needed in many applications and explicit results in two or more dimensions are still few. Secondly, the geometric Brownian motion is a stochastic process which seems to become increasingly important, especially in economics applications. Thirdly, and perhaps the most important reason, this paper was written to show the usefulness of the method of similarity solutions for solving a partial differential equation such as the Kolmogoroff backward equation. This technique is not very popular, particularly in comparison with the method of separation of variables, and we feel that it deserves to be more widely used.

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