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Constructions of Lipschitzian Mappings with Non Zero Minimal Displacement in Spaces $L^1(0,1)$ and $L^2(0,1)$

ABSTRACT. The study of minimal displacement problem was initiated by Goe- bel in 1973 [3] and, while some further results have been obtained by Franchetti [1], Furi and Martelli [2], Reich [6] and [7], several major questions remain open. The aim of this paper is to show constructions of lipschitzian mappings with positive minimal displacement in spaces $L^1\,(0,1)$ and $L^2\,(0,1)$ which can be used as the first estimates from below of minimal displacement characteristic of X in those spaces.

Introduction. Let B, S be respectively, the unit ball and sphere in an infinitely dimensional Banach space X with norm $\|\cdot\|$. For any $k \geq 0$, let $L(k)$ denote the class of Lipschitz mappings $T : B \to B$ with constant k.

By $\psi_X(k)$ we will denote the minimal displacement characteristic of X

$$\psi_X(k) = \sup\left[\inf_{x\in B} \|x - Tx\|\right]$$

where supremum is taken over all mappings T belonging to $L(k)$. It is known that for any space X

$$\psi_X(k) \leq 1 - \frac{1}{k} \quad \text{for } k \geq 1.$$

There are some "square" spaces like $c_0, C[0,1]$ for which $\psi_X(k) = 1 - 1/k$. In the case of space l^1 we know only that $\psi_X(k) < 1 - 1/k$ and that $\psi_{l^1}(k) \le \psi_{L^1(0,1)}(k)$ but it is still unknown if $\psi_{L^1(0,1)}(k) = 1 - 1/k$ or not. Since 1973 [3] evaluation for Hilbert space H

$$\psi_H(k) \le (1 - 1/k)\sqrt{k/(k+1)}$$

has not been improved neither its exactness was shown. Our construction in the Hilbert space $L^2(0,1)$ which can be used as the estimate from below of $\psi_H(k)$ is far from the above and probably far from the real value of $\psi_H(k)$.

Construction in $L^1(0,1)$. Let us consider the unit ball B in $L^1(0,1)$. For any $f \in B$ and $k \ge 1$ define t_f as the solution of the equation

$$\int_0^t (1 + k|f(s)|)\, ds = 1$$

with respect to t. Set

$$(Tf)(t) = \begin{cases} 1 + k|f(t)| & \text{for } t \le t_f \\ 0 & \text{for } t > t_f. \end{cases}$$

Obviously $T: B \to B$ (more precisely $T: B \to S$). Suppose $f, g \in B$ with $t_f \le t_g$. Then

$$\begin{aligned} \|Tf - Tg\| &= \int_0^1 |(Tf)(t) - (Tg)(t)|\, dt \\ &= \int_0^{t_f} |k|f(t)| - k|g(t)||\, dt + \int_{t_f}^{t_g} (1 + k|g(t)|)\, dt \\ &\le k\int_0^{t_f} |f(t) - g(t)|\, dt + 1 - \int_0^{t_f} (1 + k|g(t)|)\, dt \\ &\le k\|f - g\| + \int_0^{t_f} (k|f(t)| - k|g(t)|)\, dt \le 2k\|f - g\| \end{aligned}$$

which shows that $T \in L(2k)$. Now we can calculate minimal displacement of T.

$$\|Tf - f\| = \int_0^1 |(Tf)(t) - f(t)|\, dt = \int_0^{t_f} |1 + k|f(t)| - f(t)|\, dt + \int_{t_f}^1 |f(t)|\, dt$$

$$\geq \int_0^{t_f} (1 + (k-1)|f(t)|)\, dt = t_f + (k-1)\int_0^{t_f} |f(t)|\, dt.$$

Because

$$1 = \int_0^{t_f} (1 + k|f(t)|)\, dt$$

so we obtain

$$\int_0^{t_f} |f(t)|\, dt = (1 - t_f)/k\,.$$

Finally we get

$$\begin{aligned} \|Tf - f\| &\geq t_f + (k-1)\int_0^{t_f} |f(t)|\, dt = t_f + (k-1)(1-t_f)/k \\ &= t_f/k + (k-1)/k \geq 1 - 1/k\,. \end{aligned}$$

Which means

$$\psi_{L^1(0,1)}(2k) \geq 1 - 1/k$$

so

$$\psi_{L^1(0,1)}(k) \geq 1 - 2/k\,.$$

This result can be slightly improved, by taking a tangent line to the graph from 1, because function ψ_X is concave with respect to 1 (see [3]). After easy calculations we get

$$\psi_{L^1(0,1)}(k) \geq \begin{cases} (3 - 2\sqrt{2})(k-1) & \text{for } 1 \leq k \leq 2 + \sqrt{2} \\ 1 - \frac{2}{k} & \text{for } k > 2 + \sqrt{2}\,. \end{cases}$$

Now, we show what happens with a construction similar to the above in the space $L^2(0,1)$.

Construction in $L^2(0,1)$. Let B and S denote, respectively, the unit ball and sphere in the Hilbert space $L^2(0,1)$ with standard norm and inner product. As in the previous construction, for $f \in B$ and $k \geq 1$ define t_f as the solution of the equation

$$\int_0^t (1 + k|f(s)|)^2 \, ds = 1$$

with respect to t and set

$$(Tf)(t) = \begin{cases} 1 + k|f(t)| & \text{for } t \leq t_f \\ 0 & \text{for } t > t_f. \end{cases}$$

Obviously $T : B \to B$ (more precisely $T : B \to S$). Suppose that $f, g \in B$ with $t_f \leq t_g$. Then

$$\begin{aligned} \|Tf - Tg\|^2 &= \int_0^1 |(Tf)(t) - (Tg)(t)|^2 \, dt \\ &= \int_0^{t_f} (k|f(t)| - k|g(t)|)^2 \, dt + \int_{t_f}^{t_g} (1 + k|g(t)|)^2 \, dt \\ &\leq k^2 \int_0^{t_f} (f(t) - g(t))^2 \, dt + 1 - \int_0^{t_f} (1 + k|g(t)|)^2 \, dt \\ &\leq k^2 \|f - g\|^2 + \int_0^{t_f} \left((1 + k|f(t)|)^2 - (1 + k|g(t)|)^2 \right) dt \\ &\leq k^2 \|f - g\|^2 + 2k \int_0^{t_f} |f(t) - g(t)| \, dt + k^2 \int_0^{t_f} \left((f(t))^2 - (g(t))^2 \right) dt. \end{aligned}$$

Because

$$\begin{aligned} \int_0^{t_f} |f(t) - g(t)| \, dt &\leq \sqrt{\int_0^{t_f} (f(t) - g(t))^2 \, dt} \sqrt{\int_0^{t_f} dt} \\ &\leq \sqrt{t_f} \|f - g\| \leq \|f - g\| \end{aligned}$$

and

$$\int_0^{t_f}\left((f(t))^2-(g(t))^2\right)dt \le \sqrt{\int_0^{t_f}(f(t)-g(t))^2\,dt}\sqrt{\int_0^{t_f}(f(t)+g(t))^2\,dt}$$
$$\le \|f-g\|\,\|f+g\| \le 2\,\|f-g\|\,.$$

So we finally get

$$\|Tf-Tg\|^2 \le k^2\,\|f-g\|^2+2k\,\|f-g\|+2k^2\,\|f-g\|$$
$$= k^2\,\|f-g\|^2+2k\,(k+1)\,\|f-g\|\,.$$

This shows that T is uniformly continuous. However (as my be checked) T is not lipschitzian. Nerveless, T may be used to produce a lipschitzian mapping but let us first calculate the minimal displacement of T.

$$\|Tf-f\|^2 = \int_0^1 ((Tf)(t)-f(t))^2\,dt$$

$$= \int_0^{t_f}(1+k\,|f(t)|-f(t))^2\,dt+\int_{t_f}^1 (f(t))^2\,dt$$

$$\ge \int_0^{t_f}(1+(k-1)\,|f(t)|)^2\,dt$$

$$= \int_0^{t_f}dt+2\,(k-1)\int_0^{t_f}|f(t)|\,dt+(k-1)^2\int_0^{t_f}(f(t))^2\,dt.$$

Because

$$1=\int_0^{t_f}(1+k\,|f(t)|)^2\,dt = t_f+2k\int_0^{t_f}|f(t)|\,dt+k^2\int_0^{t_f}(f(t))^2\,dt$$

so

$$\frac{k^2}{(k-1)^2}\|Tf-f\|^2 \ge \frac{k^2}{(k-1)^2}t_f+\frac{2k^2}{k-1}\int_0^{t_f}|f(t)|\,dt+k^2\int_0^{t_f}(f(t))^2\,dt$$

$$\ge t_f+2k\int_0^{t_f}|f(t)|\,dt+k^2\int_0^{t_f}(f(t))^2\,dt=1\,.$$

Which finally shows that

$$\|Tf - f\| \geq 1 - 1/k\,.$$

Now we can modify the mapping T to obtain a lipschitzian mapping. Let us take $\varepsilon > 0$ and choose a set $W \subset B$ with the following properties

(i) $\forall f, g \in W \quad f \neq g \quad \|f - g\| \geq \varepsilon$

(ii) $\forall f \in B \quad \operatorname{dist}(f, W) \leq \varepsilon, \quad$ where $\operatorname{dist}(f, W) = \inf_{g \in W} \|f - g\|$.

Let $T_1 = T_{|W}$. We claim T_1 is lipschitzian on W.
Indeed for any $f, g \in W$ we have

$$\|T_1 f - T_1 g\| \leq \sqrt{k^2 \|f-g\|^2 + 2k(k+1)\|f-g\|} \leq \sqrt{k^2 + \frac{2k(k+1)}{\varepsilon}}\,\|f-g\|\,.$$

By Kirzbraun's theorem $T_1 : W \to S$ may be extended to a mapping $T_2 : B \to B$ with the same Lipschitz constant. It is possible to calculate minimal displacement of T_2

$$\begin{aligned}\|f - T_2 f\| &\geq \|f_1 - T_2 f_1\| - (\|f - f_1\| + \|T_2 f_1 - T_2 f\|)\\ &\geq 1 - \frac{1}{k} - \varepsilon - \sqrt{k^2 + \frac{2k(k+1)}{\varepsilon}}\,\varepsilon\\ &= 1 - \frac{1}{k} - \varepsilon\left(1 + \sqrt{k^2 + \frac{2k(k+1)}{\varepsilon}}\right),\end{aligned}$$

where $f_1 \in W$ and $\|f - f_1\| \leq \varepsilon$. We obtain that

$$\psi_{L^2(0,1)}\left(\sqrt{k^2 + \frac{2k(k+1)}{\varepsilon}}\right) \geq 1 - \frac{1}{k} - \varepsilon\left(1 + \sqrt{k^2 + \frac{2k(k+1)}{\varepsilon}}\right)$$

which implies

$$\psi_{L^2(0,1)}(k) \geq 1 - \frac{2+\varepsilon}{\sqrt{1 + \varepsilon(\varepsilon+2)k^2} - 1} - \varepsilon(k+1)$$

for sufficiently large k.

This estimate strongly depends on the choice of ε. For instance for $k = 50$ almost optimal value of ε is $\varepsilon = 0.005$ and then $\psi_{L^2(0,1)}(50) > 0.25$.

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