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BILLY E. RHOADES

Using General Principles to Prove Fixed Point Theorems

ABSTRACT. Beginning in the 1970's many fixed point theorems have been proved using the same proof technique. The procedure is to define $\{x_n\}$ by $x_0 \in X$, $x_{n+1} = Tx_n$, and then observe that, setting $x = x_n$, $y = x_{n+1}$ in the contractive definition yields an inequality of the form $d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$ for some $0 \leq k \leq 1$. Sehie Park established some general fixed point theorems based on this observation.

This paper will begin with this work of Sehie Park, and then discuss several later papers by the author using general principles for a single map, a pair of maps, and will include some of the recent joint work of the author and Gwon Jeong for contractive definitions involving four maps.

Since the 1960's many fixed point theorems have been proved using the same proof technique that can be used to prove Banach's theorem. For one selfmap T, the procedure is to define a sequence $\{x_n\}$ by $x_0 \in X, x_{n+1} = Tx_n$, and then observe that, by setting $x = x_n$, $y = x_{n+1}$ in the contractive definition, one obtains an inequality of the form $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$ where $0 \leq k < 1$. It then follows that $\{x_n\}$ is Cauchy. Some kind of completeness condition is hypothesized to give convergence to a point z. In every case, the contractive condition is strong enough to imply that z is the unique fixed point of the map.

It makes sense, therefore, to hypothesize the procedure. Then all of these fixed point theorems are corollaries of that procedure.

In this paper we trace the history of this general principle, from single maps, to pairs of maps, and to more than two maps. In most cases we also provide an application of the procedure.

We begin with a 1979 result of the author and Troy Hicks. Let (X,d) be a complete metric space, T a selfmap of X, and $O(x) := \{x, Tx, T^2x, \dots\}$.

Theorem HR [7]. *Let $0 \leq h < 1$. Suppose there exists an x in X such that*

$$\text{(A)} \qquad d(Ty, T^2y) \leq hd(y, Ty) \quad \text{for each } y \in O(x).$$

Then

(i) $\lim_n T^n x = z$ *exists, and*

(ii) $d(T^n x, z) \leq \frac{h^n}{1-h} d(x, Tx)$.

(iii) *z is a fixed point of T if and only if $G(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at z.*

In applying Theorem HR to specific situations, it is often the case that the contractive definition is strong enough that condition (iii) is not needed. As an illustration, the author in [21] partially ordered 125 contractive definitions for a single map. Definition (21) of that paper is

$$\text{(21)} \qquad \begin{aligned} d(Tx, Ty) \leq k \max \Big\{ & d(x,y), d(x,Tx), d(y,Ty), \\ & [d(x,Ty) + d(y,Tx)]/2 \Big\}, \quad 0 \leq k < 1. \end{aligned}$$

Setting $y = Tx$ yields (A). Thus (i) holds. Substituting into (21) with $x = x_n$, $y = z$, one obtains

$$\begin{aligned} d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq k \max \Big\{ & d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \\ & [d(x_n, Tz) + d(z, x_{n+1})]/2 \Big\}. \end{aligned}$$

Taking the limit of both sides of the above inequality as $n \to \infty$ yields $d(z, Tz) \leq kd(z, Tz)$, which implies that $z = Tz$. Also (21) implies that the fixed point is unique.

Applications of Theorem HR to some other contractive definitions appear in [22].

Sehie Park also established some general fixed point theorems based on this observation. Theorem 2 of Park [18] strengthens Theorem HR by replacing the completeness of X with the completeness of $O(x)$.

Theorem P1 [18, Theorem 1]. *Let T be a selfmap of a metric space (X,d). If*

(i) *there exist a point $u \in X$ such that the orbit $O(u)$ has a cluster point $z \in X$,*

(ii) *T is orbitally continuous at z and Tz, and*

(iii) *T satisfies $d(Tx,Ty) < d(x,y)$ for all $x, y = Tx \in \overline{O}(u), x \neq y$,*

then z is a fixed point of T.

Some applications of Theorem P1 appear in [18] and [22].

A modest extension of Theorem HR is the following.

Proposition 1 [23]. *Let T be a selfmap of a metric space (X,d). Let $0 \leq h < 1$. Suppose there exists a point $x \in X$ such that*

(A) *$x_n := T^n x$ has a convergent subsequence with limit $z \in X$, and, for this x,*

(B) *$d(Ty, T^2y) \leq hd(y,Ty)$ for each $y \in O(x)$.*

Then, for this x,

(i) $\lim T^n x = z$,

(ii) $d(T^n x, z) \leq \frac{h^n}{1-h} d(x,Tx)$,

and

(iii) *z is a fixed point of T in X if and only if $G(x) := d(x,Tx)$ is lower semicontinuous at z.*

An application of Proposition 1 is the following.

Corollary 1 [8]. *Let T be a selfmap of a metric space (X,d) satisfying:*

(i) *for some $\alpha, \beta, \in [0,1)$ with $\alpha + \beta < 1$,*

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx) + d(x,y)} + \beta d(x,y)$$

for all $x, y \in X, x \neq y$.

(ii) *There exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X.*

Then T has a unique fixed point Z in X.

Proof. Set $x_0 = x$ and $y = Tx$ in (i). We may assume that $x \neq Tx$, since, otherwise, x is a fixed point of T. From (i)

$$d(Tx, T^2x) \leq \frac{\beta}{1-\alpha} d(x,Tx).$$

We may also assume that $T^n x \neq T^{n+1}x$ for each $n \geq 0$. For, otherwise, T has a fixed point. Thus the above inequality is true for each $y \in O(x)$ and (B) of Proposition 1 is satisfied. Thus Proposition 1 applies and $\lim T^n x = z$.

Suppose that $z \neq Tz$. Set $x = T^{n-1}x, y = z$ in (i) to get

$$d(T^n x, Tz) \leq \frac{\alpha d(T^{n-1}x, T^n x) d(z, Tz)}{d(T^{n-1}x, Tz) + d(z, T^n x) + d(T^{n-1}x, z)} + \beta d(T^{n-1}x, z).$$

Taking the limit as $n \to \infty$ gives $d(z, Tz) \leq 0$, a contradiction. Therefore z is a fixed point of T. Uniqueness follows from (i).

Other applications of Proposition 1 appear in [23].

Unfortunately, not every contractive condition involving a single map is a special case of Proposition 1. A simple example is the definition of Ćirić [4]:

$$(24) \quad d(Tx, Ty) \leq k \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}$$

for each x, y in X, where $0 \leq k < 1$, and there are others. However, many definitions, involving a rational expression, can be exploited by the techniques discussed in this paper.

Park [19] established the following general principles for a pair of maps.

Theorem P2 [19, Theorem 3.1]. *Let S and T be selfmaps of a metric space (X, d). If there exists a sequence $\{x_i\} \subset X$, where $x_{2i+1} := Sx_{2i}$, $x_{2i+2} := Tx_{2i+1}$, such that $\overline{\{x_i\}}$ is complete, and if there exists a $\lambda \in [0, 1)$ such that*

$$(1) \qquad d(Sx, Ty) \leq \lambda d(x, y)$$

for each distinct $x, y \in \overline{\{x_i\}}$ satisfying either $x = Ty$ or $y = Sx$, then either:

(i) S or T has a fixed point in $\{x_i\}$, or

(ii) $\{x_i\}$ converges to some $z \in X$ and

$$d(x_i, z) \leq \frac{\lambda^i d(x_0, x_1)}{1 - \lambda} \text{ for } i > 0.$$

Further, if either S or T is continuous at z and (1) holds for any distinct $x, y \in \overline{\{x_i\}}$, then z is a common fixed point of S and T.

Corollary 2 [1, Theorem 2]. *Let S,T be selfmaps of a complete metric space (X,d) satisfying*

$$(2)\qquad [d(Sx,Ty)]^2 \le a \max\Big\{d(x,Sx)d(y,Ty), d(x,y)d(x,Sx), d(x,y)d(y,Ty), c\, d(x,Ty)d(y,Sx)\Big\}$$

for all $x,y \in X$, where $0 < a < 1$, $0 \le c < 1$. Then S and T have a unique common fixed point.

Proof. Suppose p is a fixed point of T. We shall show that it is also a fixed point of S.

Setting $x = y = p$ in (2), we obtain $[d(Sp,p)]^2 \le 0$, and $Sp = p$. Similarly, if p is a fixed point of S, it is also a fixed point of T. Therefore condition (i) of Theorem P2 does not hold and (ii) applies.

Placing $x = x_{2n}$, $y = z$ in (2), we have

$$[d(Sx_{2n},Tz)]^2 \le a \max\Big\{d(x_{2n},x_{2n+1})d(z,Tz), d(x_{2n},z)d(x_{2n},x_{2n+1}), d(x_{2n},z)d(z,Tz), c\, d(x_{2n},Tz)d(z,x_{2n+1})\Big\}.$$

Taking the limit of the above inequality as $n \to \infty$ yields $[d(z,Tz)]^2 \le a[d(z,Tz)]^2$, and z is a fixed point of T, hence of S.

Condition (2) implies uniqueness of the fixed point.

Corollary 3 [5, Theorem 3]. *Let S,T be selfmaps of a complete metric space (X,d) satisfying*

$$(3)\qquad d(Sx,Ty) \le \frac{c\, d(x,Sx)d(y,Ty) + b\, d(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

for each $x,y \in X$ for which $d(x,Sx) + d(y,Ty) \ne 0$, and $d(Sx,Ty) = 0$ otherwise, where $b \ge 0, 1 < c < 2$. Then S and T have a unique common fixed point which is the unique fixed point of S and of T.

Proof. Suppose p is a fixed point of T and $p \ne Sp$. From (3), with $x = y = p$, we have $d(Sp,p) \le 0$, a contradiction. Therefore $p = Sp$. Similarly, $p = Sp$ implies that $p = Tp$.

Therefore condition (i) of Theorem P2 does not apply, and (ii) is satisfied.

Assume that $z \ne Tz$. With $x = x_{2n}, y = z$ in (3), we obtain

$$d(x_{2n+1},Tz) \le \frac{cd(x_{2n},x_{2n+1})d(z,Tz) + bd(x_{2n},Tz)d(z,x_{2n+1})}{d(x_{2n},x_{2n+1}) + d(z,Tz)}.$$

Taking the limit as $n \to \infty$ yields $d(z,Tz) \le 0$, a contradiction. Therefore z is a fixed point of T, hence a fixed point of S.

Condition (3) implies the uniqueness properties.

Corollary 4 [6, Theorem 1]. *Suppose that S and T are selfmaps of a complete metric space (X,d) such that*

$$(4) \qquad d(Sx,Ty) \le \frac{bd(x,Sx)d(x,Ty)+cd(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}$$

for each x,y in X, such that $d(x,Ty+d(y,Sx) \neq 0$ and $d(Sx,Ty)=0$ otherwise, where $b,c \ge 0, bc < 1$. Then S and T have a unique common fixed point.

Proof. Suppose that p is a fixed point of T and $p \neq Sp$. Using (4) with $x=y=p$ gives

$$d(Sp,p) \le \frac{0+0}{0+d(p,Sp)} = 0,$$

a contradiction. Similarly, if $p = Sp$, then it follows that $p = Tp$.

Therefore (ii) of Theorem P2 applies. Suppose that $z \neq Tz$. In (4) set $x = x_{2n}, y = z$ to obtain

$$d(x_{2n+1},Tz) \le \frac{bd(x_{2n},x_{2n+1})d(x_{2n},Tz)+cd(z,Tz)d(z,x_{2n+1})}{d(x_{2n},Tz)+d(z,x_{2n+1})}.$$

Taking the limit as $n \to \infty$ yields $d(z,Tz) \le 0$, a contradiction. Therefore $z = Tz$, and hence $z = Sz$. Condition (3) implies uniqueness.

Corollary 5 [13, Theorem 1]. *Let $T_i (i = 1,2,\ldots,k)$ be a finite family of mappings of X into itself. Suppose $T_iT_j = T_jT_i, i,j = 1,2,\ldots,k$. Suppose that there exist integers $m_i, n_j, i,j = 1,2,\ldots,k$ such that, for each $x,y \in X$, $S := T_1^{m_1}T_2^{m_2}\cdots T_k^{m_k}$ and $T := T_1^{n_1}T_2^{n_2}\cdots T_k^{n_k}$ satisfy*

$$(5) \qquad d(Sx,Ty) \le k\left\{\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}\right\}.$$

Then $\{T_i\}_{i=1}^k$ has a common fixed point.

Proof. We first note that the statement of Theorem 1 of [13] is incomplete, since one needs to place the restriction that (5) is true for those values of x and y for which $d(x,Ty)+d(y,Sx) \neq 0$.

Then the proof that (5) implies a unique common fixed point for S and T is the same as the proof of Corollary 4. Therefore S and T have a unique common fixed point. A straightforward argument then implies that the sequence $\{T_i\}_{i=1}^k$ has a common fixed point.

Corollary 6 [12, Theorem 1]. *Let (X,d) be a complete metric space, $\{T_i\}_{i=1}^{\infty}$ a sequence of selfmaps of X satisfying*

$$(6)\qquad d(T_i^p x, T^{jq} y) \le k\left\{\frac{d(x,T_i^p x)d(x,T_j^q y)+d(y,T_j^q y)d(y,T_i^p x)}{d(x,T_j^q y)+d(y,T_i^p x)}\right\}$$

for all $x,y \in X$, where $p,q \in \mathbf{N}, 0 \le k < 1$. Then $\{T_i\}$ has a unique common fixed point.

Proof. Again the statement of the result needs to be amended to exclude the values of x and y for which $d(x,T_j^q y)+d(y,T_i^p x)=0$.

If we define $S=T_i^p, T=T_j^q$, then (6) becomes (5).

Corollary 7 [16, Theorem 1]. *Let (X,d) be orbitally complete, T_1, T_2 orbitally continuous selfmaps of X satisfying*

$$\begin{aligned}&\min\Big\{d(T_1x,T_2y),d(x,T_1x),d(y,T_2y)\Big\}-\min\Big\{d(x,T_2y),d(y,T_1x)\Big\}\\ (7)\qquad &\le q\max\left\{\frac{d(x,T_2y)[d(x,y)+d(x,T_1x)+d(y,T_1x)]}{2[d(x,T_2y)+d(y,T_1x)]},\right.\\ &\qquad\qquad\left.\frac{d(y,T_1x)[d(x,y)+d(y,T_2y)+d(x,T_2y)]}{2[d(x,T_2y)+d(y,T_1x)]}\right\}\end{aligned}$$

for all $x,y \in X, q \in (0,1)$. Then T_1 and T_2 have a common fixed point.

Proof. The statement needs to be amended to exclude those x and y for which $d(x,T_2y)+d(y,T_1x)=0$. The conclusion needs to be amended to add: either T_1 or T_2 has a fixed point or T_1 and T_2 have a common fixed point.

Now apply Theorem P2. If (ii) holds, then, since $\lim x_{2n}=z, x_{2n+1}=z$, and T_1 and T_2 are orbitally continuous, z is a common fixed point of T_1 and T_2.

Corollary 8 [17, Theorem 1]. *Let S,T be selfmaps of a complete metric space (X,d) satisfying*

$$(8)\qquad \begin{aligned}[d(Sx,Ty)]^2 \le{}& \alpha\Big[d(x,Sx)d(y,Ty)+d(x,Ty)d(y,Sx)\Big]\\ &+\beta\Big[d(x,Sx)d(y,Sx)+d(x,Ty)d(y,Ty)\Big]\end{aligned}$$

for all $x,y \in X, \alpha,\beta > 0, \alpha+2\beta < 1$. Then S and T have a unique common fixed point.

Proof. Suppose p is a fixed point of T. Substituting $x=y=p$ into (8) gives $\Big[d(Sp,p)\Big]^2 \le \beta\Big[d(p,Sp)\Big]^2$, which implies that $p=Sp$. Similarly, $p=Sp$ implies $p=Tp$.

From Theorem P2 (ii) applies. Setting $x = s_{2n}, y = z$ in (8) yields

$$\Big[d(x_{2n+1}, Tz)\Big]^2 \leq \alpha\Big[d(x_{2n}, x_{2n+1})d(z, Tz) + d(x_{2n}, Tz)d(z, x_{2n+1})\Big] + \beta\Big[d(x_{2n}, x_{2n+1})d(z, x_{2n+1}) + d(x_{2n}, Tz)d(z, Tz)\Big].$$

Taking the limit as $n \to \infty$ gives $\Big[d(z, Tz)\Big]^2 \leq 0$, or $z = Tz$. Thus also $z = Sz$. Uniqueness of the fixed point follows from (8).

Corollary 9 [24, Theorem 2]. *Let (X, d) be a complete metric space. Let $T_i : X \to X, i = 1,2,3,4$ satisfying:*

$$\begin{aligned}\Big[d(T_1T_2x, T_3T_4y)\Big]^2 &\leq \alpha_1[d(x,y)]^2 + \alpha_2 d(x, T_1T_2x)d(y, T_3T_4y) \\ &+ \alpha_3 d(x, T_1T_2x)d(x, T_3T_4y) + \alpha_4 d(x, T_1T_2x)d(y, T_1T_2x) \\ &+ \alpha_5 d(y, T_3T_4y)d(x, T_3T_4y) + \alpha_6 d(y, T_3T_4y)d(y, T_1T_2x) \\ &+ \alpha_7 d(x, T_3T_4y)d(y, T_1T_2x) + \alpha_8 d(x,y)d(x, T_1T_2x) \\ &+ \alpha_9 d(x,y)d(y, T_1T_2x) + \alpha_{10} d(x,y)d(y, T_3T_4y) \\ &+ \alpha_{11} d(x,y)d(x, T_3T_4y)\end{aligned} \tag{9}$$

for each $x, y \in X$, where $\alpha_i \geq 0, \sum_{i=1}^{11} \alpha_i < 1$ and $\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1$. Further, assume that $T_1T_2 = T_2T_1$ and $T_3T_4 = T_4T_3$. Then the T_i have a unique common fixed point in X.

Proof. Define $S = T_1T_2, T = T_3T_4$. Then (8) becomes
(9´)

$$\begin{aligned}\Big[d(Sx, Ty)\Big]^2 &\leq \alpha_1[d(x,y)]^2 + \alpha_2 d(x, Sx)d(y, Ty) + \alpha_3 d(x, Sx)d(x, Ty) \\ &+ \alpha_4 d(x, Sx)d(y, Sx) + \alpha_5 d(y, Ty)d(x, Ty) \\ &+ \alpha_6 d(y, Ty)d(y, Sx) + \alpha_7 d(x, Ty)d(y, Sx) + \alpha_8 d(x,y)d(x, Sx) \\ &+ \alpha_9 d(x,y)d(y, Sx) + \alpha_{10} d(x,y)d(y, Ty) + \alpha_{11} d(x,y)d(x, Ty).\end{aligned}$$

Suppose that p is a fixed point of T. Then, setting $x = y = p$ in (9′) gives $\Big[d(Sp, p)\Big]^2 \leq \alpha_4\Big[d(p, Sp)\Big]^2$, and $p = Sp$. Similarly $p = Sp$ implies $p = Tp$.

Therefore condition (ii) of Theorem P2 applies. Setting $x = x_{2n}, y = z$

in (9′) yields

$$\begin{aligned}\left[d(x_{2n+1},Tz)\right]^2 &\le \alpha_1[d(x_{2n},z)]^2+\alpha_2 d(x_{2n},x_{2n+1})d(z,Tz)\\ &+\alpha_3 d(x_{2n},x_{2n+1})d(x_{2n},Tz)+\alpha_4 d(x_{2n},x_{2n+1})d(z,x_{2n+1})\\ &+\alpha_5 d(z,Tz)d(x_{2n},Tz)+\alpha_6 d(z,Tz)d(z,x_{2n+1})\\ &+\alpha_7 d(x_{2n},Tz)d(z,x_{2n+1})+\alpha_8 d(x_{2n},z)d(x_{2n},x_{2n+1})\\ &+\alpha_9 d(x_{2n},z)d(z,x_{2n+1})+\alpha_{10} d(x_{2n},z)d(z,Tz)\\ &+\alpha_{11} d(x_{2n},z)d(x_{2n},Tz).\end{aligned}$$

Taking the limit as $n \to \infty$ yields $\left[d(z,Tz)\right]^2 \le \alpha_5\left[d(z,Tz)\right]^2$, which implies that $z = Tz$, and hence that $z = Sz$.

A standard argument yields the uniqueness of z as the common fixed point of T_i, $i = 1,2,3,4$.

Corollary 10 [3]. *Let (X,d) be a complete metric space, $T_i : X \to X$, $i = 1,2,3,4$, satisfying*

$$\begin{aligned}\left[d(T_1T_2x,T_4T_3y)\right]^2 &\le \alpha_1\left[d(x,y)\right]^2+\alpha_2 d(x,T_1T_2x)d(y,T_4T_3y)\\ &+\alpha_3 d(x,T_4T_3y)d(y,T_1T_2x)+\alpha_4 d(x,y)d(x,T_1T_2x)\\ &+\alpha_5 d(x,y)d(y,T_4T_3y)\end{aligned}\tag{10}$$

for each $x,y \in X$, $\alpha_i \ge 0, \sum_{i=1}^5 \alpha_i < 1, \alpha_1+\alpha_3 < 1$. Further assume that $T_1T_2 = T_2T_1$ and $T_3T_4 = T_4T_3$. Then $T_i, i = 1,2,3,4$ have a unique common fixed point in X.

Proof. Note that (10) is a special case of (9).

Corollary 11 [20, Theorem 1]. *Let (X,d) be a complete metric space. Let $S,T,P : X \to X$ satisfying*

$$\begin{aligned}\left[d(SPx,TPy)\right]^2 \le a\left[d(x,y)\right]^2 &+ bd(x,SPx)d(y,TPy)\\ &+ cd(x,TPy)d(y,SPx)\end{aligned}\tag{11}$$

for each $x,y \in X$, where $a,b,c \ge 0, a+b < 1, a+c < 1$. Assume either that S commutes with P or T commutes with P. Then S,T, and P, have a unique common fixed point in X.

Proof. Define $S = SP, T = TP$. Then (11) is a special case of (9′).

Corollary 12 [2]. *Let S, T, P be selfmaps of a complete metric space (X, d) satisfying*
(12)
$$d(SPx, TPy) \leq \frac{\alpha d(y, TPy)\Big[1 + d(x, SPx)\Big]}{1 + d(x, y)} + \beta\Big[d(x, SPx) + d(y, TPy)\Big] + \gamma\Big[d(x, TPy) + d(y, SPx)\Big] + \delta d(x, y)$$
for each $x, y \in X$, where
$$0 \leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} < 1, \quad \beta + \gamma < 1, \quad 2\gamma + \delta < 1, \quad \gamma \geq 0.$$
Further, assume that $SP = PS$ and $TP = PT$. Then S, P, and T have a unique common fixed point in X.

Proof. Define $A = SP, B = TP$. Then (12) becomes
$$d(Ax, By) \leq \frac{\alpha d(y, By)\Big[1 + d(x, Ax)\Big]}{1 + d(x, y)} + \beta[d(x, Ax) + d(y, By)] + \gamma\Big[d(x, By) + d(y, Ax)\Big] + \delta d(x, y). \tag{12´}$$

Set $y = Ax$ in $(12')$ to obtain
$$d(Ax, BAx) \leq \alpha d(Ax, BAx) + \beta\Big[d(x, Ax) + d(Ax, BAx)\Big] + \gamma\Big[d(x, BAx) + 0\Big] + \delta d(x, Ax),$$
or
$$d(Ax, BAx) \leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} d(x, Ax),$$
and (1) of Theorem P2 is satisfied.

Suppose that p is a fixed point of B. In $(12')$ set $x = y = p$ to obtain
$$d(Ap, p) \leq 0 + \beta d(p\ Ap) + \gamma\Big[0 + d(p, Ap)\Big] + 0,$$
which implies that $p = Ap$. Similarly, $p = Ap$ implies that $p = Bp$.

Therefore condition (ii) of Theorem P2 applies. Setting $x = x_{2n}, y = z$ in $(12')$ we have
$$d(x_{2n+1}, Bz) \leq \frac{\alpha d(z, Bz)\Big[1 + d(x_{2n}, x_{2n+1})\Big]}{1 + d(x_{2n}, z)} + \beta\Big[d(x_{2n}, x_{2n+1}) + d(z, Bz)\Big] + \gamma\Big[d(x_{2n}, Bz) + d(z, x_{2n+1})\Big] + \delta d(x_{2n}, z).$$

Taking the limit as $n \to \infty$ yields $d(z, Bz) \le (\alpha + \beta + \gamma)d(z, Bz)$, which implies that $z = Bz$, and hence that $z = Az$.

A standard argument gives uniqueness of the common fixed point for S, P, and T.

Theorem P3 [19, Theorem 2.1]. *Let S and T be selfmaps of a metric space X. If*

(i) *there exists a sequence $\{x_i\} \subset X$ with $x_{2i+1} := Sx_{2i}, x_{2i+2} := Tx_{2i+1}$, such that $\{x_i\}$ has a cluster point $z \in X$,*

(ii) *S, T, ST and TS are continuous at z, and*

(iii) *S and T satisfy $d(Sx, Ty) < d(x, y)$ for distinct $x, y \in \overline{\{x_i\}}$ satisfying $x = Ty$ or $y = Sx$, then either*

(1) *S or T has a fixed point in $\{x_i\}$, or*

(2) *z is a common fixed point of S and T and $\lim x_n = z$.*

Other applications of Theorems P2 and P3 appear in [19], [22], and [23].

The following is a modest extension of Theorem P2 to the situation in which $\overline{\{x_i\}}$ is not complete.

Proposition 2 [23, Proposition 3]. *Let S and T be selfmaps of a metric space (X, d). Suppose that there exists a point x in X such that the sequence $\{x_i\}$ defined by $x_0 = x, x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ has a convergent subsequence with limit z in X. Suppose there exists a $\lambda \in [0, 1)$ such that (1) is satisfied for each distinct $x, y \in \{x_i\}$ satisfying either $x = Ty$ or $y = Sx$. Then either*

(i) *S or T has a fixed point in $\{x_i\}$ or*

(ii) *$\{x_i\}$ converges to z and $d(x_i, z) \le \lambda^i d(x_0, x_1)/(1 - \lambda)$ for $i > 0$.*

Corollary 13 [14, Theorem 1]. *Let (X, d) be a metric space, T_1, T_2 selfmaps of X such that*

$$d(T_1^r x, T_2^s y) \le \frac{\alpha d(x, T_1^r x) d(y, T_2^s y)}{d(x, T_2^s y) + d(y, T_1^r x) + d(x, y)} + \beta d(x, y) \tag{13}$$

for all x, y in $X, x \ne y$, where $r, s > 0$ are fixed integers and $\alpha, \beta \ge 0$ are such that $\alpha + \beta < 1$. If, for some x in X the sequence $\{x_i\}$, defined by $x_{2i+1} = T_1^r x_{2n}, x_{2n+2} = T_2^s x_{2n+1}$ has a convergent subsequence with limit point in X, then T_1 and T_2 have a unique common fixed point z in X.

Proof. The authors impose the condition $x \ne y$ to ensure that the denominator in (13) does not vanish. A better condition to impose would be to have (13) hold for all points x, y such that the denominator does not vanish.

For notational simplicity, define $S = T_1^r, T = T_2^s$. Then (13) becomes

$$(13') \qquad d(Sx,Ty) \leq \frac{\alpha d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} + \beta d(x,y),$$

Now evaluate (13′) at $x_0 = x, y = Sx$ to obtain

$$d(Sx,TSx) \leq \frac{\alpha d(x,Sx)d(Sx,TSx)}{d(x,TSx)+d(x,Sx)} + \beta d(x,Sx),$$

which implies that $d(Sx,TSx) \leq \beta d(x,Sx)/(1-\alpha)$, and (1) is satisfied. Therefore either condition (i) or (ii) of Proposition 2 holds.

We shall first show that, if S or T has a fixed point, then it is the unique common fixed point of S and T. Suppose that z is a fixed point of S, and assume that $z \neq Tz$. Then, from (13′),

$$d(z,Tz) = d(Sz,Tz) \leq \frac{\alpha d(z,Sz)d(z,Tz)}{d(z,Tz)+d(z,Sz)+0} = 0,$$

a contradiction. Therefore z is also a fixed point of T. Similarly, if z is a fixed point of T, it is also a fixed point of S. Condition (13′) implies uniqueness.

Suppose that conclusion (ii) of Proposition 2 is satisfied and assume that $z \neq Tz$. Substituting $x = x_{2n}, y = z$ in (13′), we have

$$d(Sx_{2n},Tz) \leq \frac{\alpha d(x_{2n},x_{2n+1})d(z,Tz)}{d(x_{2n},Tz)+d(z,Sx_{2n})+d(x_{2n},z)} + \beta d(x_{2n},z).$$

Taking the limit as $n \to \infty$ yields $d(z,Tz) \leq 0$, a contradiction. Therefore $z = Tz$, and, by what we have already shown, $z = Sz$. Moreover z is unique.

A standard argument then shows that z is the unique common fixed point of T_1 and T_2.

Other applications of Proposition 2 appear in [23].

Theorems involving more than two maps require some sort of commutativity condition in order to ensure the existence of a common fixed point. It appears not to be possible to obtain three-function or four-function analogs of the above results. However, one can make some general statements and we shall now turn to them.

We shall begin with the standard statements that (X,d) is a metric space and A,B,S,T are selfmaps of X such that

$$(14) \qquad A(X) \subset T(X) \text{ and } B(X) \subset S(X).$$

Condition (14) guarantees that it is possible to define a sequence $\{x_n\}$ as follows. Pick $x_0 \in X$, choose x_1 so that $Tx_1 = Ax_0$, x_2 so that $Sx_2 = Bx_1$, and, in general, define $\{x_n\}$ so that $Tx_{2n+1} = Ax_{2n}, Sx_{2n+2} = Bx_{2n+1}$. Now define $\{y_n\}$ by $y_{2n} = Sx_{2n}, y_{2n+1} = Tx_{2n+1}$.

From [9] we have the following.

Lemma 1. *Let A, B, S, and T be selfmaps of a metric space (X, d) satisfying (14). Assume that $\{y_n\}$ is complete. Suppose that there exists a $\lambda \in [0, 1)$ such that*

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \text{ for all } y_n \neq y_{n+1}. \tag{15}$$

Then, either

(a) *A and S have a common coincidence point,*
(b) *B and T have a common coincidence point,*
(c) *A, S, and T have a common coincidence point,*
(d) *B, S, and T have a common coincidence point, or*
(e) *$\{y_n\}$ converges to a point $z \in X$, and*

$$d(y_i, z) \leq \frac{\lambda^i}{1-\lambda} d(y_0, y_1) \text{ for each } i > 0.$$

Proof. Suppose that $y_{2n} = y_{2n+1}$ for some n. Then $Sx_{2n} = Tx_{2n+1} = Ax_{2n}$, and (a) is satisfied. If also $x_{2n} = x_{2n+1}$, then $Tx_{2n} = Tx_{2n+1}$, and (c) is satisfied. If $y_{2n+1} = y_{2n+2}$ for some n, then a similar argument yields (b) or (d).

Suppose now that $y_n \neq y_{n+1}$ for each n. Then, from (15),

$$d(y_n, y_{n+1}) \leq \lambda^n d(y_0, y_1), \tag{16}$$

and hence $\{y_n\}$ is Cauchy. Since the orbit is complete, there exists a point z with $\lim y_n = z$.

For any positive integers i, n, using the triangular inequality and (16),

$$\begin{aligned} d(y_i, y_{n+i}) &\leq \sum_{k=0}^{n-1} d(y_{i+k}, y_{i+k+1}) \leq \sum_{k=0}^{n-1} \lambda^{i+k} d(y_0, y_1) \\ &= \lambda^i d(y_0, y_1) \sum_{k=0}^{n-1} \lambda^k = \frac{\lambda^i(1-\lambda^n)}{1-\lambda} d(y_0, y_1) \leq \frac{\lambda^i d(y_0, y_1)}{1-\lambda}. \end{aligned}$$

Taking the limit as $n \to \infty$ gives $d(y_i, z) \leq \lambda^i d(y_0, y_1)/(1-\lambda)$.

A pair of maps S and T are said to be compatible (see, e.g., Jungck [10]) if, whenever $\{x_n\} \subset X$ is such that $\lim Tx_n = \lim Sx_n = t \in X$, then $\lim d(STx_n, TSx_n) = 0$. Two maps are said to be weakly compatible if they commute at coincidence points (see, e.g., Jungck and Rhoades [11]).

Corollary 14 [26, Theorem 3.1]. *Let A, B, S and T be self-mappings on a complete metric space (X, d) satisfying conditions* (14) *and*

$$
\begin{aligned}
[d(Ax, By)]^2 \leq & c_1 \max\left\{\left[d(Sx, Ax)\right]^2, \left[d(Ty, By)\right]^2, \left[d(Sx, Ty)\right]^2\right\} \\
& + c_2 \max\left\{d(Sx, Ax)d(Sx, By)d(Ax, Ty)d(By, Ty)\right\} \\
& + c_3 d(Sx, By)d(Ty, Ax)
\end{aligned} \tag{17}
$$

for all x, y in X, where $c_1, c_2, c_3 \geq 0, c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$. Suppose that one of the maps is continuous. If A and B are compatible with S and T respectively, then A, B, S and T have a common fixed point z. Further z is the unique common fixed point of A and S and of B and T.

Proof. As in [26], we obtain $d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n)$, where $h^2 = (2c_1 + 3c_2)/(2 - c_2) < 1$. Thus, if there exists an n such that $y_n = y_{n+1}$, then $y_{n+k} = y_n$ for all $k \geq 0$. Hence there exists a value p such that $Ap = Sp$ and there exists a q such that $Bq = Tq$. Moreover $Ap = Bq$.

Substituting $x = Sp, y = q$ into (17) we have

$$
\begin{aligned}
\left[d(ASp, Bq)\right]^2 \leq & c_1 \max\left\{\left[d(S^2p, ASp)\right]^2, \left[d(Tq, Bq)\right]^2, \left[d(S^2p, Tq)\right]^2\right\} \\
& + c_2 \max\left\{d(S^2p, ASp)d(S^2p, Bq)d(ASp, Tq)d(Bq, Tq)\right\} \\
& + c_3 d(S^2p, Bq)d(Tq, ASp).
\end{aligned}
$$

Since S and A are compatible, they commute at coincidence points. Therefore $S^2p = SSp = SAp = ASp$, and the above inequality becomes

$$
\left[d(ASp, Bq)\right]^2 \leq c_1 \left[d(S^2p, Tq)\right]^2 + c_3 d(S^2p, Bq)d(Tq, ASp),
$$

and

$$
\left[d(ASp, Bq)\right]^2 \leq \frac{c_3}{1 - c_1} d(S^2p, Bq)d(Tq, ASp) = \frac{c_3}{1 - c_1}\left[d(ASp, Bq)\right]^2
$$

which implies that $ASp = Bq = Sp$ and Sp is a fixed point of A.

Now we set $x = p, y = Tq$ in (17) to obtain

$$
\begin{aligned}
\left[d(Ap, BTq)\right]^2 \leq & c_1 \max\left\{\left[d(Sp, Ap)\right]^2, \left[d(T^2q, BTq)\right]^2, \left[d(Sp, T^2q)\right]^2\right\} \\
& + c_2 \max\left\{d(Sp, Ap)d(Sp, BTq), d(Ap, T^2q)d(Bq, Tq)\right\} \\
& + c_3 d(Sp, BTq)d(T^2q, Ap).
\end{aligned}
$$

Since B and T are compatible, $T^2q = T(Tq) = TBq = BTq$, and the above inequality becomes

$$\left[d(Ap, BTq)\right]^2 \le c_1\left[d(Sp, T^2q)\right]^2 + c_3 d(Sp, BTq)d(T^2q, Ap)$$
$$= (c_1 + c_3)\left[d(Ap, BTq)\right]^2,$$

which implies that $Ap = BTq = BAp$, and $Ap = Sp$ is a fixed point of B.

Now $S^2p = S(Sp) = S(Ap) = A(Sp) = Sp$ and Sp is a fixed point of S. Similarly, Sp is a fixed point of T. Therefore Sp is a common fixed point of A, B, S, and T.

Now assume that $y_n \ne y_{n+1}$ for any n. Then condition (e) of Lemma 1 holds; i.e., $\lim Sx_{2n} = \lim Tx_{2n+1} = \lim Ax_{2n} = \lim Bx_{2n-1} = z$. Assume that S is continuous. In (17) set $x = Sx_{2n}, y = x_{2n-1}$ to obtain

$$\left[d(ASx_{2n}, Bx_{2n-1})\right]^2$$
$$\le c_1 \max\left\{\left[d(S^2x_{2n}, ASx_{2n})\right]^2, \left[d(Tx_{2n-1}, Bx_{2n-1})\right]^2, \left[d(S^2x_{2n}, Tx_{2n-1})\right]^2\right\}$$
$$+ c_2 \max\left\{d(S^2x_{2n}, ASx_{2n})d(S^2x_{2n}, Bx_{2n-1}), d(ASx_{2n}, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\right\}$$
$$+ c_3 d(S^2x_{2n}, Bx_{2n-1})d(Tx_{2n-1}, ASx_{2n}).$$

Since A and S are compatible, and $\lim Ax_{2n} = \lim Sx_{2n} = z$, $\lim d(ASx_{2n}, SAx_{2n}) = 0$. Since S is continuous, $\lim SAx_{2n} = Sz$, and hence $\lim Asx_{2n} = z$. Taking the limit of the above inequality as $n \to \infty$ yields $\left[d(Sz, z)\right]^2 \le (c_1 + c_3)\left[d(Sz, z)\right]^2$, which implies that $Sz = z$. Using (17) with $x = z, y = x_{2n-1}$, we have

$$\left[d(Az, Bx_{2n-1})\right]^2 \le c_1 \max\left\{\left[d(Sz, Az)\right]^2, \left[d(Tx_{2n-1}, Bx_{2n-1})\right]^2, \left[d(Sz, Tx_{2n-1})\right]^2\right\}$$
$$+ c_2 \max\left\{d(Sz, Az)d(Sz, Bx_{2n-1}), d(Az, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\right\}$$
$$+ c_3 d(Sz, Bx_{2n-1})d(Tx_{2n-1}, Az).$$

Taking the limit as $n \to \infty$ yields

$$\left[d(Az,z)\right]^2 \le c_1\left[d(Sz,Az)\right]^2 = c_1\left[d(z,Az)\right]^2,$$

and $Az = z$.

From condition (14) there exists a point u in X such that $z = Az = Sz = Tu$. Using (17) with $x = z, y = u$ gives

$$\begin{aligned}\left[d(Az,Bu)\right]^2 &\le c_1 \max\left\{\left[d(Sz,Az)\right]^2, \left[d(Tu,Bu)\right]^2, \left[d(Sz,Tu)\right]^2\right\}\\ &+ c_2 \max\left\{d(Sz,Az)d(Sz,Bu), d(Az,Tu)d(Bu,Tu)\right\}\\ &+ c_3 d(Sz,Bu)d(Tu,Az) = c_1\left[d(z,Bu)\right]^2,\end{aligned}$$

and $Az = z = Bu$. Since B and T are compatible, $Tz = TBu = BTu = Bz$. Using (17) with $x = y = z$, we have

$$\begin{aligned}\left[d(Az,Bz)\right]^2 &\le c_1 \max\left\{\left[d(Sz,Az)\right]^2, \left[d(Tz,Bz)\right]^2, \left[d(Sz,Tz)\right]^2\right\}\\ &+ c_2 \max\left\{d(Sz,Az)d(Sz,Bz), d(Az,Tz)d(Bz,Tz)\right\}\\ &+ c_3 d(Sz,Bz)d(Tz,Az) = (c_1+c_3)\left[d(Az,Bz)\right]^2\end{aligned}$$

and $Az = Bz$. Thus z is a common fixed point of A, B, S, and T.

The proof for T continuous is similar.

Suppose that A is continuous. Setting $x = Ax_{2n}, y = x_{2n-1}$ is (17) gives

$$\begin{aligned}\left[d(A^2x_{2n},Bx_{2n-1})\right]^2 \le c_1 \max\Big\{&\left[d(SAx_{2n},A^2x_{2n})\right]^2, \left[d(Tx_{2n-1},Bx_{2n-1})\right]^2,\\ &\left[d(SAx_{2n},Tx_{2n-1})\right]^2\Big\}\\ + c_2 \max\Big\{&d(SAx_{2n},A^2x_{2n})d(SAx_{2n},Bx_{2n-1}),\\ &d(A^2x_{2n},Tx_{2n-1})d(Bx_{2n-1},Tx_{2n-1})\Big\}\\ + c_3 d(SAX_{2n},&Bx_{2n-1})d(Tx_{2n-1},A^2x_{2n}).\end{aligned}$$

Taking the limit as $n \to \infty$ yields $\left[d(Az,z)\right]^2 \le (c_1+c_3)\left[d(Az,z)\right]^2$, and $Az = z$.

Using condition (14) there exists a point v in X such that $Tv = Az = z$. Using (17) with $x = Ax_{2n}, y = z$, we obtain

$$\Big[d(A^2x_{2n}, Bv)\Big]^2 \leq c_1 \max\Big\{\Big[d(SAx_{2n}, A^2x_{2n})\Big]^2, \Big[d(Tv, Bv)\Big]^2, \Big[d(SAx_{2n}, Tv)\Big]^2\Big\}$$
$$+ c_2 \max\Big\{d(SAx_{2n}, A^2x_{2n})d(SAx_{2n}, Bv), d(A^2x_{2n}, Tv)d(Bv, Tv)\Big\}$$
$$+ c_3 d(SAx_{2n}, Bv)d(Tv, A^2x_{2n}).$$

Taking the limit as $n \to \infty$ yields $\Big[d(Az, Bv)\Big]^2 \leq c_1\Big[d(z, Bv)\Big]^2$ and $z = Bv = Tv$. Since B and T are compatible, $Bz = BTv = TBv = Tz$.

Using (17) with $x = x_{2n}, y = z$ gives

$$\Big[d(Ax_{2n}, Bz)\Big]^2 \leq c_1 \max\Big\{\Big[d(Sx_{2n}, Ax_{2n})\Big]^2, \Big[d(Tz, Bz)\Big]^2, \Big[d(Sx_{2n}, Tz)\Big]^2\Big\}$$
$$+ c_2 \max\Big\{d(Sx_{2n}, Ax_{2n})d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)d(Bz, Tz)\Big\}$$
$$+ c_3 d(Sx_{2n}, Bz)d(Tz, Ax_{2n}).$$

Taking the limit as $n \to \infty$ we have $\Big[d(z, Bz)\Big]^2 \leq (c_1 + c_3)\Big[d(z, Bz)\Big]^2$, and $Bz = z$.

Using condition (14) there exists a point w in X such that $Sw = Bz = z$. Using (17) with $x = w, y = z$ gives

$$\Big[d(Aw, z)\Big]^2 \leq c_1 \max\Big\{\Big[d(Sw, Aw)\Big]^2, \Big[d(Tz, Bz)\Big]^2, \Big[d(Sw, Tz)\Big]^2\Big\}$$
$$+ c_2 \max\Big\{d(Sw, Aw)d(Sw, Bz), d(Aw, Tz)d(Bz, Tz)\Big\}$$
$$+ c_3 d(Sw, Bz)d(Tz, Aw) = c_1\Big[d(Aw, z)\Big]^2,$$

and $Aw = z$. Since A and S are compatible and $Aw = Sw$, we have $ASw = SAw$, or $Az = Sz$. Thus z is a common fixed point of A, B, S, and T.

The proof for B continuous is similar.

Uniqueness of the fixed point follows from (17).

Other applications appear in [9].

There are two contractive forms for three maps. One is obtained by setting $T = S$ and the other is obtained by setting $B = A$. Also for three maps we can prove slightly more general results. For the situation in which $T = S$, set $x_0 \in X$ and define $\{x_n\}$ by $Ax_{2n} = Sx_{2n+1}, Bx_{2n+1} = Sx_{2n+2}, y_{2n} := Sx_{2n}, y_{2n+1} := Sx_{2n+1}$.

Lemma 2 [9]. *Let A, B, S be selfmaps of a metric space X such that $A(X) \cup B(X) \subset S(X)$. Suppose there exists a $\lambda \in [0,1)$ such that*

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \quad \text{for} \quad y_n \neq y_{n+1}.$$

Assume that $\{y_n\}$ is complete. Then, either

(a) *A and S have a coincidence point,*

(b) *B and S have a coincidence point, or*

(c) *$\{y_n\}$ converges to a point $z \in X$ and*

$$d(y_n, z) \leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \quad \text{for each } i > 0.$$

Corollary 15 [25, Theorem 1]. *Let (X,d) be a complete metric space, S, A, T three continuous selfmaps of X such that $SA = AS$, $AT = TA$, $S(X) \subset A(X)$, $T(X) \subset A(X)$, and satisfying*

$$\begin{aligned}\Big[d(Sx,Ty)\Big]^2 &\leq a_1 d(Ax,Sx)d(Ay,Ty) + a_2 d(Ax,Ty)d(Ay,Sx) \\ &+ a_3 d(Ax,Sx)d(Ax,Ty) + a_4 d(Sx,Ay)d(Ty,Ay) \\ &+ a_5\Big[d(Ax,Ay)\Big]^2\end{aligned} \tag{18}$$

for each $x, y \in X$, where $a_i \geq 0$, $a_1 + a_4 = a_5 < 1$, $2a_1 + 3a_3 + 2a_5 < 2$. Then S, A, and T have a unique common fixed point in X.

Proof. As in [25], $d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n)$ for each $n \geq 1$ and condition (14) is satisfied.

If there exists an n for which $y_n = y_{n+1}$, then $y_{n+k} = y_n$ for each $k \geq 0$. Suppose that p is a coincidence point for A and S. Then, using (18) with $x = y = p$ gives $[d(Sp,Tp)]^2 \leq 0$ and p is also a coincidence point for T. Similarly, if p is a coincidence point for A and T, then it is also a coincidence point for S. Thus A, S, and T have a common coincidence point.

Suppose that condition (c) of Lemma 3 is satisfied. Then $\lim Ax_n = z$. Since $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequences of $\{Ax_n\}$, they also converge to z. Since A, S, and T are continuous, and A commutes with S and T, $Az = \lim ASx_{2n} = \lim SAx_{2n} = Sz$. Also, $Az = \lim ATx_{2n+1} = \lim TAx_{2n+1} = Tz$, and z is a common coincidence point of A, S, and T. Thus, in all cases we obtain a common coincidence point.

The remainder of the proof follows as in [25].

Other applications of Lemma 2 appear in [9].

The ideas of this paper can obviously be extended to theorems dealing with multivalued maps, theorems involving d-complete spaces, and to probabilistic metric spaces.

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Department of Mathematics
Indiana University
Bloomington, IN 47405-5701, U.S.A.
e-mail: rhoades@ucs.indiana.edu

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