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A New Constant in Banach Spaces and Stability of the Fixed Point Property

ABSTRACT. We define a new geometric coefficient $\lambda(X)$ for a Banach space X and we prove that a Banach space Y has the weak fixed point property for nonexpansive mappings if for some isomorphic space X we have

$$d(X,Y) < \frac{1+\sqrt{1+4\left(\frac{\lambda(X)}{WCS(X)}+1\right)}}{2\left(\frac{\lambda(X)}{WCS(X)}+1\right)}$$

where $WCS(X)$ is the Bynum's weakly convergent sequence coefficient.

Introduction. Let (M,d) be a metric space. A mapping $T: M \to M$ is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for every $x,y \in M$. A Banach space X is said to have the weak fixed point property (w-FPP) for nonexpansive mappings if every nonexpansive mapping T defined from a nonempty convex weakly compact subset C of X into C has a fixed point. Although isolated results about fixed points for nonexpansive mappings

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appeared earlier, the active research in this branch of nonlinear functional analysis began after the publication in 1965 of two papers by Browder [B] and Kirk [K] in which they respectively proved that every uniformly convex Banach space and every Banach space with a weak normal structure have the w-FPP.

In 1981 Alspach [A] proved that $L_1[0,1]$ fails to have the w-FPP. This example and a result given by V. Dulst [D], who proved that every separable Banach space can be renormed in such a way that the new norm satisfies the Opial condition, allows us to check that the w-FPP is not preserved under isomorphisms. This fact raises the following natural question which is known as the stability problem of the w-FPP for nonexpansive mappings: Let X be a Banach space with the w-FPP. What is the greatest number $k = k(X)$ such that, if Y is an isomorphic space and the Banach-Mazur distance between X and Y is less than k, then Y has the w-FPP? Although the value of $k(X)$ is unknown for most classic Banach space (including Hilbert spaces), several authors have found lower bound for $k(X)$ using either geometric coefficients or special geometric properties which are satisfied by X.

A fixed point theorem. We start recalling the definition of the weakly convergent sequence coefficient [By], for a Banach space X without the Schur property.

$$WCS(X) = \inf\left\{ \frac{\operatorname{diam}_a\{x_n\}}{r_a\{x_n\}} : \{x_n\} \text{ is a weakly convergent sequence which is not norm convergent} \right\}$$

In [ADL] the authors proved that $WCS(X)$ can be also defined by a different expression:

$$WCS(X) = \inf\left\{ \frac{\lim_{n,m;n\neq m} \|x_n - x_m\|}{\lim \|x_n\|} : \{x_n\} \text{ is a weakly null sequence and } \lim_{n,m;n\neq m} \|x_n - x_m\|,\ \lim \|x_n\| \text{ exist} \right\}$$

Definition. Let X be a Banach space without the Schur property. We define the coefficient

$$\lambda(X) = \sup\{ \limsup \|x_n + x\| - \lim_{n,m;\, n\neq m} \|x_n - x_m\| \}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and all weakly null sequences such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists and $\lim_n \|x_n\| = 1$.

We should note that, if $\{x_n\}$ is a weakly null sequence in the unit ball of X such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists, then:

$$\lim_n \|x_n\| \leq \lim_{n,m;n\neq m} \|x_n - x_m\| \leq \sup_{\|x\|\leq 1} \limsup_n \|x_n + x\|.$$

Therefore, the coefficient $\lambda(X)$ always verifies $0 \leq \lambda(X) \leq 1$.

Theorem 1. *If X and Y are isomorphic Banach spaces with*

$$d(X,Y) < \frac{1+\sqrt{1+4\left(\frac{\lambda(X)}{WCS(X)}+1\right)}}{2\left(\frac{\lambda(X)}{WCS(X)}+1\right)}$$

then Y has the w-FPP. In particular, X has the w-FPP if $\lambda(X) < 1$.

Proof. Without loss of generality we may assume that $Y = (X, |\cdot|)$ and

$$\|x\| \leq |x| \leq d(X,Y)\|x\| \quad \text{for all } x \in X,$$

where $\|\cdot\|$ denotes the original norm on X.

On the contrary, suppose that Y fails to have the w-FPP. Then, by standard arguments, there exist a weakly compact convex subset K of Y with diam(K) =1 and a fixed point free nonexpansive map $T : K \to K$ such that K is minimal for T and contains an approximated fixed point sequence $\{x_n\}$ which is weakly null.

We denote by $[X]$ the quotient space $l_\infty(X)/c_0(X)$ endowed with the quotient norm $\|[z_n]\| = \limsup |z_n|$, where $[z_n]$ denotes the equivalent class of $\{z_n\} \in l_\infty(X)$. Let $[K] = \{[x_n] : x_n \in K, \text{ for } n = 1,2,\ldots\}$. Then $[T][x_n] = [Tx_n]$ is a well defined nonexpansive self mapping in $[K]$.

We consider the set

$$[W] = \{[z_n] \in [K] : |[z_n] - [x_n]| \leq 1 - t;\ \limsup_n \limsup_m |z_n - z_m| \leq t\}$$

where $t = \frac{-1+\sqrt{1+4\left(\frac{\lambda(X)}{WCS(X)}+1\right)}}{2\left(\frac{\lambda(X)}{WCS(X)}+1\right)} \in (0,1)$.

It is not difficult to check that $[W]$ is $[T]$-invariant, closed, convex and nonempty, since $[tx_n] \in [W]$. Thus, by Lin's Lemma [L] we know that

$$\sup\{\, |[w_n] - x| \, : \, [w_n] \in [W]\} = 1 \quad \text{for every } \; x \in K.$$

However, we take $[w_n] \in [W]$. Since K is weakly compact, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ weakly convergent, say to an element $w_0 \in K$. By passing to a subsequence, we can assume that $\lim_k |w_{n_k}|$ exists and it is equal to $\limsup_n |w_n| = |[w_n]|$. Also, we can suppose that the following limits exist: $\lim_k |w_{n_k} - w_0|$, $\lim_k \|w_{n_k} - w_0\|$, $\lim_{k,l;\, k\neq l} \|w_{n_k} - w_{n_l}\|$.

Since the sequence $\{w_{n_k} - x_{n_k}\}$ converges weakly to w_0, we have

$$|w_0| \leq \liminf_k |w_{n_k} - x_{n_k}| \leq \limsup_n |w_n - x_n| \leq 1 - t.$$

On the other hand, for a fixed k the sequence $\{w_{n_k} - w_{n_l}\}_l$ converges weakly to $w_{n_k} - w_0$ so, we get: $\lim_k |w_{n_k} - w_0| \leq \limsup_k \liminf_l |w_{n_k} - w_{n_l}| \leq t$. In addition, by using Goebel-Karlovitz Lemma [GK], we have

$$\lim_k |w_{n_k} - w_0| \geq \lim_k |x_{n_k} - w_0| - \limsup_k |w_{n_k} - x_{n_k}| \geq 1 - (1-t) = t.$$

Thus $t = \lim_k |w_{n_k} - w_0|$. Define $c = \lim_k \|w_{n_k} - w_0\|$. Then

$$c \geq \frac{1}{d(X,Y)} \lim_k |w_{n_k} - w_0| = \frac{t}{d(X,Y)}$$

and therefore $|w_0| \leq 1 - t \leq t/d(X,Y) \leq c$, because the definition of t and the condition on $d(X,Y)$ imply $t\left(1 + \frac{1}{d(X,Y)}\right) \geq 1$. Moreover, $\{w_{n_k} - w_0\}$ is a weakly null sequence in X. Thus

$$c \leq \frac{\lim_{k,l;k\neq l} \|w_{n_k} - w_{n_l}\|}{WCS(X)} \leq \frac{\limsup_k \limsup_l |w_{n_k} - w_{n_l}|}{WCS(X)} \leq \frac{t}{WCS(X)}.$$

Hence

$$\begin{aligned}
|[w_n]| = \lim_k |w_{n_k}| &\leq d(X,Y) c \lim_k \left\| \frac{w_{n_k} - w_0}{c} + \frac{w_0}{c} \right\| \\
&\leq d(X,Y) c \left(\lambda(X) + \frac{\lim_{k,l;k\neq l} \|w_{n_k} - w_{n_l}\|}{c} \right) \\
&\leq d(X,Y) t \left(\frac{\lambda(X)}{WCS(X)} + 1 \right) < 1
\end{aligned}$$

and this contradiction proves the result.

Connections with other geometric coefficients. We establish some relations between the coefficient $\lambda(X)$ and some other geometric coefficients defined for a Banach space X.

We recall the definition of the coefficient $R(X)$ introduced by Garcia-Falset [G] : $R(X) = \sup\{\limsup_n \|x_n + x\| : x_n \rightharpoonup 0, \|x_n\|, \|x\| \leq 1\}$.

It is known that, if X and Y are isomorphic Banach spaces such that either $d(X,Y) < WCS(X)$ [By] or $d(X,Y) < 2/R(X)$ [G], then Y has the w-FPP. In particular, X has the w-FPP if either $WCS(X) > 1$ or $R(X) < 2$.

On the other hand, the definition of $\lambda(X)$ is invariant if we replace the condition $\lim \|x_n\| = 1$ by $\|x_n\| = 1$ for every $n \in \mathrm{N}$. Then, it is easy to check the inequality:

$$\lambda(X) \leq R(X) - WCS(X).$$

Hence Theorem 1 improves Theorem 5 in [By] and Theorem 3 in [G].

We give an example where Theorem 1 assures the w-FPP, but the results in [By] and [G] cannot be applied.

Example. Consider the Bynum's spaces $l_{2,1}$ and $l_{2,\infty}$, i.e., the space l_2 with the norms $\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2$ or $\|x\|_{2,\infty} = \max\{\|x^+\|_2, \|x^-\|_2\}$, where $x^+(n) = \max\{x(n), 0\}$ and $x^-(n) = \max\{-x(n), 0\}$ are respectively the positive and the negative part of the vector x.

We define the Banach space X as the euclidean direct sum of $l_{2,1}$ and $l_{2,\infty}$ $X = l_{2,1} \oplus_2 l_{2,\infty}$. Since $R(l_{2,1}) = 2$ and $WCS(l_{2,\infty}) = 1$, $R(X) = 2$ and $WCS(X) = 1$.

Let (x, y) be a vector in X, $x \in l_{2,1}$, $y \in l_{2,\infty}$, such that $\|(x, y)\| \leq 1$ and let $\{(x_n, y_n)\}$ be a weakly null sequence in X such that $\|(x_n, y_n)\| = 1$ for every $n \in \mathrm{N}$. Then $\{x_n\}$, $\{y_n\}$ are weakly null sequences in $l_{2,1}$, $l_{2,\infty}$ respectively with $\|x_n\|_{2,1}^2 + \|y_n\|_{2,\infty}^2 = 1$.

Since $\{x_n\}$ and $\{y_n\}$ are weakly null sequences in l_2, it is not difficult to prove (see, e.g. [DJ, Proposition 2]) that we can assume, without loss of generality, that the following conditions are all fulfilled:

1) $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \operatorname{supp} y_n \cap \operatorname{supp} y_m = \emptyset$
2) $\operatorname{supp} x \cap \operatorname{supp} x_n = \operatorname{supp} y \cap \operatorname{supp} y_n = \emptyset$
3) $\|x_n^+\|_2 = \|x_m^+\|_2$, $\|x_n^-\|_2 = \|x_m^-\|_2$, $\|y_n^+\|_2 = \|y_m^+\|_2$, $\|y_n^-\|_2 = \|y_m^-\|_2$ for every $n, m \in \mathrm{N}$, $n \neq m$.

Therefore:

$$\begin{aligned}
\|(x, y) + (x_n, y_n)\|^2 &= \|x_n + x\|_{2,1}^2 + \|y_n + y\|_{2,\infty}^2 \\
&\leq (\|x\|_{2,1} + \|x_n\|_{2,1})^2 + \max\{\|y_n^+\|_2^2 + \|y^+\|_2^2, \|y_n^-\|_2^2 + \|y^-\|_2^2\} \\
&\leq (\|x\|_{2,1} + \|x_n\|_{2,1})^2 + \|y_n\|_{2,\infty}^2 + \|y\|_{2,\infty}^2 \leq 2 + 2\|x\|_{2,1}\|x_n\|_{2,1}.
\end{aligned}$$

Then

$$\|(x,y)+(x_n,y_n)\| \le \sqrt{2}\sqrt{1+\|x\|_{2,1}\|x_n\|_{2,1}} \le \sqrt{2}\sqrt{1+\|x_n\|_{2,1}}.$$

On the other hand,

$$\begin{aligned}\|(x_n,y_n)-(x_m,y_m)\|^2 &= \|x_n-x_m\|_{2,1}^2+\|y_n-y_m\|_{2,\infty}^2\\ &= \left((\|x_n^+\|_2^2+\|x_m^-\|_2^2)^{1/2}+(\|x_n^-\|_2^2+\|x_m^+\|_2^2)^{1/2}\right)^2\\ &\quad+\max\{\|y_n^+\|_2^2+\|y_m^-\|_2^2,\|y_n^-|_2^2+\|y_m^+\|_2^2\}\\ &\ge 4(\|x_n^+\|_2^2+\|x_n^-\|_2^2)+\max\{\|y_n^+\|_2^2,\|y_n^-\|_2^2\}\\ &\ge 2\|x_n\|_{2,1}^2+\|y_n\|_{2,\infty}^2=1+\|x_n\|_{2,1}^2.\end{aligned}$$

Thus

$$\|(x_n,y_n)-(x_m,y_m)\|^2 \ge \sqrt{1+\|x_n\|_{2,1}^2}.$$

Therefore

$$\begin{aligned}\lambda(X) &\le \sqrt{2}\sqrt{1+\|x_n\|_{2,1}}-\sqrt{1+\|x_n\|_{2,1}^2}\\ &\le \sup_{t\in[0,1]}\left(\sqrt{2}\sqrt{1+t}-\sqrt{1+t^2}\right)<1.\end{aligned}$$

Stability in Banach spaces with the property (M). We shall now consider a Banach space X with the property (M) introduced by Kalton. We recall that a Banach space has the property (M) if for every weakly null sequence $\{x_n\}$ the function $\Phi(x)=\limsup_n\|x_n-x\|$ depends only on $\|x\|$.

If X has the property (M) and $\{x_n\}$ is a weakly null sequence, it was proved in [GS] that $\sup\limits_{\|x\|\le 1}\limsup\limits_n\|x_n-x\|=\limsup\limits_n\limsup\limits_m\|x_n-x_m\|$. And it is easy to check that $\limsup_n\limsup_m\|x_n-x_m\|=\lim_{n,m;n\neq m}\|x_n-x_m\|$ if $\lim_{n,m;n\neq m}\|x_n-x_m\|$ exists.

Now, it is clear that $\lambda(X)=0$ if X has the property (M). In this case we obtain from Theorem 1 the following result which was first proved in [GS]:

Corollary. *If X is a Banach space with the property (M) and Y is an isomorphic Banach space such that $d(X,Y)<(1+\sqrt{5})/2$ then Y has the w-FPP.*

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