ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA
VOL. LI. 2, 20 SECTIO A 1997

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Fixed Points in Homeomorphically Convex Sets

ABSTRACT. We obtain new fixed point theorems for the admissible class $\mathfrak{A}_c^\kappa$ of multimaps defined on admissible subsets X (in the sense of Klee) of not-necessarily locally convex topological vector spaces. It is shown also that X can be homeomorphically convex.

1. Introduction and preliminaries. In this paper we obtain new fixed point theorems for the admissible class $\mathfrak{A}_c^\kappa$ of multimaps defined on admissible subsets (in the sense of Klee) of not-necessarily locally convex topological vector spaces. Our new results properly generalize a large number of historically well-known theorems.

A *multimap*, or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

Given two maps $T : X \multimap Y$ and $S : Y \multimap Z$, their *composite* $ST : X \multimap Z$ is defined by $(ST)(x) = S(T(x))$ for $x \in X$.

1991 *Mathematics Subject Classification.* Primary 47H10, 54C60; Secondary 54H25, 55M20.

Key words and phrases. Multimap (closed, compact, u.s.c., l.s.c., continuous), acyclic, polytope, admissible class of multimaps, admissible set (in the sense of Klee), the Schauder fixed point theorem.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x,y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^{-1}(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that composites of u.s.c. maps are u.s.c.; the image of a compact set under an u.s.c. map with compact values is compact; and every u.s.c. map T with closed values is closed.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with compact acyclic values.

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co denotes the convex hull. A *polytope* is a convex hull of a nonempty finite subset of a t.v.s., or a compact convex subset of a finite dimensional subspace.

For any topological spaces X and Y and a given class $\mathbb{X}$ of maps, $\mathbb{X}(X,Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued;
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P,P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex sets), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in t.v.s., admissible maps in the sense of Górniewicz, permissible maps of Dzedzej; for references, see [P1,5].

We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

Note that $\mathbb{K}_c^\sigma$ due to Lassonde [L] and $\mathbb{V}_c^\sigma$ due to Park *et al.*[PSW] are examples of $\mathfrak{A}_c^\sigma$. An approximable map defined by Ben-El-Mechaiekh and

Idzik [BI] belongs to $\mathfrak{A}_c^\kappa$. Moreover, any u.s.c. compact map defined on a closed subset of a locally convex t.v.s. with closed values is approximable whenever its values are all (1) convex, (2) contractible, (3) decomposable, or (4) ∞-proximally connected, see [BI].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any class $\mathfrak{A}_c^\kappa$ will be called *admissible.* For details, see [P1-3, PK1, 2].

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [K]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, and others. Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. are admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [H], Weber [W1,2], and references therein.

2. Main results.

In our previous works [P1, 2], it is shown that if X is a nonempty convex subset of a locally convex t.v.s., then any compact map in $\mathfrak{A}_c^\sigma(X,X)$ has a fixed point, and furthermore if X is compact, then any map in $\mathfrak{A}_c^\kappa(X,X)$ has a fixed point. Those two results are extended as follows:

Theorem 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $T \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Proof. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E and let $V \in \mathcal{V}$. Since $\overline{T(X)}$ is a compact subset of the admissible subset X, there exist a continuous function $f : \overline{T(X)} \to X$ and a finite dimensional subspace L of E such that $x - f(x) \in V$ for all $x \in \overline{T(X)}$ and $f(\overline{T(X)}) \subset L \cap X$. Let $M := f(\overline{T(X)})$. Then M is a compact subset of L and hence $K := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $f : \overline{T(X)} \to K$ and $T|_K : K \to \overline{T(X)}$. Since $T \in \mathfrak{A}_c^\kappa(X,X)$ and K is a compact subset of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, \overline{T(X)})$ such that $\Gamma(x) \subset T(x)$ for all $x \in K$. Then the composite $f\Gamma : K \to K$ belongs to $\mathfrak{A}_c(K,K)$ and hence, has a fixed point $x_V \in f\Gamma(x_V)$. Let $x_V = f(y_V)$ for some $y_V \in \Gamma(x_V) \subset \overline{T(X)}$. Since $\overline{T(X)}$ is compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$ and hence $\hat{x} \in K$.

Since the graph of Γ is closed in $K \times \Gamma(K)$, we have $\hat{x} \in \Gamma(\hat{x}) \subset T(\hat{x})$. This completes our proof.

Remark. As we have seen in our previous works [P1,2], Theorem 1 is a far-reaching generalization of historically well-known results due to Brouwer, Schauder, Tychonoff, Mazur, Kakutani, Hukuhara, Bohnenblust and Karlin, Fan, Glicksberg, Rhee, Himmelberg, Powers, Granas and Liu, Simons, Lassonde, and Ben-El-Mechaiekh *et al.* For the literature, see [P3,5].

A particular form of Theorem 1 for acyclic maps due to Park [P4] was applied to prove the existence of solutions of quasi-equilibrium problems.

As an application of Theorem 1, we show that the convexity of the set X in Theorem 1 is not essential. In fact, Theorem 1 holds for homeomorphically convex sets as follows:

Theorem 2. *Let E and F be t.v.s. and X a subset of E which is homeomorphic to an admissible convex subset Δ of F. Then any compact map $T \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

Proof. Let $h : \Delta \to X$ be the homeomorphism. Then the composite $h^{-1}Th : \Delta \to \Delta$ belongs to $\mathfrak{A}_c^\kappa(\Delta, \Delta)$. Since T is compact, so is $h^{-1}Th$. Therefore, by Theorem 1, there exists an $z_0 \in \Delta$ such that $z_0 \in h^{-1}Th(z_0)$ or equivalently $h(z_0) \in Th(z_0)$. Hence $\hat{x} = h(z_0)$ is a fixed point of T. This completes our proof.

Remark. Theorem 2 is motivated by recent works of Clarke, Ledyaev, and Stern [C1,2] on the existence of zeros and fixed points of multimaps in nonconvex sets.

As an application of Theorem 2, we have the following new Fan-Browder type fixed point theorem for compact maps:

Theorem 3. *Let E be a t.v.s. and X a subset of E which is homeomorphic to an admissible convex set. Let $S, T : X \multimap X$ be compact maps such that*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*
(2) *$\{\operatorname{Int} S^{-1}(y)\}_{y \in X}$ covers X.*

Then T has a fixed point.

Proof. It is well-known that, for each compact subset K of X, the map $T|_K$ has a continuous selection. Then $T \in \mathfrak{C}_c^\kappa(X, X) \subset \mathfrak{A}_c^\kappa(X, X)$. Therefore, by Theorem 2, T has a fixed point.

Remark. If X itself is compact and convex, then Theorem 3 holds without assuming the admissibility of X. This is usually called the Fan-Browder fixed point theorem and has numerous applications. For far-reaching generalizations of the theorem, see Park and Kim [PK2]. Note that Ben-El-Mechaiekh [B] obtained a particular form of Theorem 3 for a locally convex t.v.s. E.

Now, we raise the following general form of the Schauder conjecture:

Problem. *Does a convex subset of a (metrizable) t.v.s. have the fixed point property for compact maps in* $\mathfrak{A}_c^\kappa$*?*

If the answer is affirmative, then admissibility can be eliminated in Theorems 1-3.

Acknowledgement. This research is supported in part by the Non-directed Research Fund, Korea Research Foundation, 1997.

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received September 10, 1997