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Some Consequences of Fundamental Ordering Principles in Metric Fixed Point Theory

Abstract. We present some results of metric fixed point theory, which can be derived from the following fixed point theorems involving a partial ordering: Zermelo's Theorem, the Knaster-Tarski Theorem and the Tarski-Kantorovitch Theorem. Using ideas of Fuchssteiner and Mańka we also establish another generalization of Zermelo's Theorem, which enables to give constructive proofs (without a help of the Axiom of Choice) of fixed point theorems due to Caristi and Khamsi-Kreinovich.

1. Introduction. There are three fundamental fixed point principles which hold on ordered structures: the Zermelo Theorem [33] for progressive maps, the Knaster-Tarski Theorem for isotone maps (cf. [11], p. 14) and the Tarski-Kantorovitch Theorem for continuous maps (cf. [11], p. 15). All the above principles are independent of the Axiom of Choice (abbrev., AC) and therefore many constructive aspects in the metric fixed point theory can be derived from them. In particular, Fuchssteiner [13] has shown that Kirk's

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[21] fixed point theorem for nonexpansive maps can be proved constructively via the Zermelo Theorem, whereas the original proof given in [21] has relied on AC in the form of Zorn's Lemma. Other consequences of the Zermelo Theorem in metric fixed point theory are discussed, e.g., in papers of Büber and Kirk (in which a constructive proof of Soardi's [28] theorem for nonexpansive maps is given), Fuchssteiner [13] (a new proof of Sadovski's (cf. [31], p. 500) theorem for condensing maps is obtained too), and Mańka [22] (a constructive proof of Caristi's [9] theorem via a generalization of the Zermelo Theorem is given). Many results in metric fixed point theory can be also proved via the Knaster-Tarski Theorem as shown by Amann [1] (cf. also [31], p. 512). At last Baranga [4] has given a new proof of the Banach Contraction Principle using the Kleene fixed point theorem, which, in fact, is an equivalent reformulation of the Tarski-Kantorovitch Theorem.

In Sections 2 and 3 of this paper we survey results of our two earlier papers [17] and [18], in which we have discussed some consequences of the Knaster-Tarski Theorem and the Tarski-Kantorovitch Theorem, respectively. Section 4 is devoted to a study of some consequences of the Zermelo Theorem in metric fixed point theory. In particular, using ideas of Fuchssteiner [13] and Mańka [22], we generalize Zermelo's result (see Th.9; in fact, Th. 9 turns out to be equivalent to the Zermelo Theorem), which enables us to give another constructive proof of the Caristi [9] fixed point theorem with using neither AC nor any of its weaker forms. Moreover, a proof of Th. 9 seems to be easier than a proof of the corresponding result of Mańka (see Prop. 2 in [22]). As another application of Th. 9, we give a constructive proof of a fixed point theorem for dissipative maps on probabilistic metric spaces, which has been newly obtained by Khamsi and Kreinovich [20] with the help of Zorn's Lemma.

2. The Knaster-Tarski Principle. By a *chain* in a partially ordered set $(P, \preceq)$ we mean a linearly ordered nonempty subset of P. A selfmap F of P is said to be *isotone* if given $x, y \in P$, $x \preceq y$ implies that $Fx \preceq Fy$.

Theorem 1 (Knaster-Tarski). *Let $(P, \preceq)$ be a partially ordered set and $F : P \mapsto P$ isotone. Assume that there is a $b \in P$ such that $b \preceq Fb$ and every chain in $\{x \in P : b \preceq x\}$ has a supremum. Then F has a fixed point.*

In [17] we have shown (without using AC) that Th. 1 yields the Banach Contraction Principle as well as its extension to uniform spaces proved by Tarafdar [29]. However, Th. 1 let us deduce on the existence of a fixed point only, and it does not imply the convergence of successive approximations. We have also shown in [17] that Th. 1 yields the following fixed point theorem of Angelov (cf. [2] and [30]), but in this case our proof has relied on Zorn's Lemma. However, it seems to be interesting that this proof does not use any iteration method, a typical technique for proving results of such a kind.

Theorem 2 (Angelov). *Let X be a complete Hausdorff uniform space with a uniformity generated by a family of pseudometrics $\{p_\alpha : \alpha \in A\}$, and let f be a selfmap of X satisfying the inequalities*

(1) $$p_\alpha(fx, fy) \leq h_\alpha p_{j(\alpha)}(x, y) \text{ for } x, y \in X \text{ and } \alpha \in A,$$

where j is a selfmap of A and $h_\alpha \in (0,1)$ for $\alpha \in A$.
If $\sup\{h_{j^{n-1}(\alpha)} n \in \mathbb{N}\} < 1$ for each $\alpha \in A$, and there exists an $x_0 \in X$ such that

$$\sup\{p_{j^{n-1}(\alpha)}(x_0, fx_0) : n \in \mathbb{N}\} < \infty,$$

then f has a fixed point.

The introducing of a map j is motivated by applications in a theory of neutral functional differential equations (cf. [2], [3], [30]). In order to prove Th. 2 via Th. 1 we have defined in [17] the following partial ordering $\preceq$ in the Cartesian product $X \times \mathbb{R}_+^A$, where R_+^A denotes the set of all nonnegative real functions on A:

$$(x, \phi) \preceq (y, \psi) \quad \text{iff} \quad p_\alpha(x, y) \leq \phi(\alpha) - \psi(\alpha), \quad \text{for each} \quad \alpha \in A.$$

If A is a singleton, i.e., X is a metric space, then the above ordering coincides with the one introduced by Ekeland [12]. By Lemma 1 [17] a completeness of X implies that every chain in $(X \times \mathbb{R}_+^A, \preceq)$ has a supremum. Next, we have defined in [17] the following selfmap F of $X \times \mathbb{R}_+^A$ by

$$F(x, \phi) := (fx, h \cdot (\phi \circ j)), \quad \text{for} \quad (x, \phi) \in X \times \mathbb{R}_+^A,$$

where $(h \cdot (\phi \circ j))(\alpha) := h_\alpha \phi(j(\alpha))$. By Lemma 2 [17], condition (1) easily implies that F is isotone with respect to the ordering $\preceq$ defined by (2). So to prove Th. 2 via Th. 1 (with $P := X \times \mathbb{R}_+^A$, and $\preceq$ and F defined by (2) and (3), respectively), we need to show that there exists a pair $(x_0, \phi_0) \in X \times \mathbb{R}_+^A$ such that $(x_0, \phi_0) \preceq F(x_0, \phi_0)$. This problem leads to the following functional inequality with an unknown function ϕ:

(4) $$\phi(\alpha) \geq g(\alpha) + h(\alpha)\phi(j(\alpha)),$$

for $\alpha \in A$, where $g(\alpha) := p_\alpha(x_0, fx_0)$ and $h(\alpha) := h_\alpha$. If j is the identity map (as is in Tarafdar's [29] theorem and in the Banach Contraction Principle), then it is enough to put

$$\phi_0(\alpha) := \frac{p_\alpha(x_0, fx_0)}{1 - h_\alpha}, \quad \text{for} \quad \alpha \in A,$$

and then the proof of Th. 2 via Th. 1 is completed. However, in many applications j is not the identity map. Then, in order to solve (4 on A, it suffices to have a solution of (4) on each orbit of a map j according to the following

Lemma 1 (cf. [17]. *Let j be a selfmap of an abstract set A, $g : A \mapsto \mathbb{R}_+$, and $h : A \mapsto [0,1]$. If for each $\beta \in A$ inequality (4) has a solution $\phi_\beta : O_j(\beta) \mapsto \mathbb{R}_+$, where $O_j(\beta) := \{j^{n-1}(\beta) : n \in \mathbb{N}\}$, then there is a global solution of (4), i.e., a function ϕ, which satisfies (4) for all $\alpha \in A$.*

To prove Lemma 1, we have used in [17] a Hahn-Banach type argument. The proof of Lemma 1 given in [17] could be compared with a direct proof of a theorem on the existence of a non-trivial continuous functional on a normed space, the result, which is usually obtained as a corollary to the Hahn-Banach Theorem. Instead of one-dimensional subspaces (i.e., minimal, with respect to the inclusion, non-trivial linear subspaces), we have here orbits of a map j (i.e., minimal, with respect to the inclusion, j-invariant subsets of A). Instead of functionals, we consider solutions of a functional inequality. In both cases, using Zorn's Lemma let us infer that there exists an appropriate object (a functional or a function, respectively) defined on the whole space, provided that such objects can be defined on appropriate minimal sets. Now, while there is no problem with defining a continuous linear functional on any one-dimensional subspace, a corresponding problem for a functional inequality is not so trivial and it can be solved, e.g., if all the assumptions of Th. 2 are satisfied (cf. the proof of Corollary 1 in [17]).

The fact that Th. 1 implies the Banach Contraction Principle was observed earlier by Amann [1], however, under the additional hypothesis of boundedness of a metric space (X, d) considered (cf. also [31], p. 512). He used a partial ordering $\preceq$ defined in the family of all nonempty closed subsets K of X with $F(K) \subseteq K$ by $K \preceq L$ iff $K = L$ or $K \subseteq \text{cl}(T(L))$, where $\text{cl}(T(L))$ denotes the closure of the set $T(L)$.

We conclude this section with a remark that Nadler's [24] fixed point theorem, an extension of the Banach Contraction Principle to set-valued maps, can be derived from Smithson's [27] generalization of the Knaster-Tarski Principle to multifunctions on ordered structures as we have shown in [17].

3. The Tarski-Kantorovitch Principle. A selfmap F of a partially ordered set $(P, \preceq)$ is said to be *continuous with respect to an ordering* $\preceq$ if for each countable chain C having a supremum, $F(\sup C) = \sup F(C)$. Each continuous map is necessarily isotone (cf. [11], p. 15).

Theorem 3 (Tarski-Kantorovitch). *Let $(P, \preceq)$ be a partially ordered set and $F : P \mapsto P$ continuous. Assume that there is a $b \in P$ such that $b \preceq Fb$ and every countable chain in $\{x \in P : b \preceq x\}$ has a supremum. Then the set of fixed points of F is nonempty. Moreover, $\mu := \sup\{F^n(b) : n \in \mathbb{N}\}$ is a fixed point of F.*

Using the ordering $\preceq$ defined by (2) and the selfmap F of $X \times \mathbb{R}_+^A$ defined by (3), we have shown in [18] that Th. 3 yields Th. 2 (cf. the proof of Th. 3 in [18]). Additionally, the convergence of successive approximations of a map f on X can be proved via Th. 2, the fact non-obtainable via the Knaster-Tarski Principle. In fact, at first we have shown in [18] that Th. 3 implies the Banach Contraction Principle including both the existence of a fixed point and the constructive formula for it (i.e., the convergence of successive approximations). Next, we have proved Th. 2 by applying both Th. 3 and the Contraction Principle thanks to which we have avoided of using Zorn's Lemma this time. Moreover, it suffices to assume in Th. 2 that a uniform space X is *sequentially* complete. This property of X implies that every countable chain in $(X \times \mathbb{R}_+^A, \preceq)$ has a supremum (cf. Prop. 1 in [18]) as required in Th. 3. In a metric setting the reciprocal of the last implication also holds according to the following

Proposition 1 (cf. Prop. 2 in [18]). *Let (X, d) be a metric space and $\preceq$ be the ordering in $(X \times \mathbb{R}_+, \preceq)$ defined by* (2). The following conditions are equivalent.

(i) (X, d) is complete.

(ii) Every chain in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.

(iii) Every countable chain in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.

(v) Every increasing sequence in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.

In fact, condition (v) implies the sequential completeness of X as shown in [18], and we need to use the Axiom of Choice for countable families in a proof of the implication (v) $\Rightarrow$ (i).

In [18] we have also examined possibilities of deriving from Th. 3 some results for nonlinear contractions. Given a function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$, we say that a selfmap f of a metric space (X, d) is a *ϕ-contraction* if

$$d(fx, fy) \leq \phi(d(x, y)) \quad \text{for all} \quad x, y \in X.$$

Such mappings were studied by a number of authors (cf., e.g., [5], [6], [16], [17], [18], [23], [25], [34]). Actually, we have investigated in [18] consequences of Th. 3 restricted to the ordering $\preceq$ in $(X \times \mathbb{R}_+, \preceq)$ defined by (2), i.e., $(x,a) \preceq (y,b)$ iff $dd(x,y) \leq a - b$, and the operator F on $X \times \mathbb{R}_+$ defined by $F(x,a) := (fx, Ta)$, for $x \in X$ and $a \in \mathbb{R}_+$.

In view of Prop. 2 and Prop. 4 in [18], the following theorem corresponds exactly to the restriction of the Tarski-Kantorovitch Principle to the above ordering and the operator F (cf. Th. 4 in [18]).

Theorem 4. *Let f be a continuous selfmap of a complete metric space (X,d) such that, given $a, b \in \mathbb{R}_+$ and $x, y \in X$, $d(x,y) \leq a - b$ implies that $d(fx, fy) \leq Ta - Tb$, where $T : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is right continuous. If there exists $a_0 \in \mathbb{R}_+$ and $x_0 \in X$ such that $d(x_0, fx_0) \leq a_0 - Ta_0$, then f has a fixed point.*

It is natural to assume that the set $\{a \in \mathbb{R}_+ : Ta < a\}$ is non-empty; for otherwise, Th. 4 would be trivial since $d(x_0, fx_0) \leq a_0 - Ta_0$ implied then that $x_0 = fx_0$. It turns out that, in general, Th. 4 can be applied only to Banach contractions (then it suffices to put $Ta := ha$, where h is a contractive constant of f). In particular, the Rakotch [25] fixed point theorem cannot be proved via Th. 4 according to the following result (cf. Th. 6 and Rem. 4 in [18]).

Theorem 5. *Let a function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be non-decreasing, $\phi(t) < t$ for $t > 0$ and $t \mapsto \phi(t)/t$ $(t > 0)$ is non-increasing. Then the following conditions are equivalent.*

(i) *There is an $h \in (0,1)$ such that $\phi(t) \leq h\,t$ for all $t \in \mathbb{R}_+$.*

(ii) *Given a complete metric space (X,d) and a ϕ-contraction $f : X \mapsto X$, f is a Banach contraction.*

(iii) *Given a complete metric space (X,d) and a ϕ-contraction $f : X \mapsto X$, there is a $T : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that the assumptions of* Th. 4 *hold and the set $\{a \in \mathbb{R}_+ : Ta < a\}$ is non-empty.*

So in particular, if $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a function as in Rakotch's [25] theorem with the property that for each ϕ-contraction the Tarski-Kantorovitch Principle can be applied in the way described above, then each ϕ-contraction is a Banach contraction. So Th. 5 has a negative impact on the methodological possibilities of deriving from the Tarski-Kantorovitch Principle fixed point results for nonlinear contractions. Nevertheless, there exist ϕ-contractive maps (which are not Banach contractions), for which Th. 4 does apply. In particular, each continuous superadditive function ϕ such that $\lim_{t\to\infty} \phi(t)/t = 1$ generates such a ϕ-contraction (cf. Th. 7 and Rem. 5 in [18]).

4. The Zermelo Principle. A selfmap F of a partially ordered set $(P, \preceq)$ is said to be *progressive* iff $x \preceq Fx$, for all $x \in X$.

Theorem 6 (Zermelo). *Let $(P, \preceq)$ be a partially ordered set and $F : P \mapsto P$ progressive. If every non-empty well-ordered subset of P has a supremum, then F has a fixed point.*

In [19] we have shown that Th. 6 yields directly the following restriction of Caristi's [9] fixed point theorem.

Theorem 7. *Let f be a selfmap of a complete metric space (X, d) and a function $\phi : X \mapsto \mathbb{R}_+$ be continuous. If $d(x, fx) \leq \phi(x) - \phi(fx)$ for all $x \in X$, then f has a fixed point.*

In Caristi's theorem a function ϕ need not be continuous, but only lower semicontinuous. Then, however, it can be proved that every chain C (hence every non-empty well-ordered subset) in X endowed with Br≤ndsted's [7] ordering $\preceq$ defined by

$$x \preceq y \quad \text{iff} \quad d(x, y) \leq \phi(x) - \phi(y), \tag{5}$$

has an upper bound (cf. [10]), whereas a supremum of C need not exist. However, a supremum of C does exist if we assume that ϕ is continuous (cf. the proof of Th. 2 in [19]), and then Th. 6 applies. In particular, Zermelo's Theorem implies the Banach Contraction Principle (this fact was also observed by Fuchssteiner [13], but under the assumption of boundedness of X): if f is a Banach contraction with a contractive constant $h \in (0, 1)$, then f is progressive with respect to ordering (5) generated by a (continuous) function $\phi(x) := (1 - h)^{-1} d(x, fx)$ (cf. [19]).

On the other hand, the assumption of Th. 7 on continuity of ϕ is not very strong, because it turns out that, under AC, Th. 7 is equivalent to Caristi's theorem (cf. Th. 3 in [19]).

In 1988 Mańka [22] gave a wholly constructive proof of Caristi's theorem. He used his generalization of Zermelo's Theorem involving a notion of a sup-function (cf. [22] for a definition). Mańka's argument is a development of an idea of Zermelo's [32] proof. Our purpose here is to establish another extension of Zermelo's Theorem, which can be proved in a simpler way and which is strong enough to imply Caristi's theorem without a help of AC. Actually, this extension is closely related to the following Bourbaki-Kneser Principle (in a terminology of Zeidler - cf. [31], p. 504).

Theorem 8 (Bourbaki-Kneser). *Let $(P, \preceq)$ be a partially ordered set and $F : P \mapsto P$ progressive. If every chain in P has a supremum, then F has a fixed point.*

In fact, Th. 7 and Th. 8 are equivalent because of the following lemma (cf., e.g., [27]), which can be proved without AC.

Lemma 2. *Let $(P, \preceq)$ be a partially ordered set. The following conditions are equivalent:*

(i) *every chain in P has a supremum;*

(ii) *every non-empty well-ordered subset of P has a supremum.*

The following result extends Th. 8 (hence Th. 6) and it is closely related to Prop. 2 in [22]. We follow Mańka's advice at the end of his paper [22], but, in fact, the same idea (introducing a function $\mathcal{F}$ defined below) appeared earlier in Fuchssteiner [13]. Both authors used the partial ordering "to be an initial segment" (i.e., for $T, S \subseteq P$, $T \preceq S$ means that $T \subseteq S$ and each $s \in S \setminus T$ is an upper bound of T), whereas we shall simply use the set-theoretical inclusion "$\subseteq$". The family of all chains in P is denoted by $\mathcal{C}(P)$.

Theorem 9. *Let P be a partially ordered set and $F : P \mapsto P$ progressive. Assume that there exists a function $\sigma : \mathcal{C}(P) \mapsto P$ such that for each $C \in \mathcal{C}(P)$, $\sigma(C)$ is an upper bound of C. Then F has a fixed point.*

We shall derive Th. 9 from the following restriction of Th. 8 to maps, on and to a family of subsets of a set, which are progressive under the set-theoretical inclusion. The family of all subsets of a set P is denoted by 2^P.

Theorem 10. *Let P be a set, $\mathcal{P}$ be a non-empty subset of 2^P and $\mathcal{F}$ be a selfmap of $\mathcal{P}$ such that $A \subseteq \mathcal{F}(A)$ for all $A \in \mathcal{P}$. If for every chain $\mathcal{C}$ in $(\mathcal{P}, \subseteq)$, $\bigcup_{C \in \mathcal{C}} C \in \mathcal{P}$, then $\mathcal{F}$ has a fixed point.*

Proof of Theorem 9. For $C \in \mathcal{C}(P)$, define

$$\mathcal{F}(C) := C \cup F(\sigma(C)). \tag{6}$$

Since by hypothesis, for $x \in C$, $x \preceq \sigma(C) \preceq F(\sigma(C))$, we infer that $F(\sigma(C))$ is an upper bound of C and hence $\mathcal{F}(C)$ is a chain, i.e., $\mathcal{F}$ is a selfmap of $\mathcal{C}(P)$. Clearly, $\mathcal{F}$ is progressive with respect to the set-theoretical inclusion. Moreover, if $\mathcal{C}$ is a chain in $(\mathcal{C}(P), \subseteq)$ then $\bigcup_{C \in \mathcal{C}} C \in \mathcal{C}(P)$ as can be easily verified. So we may apply Th. 10 (with $\mathcal{P} := \mathcal{C}(P)$) to infer that $\mathcal{F}$ has a fixed point C_0. By (6), $F(\sigma(C_0)) \in C_0$ so by the property of σ, $F(\sigma(C_0)) \preceq \sigma(C_0)$, and, on the other hand, $\sigma(C_0) \preceq F(\sigma(C_0))$ since F is progressive. Hence, we get that $\sigma(C_0)$ is a fixed point of F.

□

Since the implications Th. 9 ⇒ Th. 8 ⇒ Th. 10 are obvious, we get (without AC) the following

Proposition 2. *Th. 6, Th. 8, Th. 9 and Th. 10 are equivalent.*

Remark 1. A counterpart of Th. 9 for $WO(P)$, the family of all non-empty well-ordered subsets of P, substituted for $\mathcal{C}(P)$ (which corresponds to Prop. 2 in [22]) cannot be proved in the above way, since condition $\cup_{C \in \mathcal{C}} C \in WO(P)$ need not hold for every chain $\mathcal{C}$ in $(WO(P), \subseteq)$.

Remark 2. Th. 9 yields directly Caristi's fixed point theorem, i.e., Th. 7 with the assumption of lower semicontinuity of ϕ substituted for continuity of ϕ. To see it, it suffices to endow a set X with ordering $\preceq$ defined by (5) and for each chain C in $(X, \preceq)$, put $\sigma(C) := \lim C$, since each chain treated as a net is convergent and its limit is an upper bound of it (cf., e.g., [10]). Clearly, f is progressive with respect to $\preceq$.

In the sequel we shall give a constructive proof of the Khamsi-Kreinovich [20] fixed point theorem for maps on probabilistic metric spaces (Menger spaces). For a definition of Menger spaces, we refer to [20] and, for a more detailed discussion, to [26]. We emphasize here that the authors of [20] assume that distribution functions are *right*-continuous, which is a deviation from the established terminology according to which distribution functions are left-continuous. In this case, however, some further changes are necessary. In particular, in Def.1 [KK] the authors should assume that for $x, y \in X$, $F_{x,y} \in \Delta^+$ (the set of all distributions F such that $F(0) = 0$) only if $x \neq y$, because their condition (2) ($F_{x,x}(t) = 1$ for $t > 0$ and $x \in X$) and right-continuity of distributions imply then that $F_{x,x}(0) = 1$, so that $F_{x,x} \notin \Delta^+$. Simultaneously, the condition that $F_{x,x}(0) = 1$ is also necessary for other purposes (see the definition of ϕ-dissipative map and condition (7) given below), so we really need to work with right-continuous distributions.

Let (X, F, t) be a Menger space, where t is a continuous T-norm, and let $\phi : X \mapsto \mathbb{R}_+$ be a lower semicontinuous function. Following [20] (with some slight changes, however) we say that a map $f : X \mapsto X$ is *ϕ-dissipative* iff there exists a function $h : \mathbb{R}_+ \mapsto (0, 1]$ such that $\lim_{\epsilon \to 0^+} h(\epsilon) = 1$, $h(a + b) \leq t(h(a), h(b))$ for all $a, b \in \mathbb{R}_+$ and

$$h(\phi(x) - \phi(fx)) \leq F_{x,fx}(\phi(x) - \phi(fx)), \quad \text{for all} \quad x \in X.$$

The above assumptions imply that h is non-increasing (cf.[KK]) and $h(0)=1$.

Theorem 11 (Khamsi-Kreinovich). *Let (X,F,t) be a complete Menger space and $f : X \mapsto X$ be ϕ-dissipative map. Then f has a fixed point.*

In their proof of Th. 11 the authors of [20] have used twice AC for countable families, and Zorn's Lemma. Our purpose here is to give a wholly constructive proof of Th. 11, without any choice. Following [20] we define the partial ordering $\preceq$ in X by

$$(7) \qquad x \preceq y \quad \text{iff} \quad h(\phi(x)-\phi(y)) \leq F_{x,y}(\phi(x)-\phi(y)).$$

(Observe that $\preceq$ is reflexive thanks to the assumption that $F_{x,x}(0)=1$.). In the sequel we shall need the following two technical lemmas. Both of them can be proved without AC.

Lemma 3. *Let (X,F,t) be a Menger space with continuous t and let $x,y \in X$. If a net $\{x_\sigma\}_{\sigma\in\Sigma}$ converges to x, then $F_{x_\sigma,y}(a) \to F_{x,y}(a)$ for every $a \in R$, which is a point of continuity of $F_{x,y}$.*

Lemma 3 can be proved by using a similar argument as in the proof of Lemma in [20].

Lemma 4. *Let $g : R \mapsto R$ be non-decreasing and $x_0, y_0 \in R$. If $g(x) \geq y_0$ for every $x > x_0$, which is a point of continuity of g, then $g(x) \geq y_0$ for all $x > x_0$.*

Lemma 4 is an immediate consequence of the well-known theorem (obtainable without AC) saying that the set of points of discontinuity of a monotonic function is at most countable.

Proposition 3. *Let (X,F,t) be a complete Menger space endowed with the partial ordering $\preceq$ defined by (7). Then every chain C in $(X,\preceq)$ is convergent as a net, and $\lim C$ is an upper bound of C.*

Proof. Let C be a chain in $(X,\preceq)$. We may treat C as a net by putting $\Sigma := C$ and $x_\sigma := \sigma$ for $\sigma \in \Sigma$. By (7), if $\alpha, \beta \in C$ and $\alpha \preceq \beta$, then $h(\phi(x_\alpha) - \phi(x_\beta)) \leq F_{x_\alpha,x_\beta}(\phi(x_\alpha)-\phi(x_\beta))$, so $\phi(x_\alpha) \geq \phi(x_\beta)$, i.e., $\{\phi(x_\sigma)\}_{\sigma\in C}$ is decreasing, hence convergent to some $r \geq 0$. We show that the net $\{x_\sigma\}_{\sigma\in C}$ is Cauchy. Let $a > 0$ and $b \in (0,1)$. By hypothesis, there is a $t_0 > 0$ such that $h(t) > 1-b$ for all $t \in [0,t_0]$. Since $\{\phi(x_\sigma)\}_{\sigma\in C}$ is Cauchy, there is a $\sigma \in C$ such that if $\sigma \preceq \alpha \preceq \beta$, then $0 \leq \phi(x_\alpha) - \phi(x_\beta) < \min\{a, t_0\}$. Hence and by monotonicity, for such α and β, $F_{x_\alpha,x_\beta}(a) \geq F_{x_\alpha,x_\beta}(\phi(x_\alpha) - \phi(x_\beta)) \geq h(\phi(x_\alpha) - \phi(x_\beta)) > 1 - b$. That means $\{x_\sigma\}_{\sigma\in C}$ is Cauchy, hence convergent to some $x_0 \in X$. Then, by hypothesis, $\phi(x_0) \leq \lim_{\sigma\in C} \phi(x_\sigma) = r$, and $r \leq \phi(x_\sigma)$ for $\sigma \in C$. Hence, if $\alpha, \beta \in C$ and $\alpha \preceq \beta$, then

$$F_{x_\alpha,x_\beta}(a_\alpha) \geq F_{x_\alpha,x_\beta}(a_{\alpha\beta}) \geq h(a_{\alpha\beta}) \geq h(a_\alpha),$$

where $a_\alpha := \phi(x_\alpha) - \phi(x_0)$ and $a_{\alpha\beta} := \phi(x_\alpha) - \phi(x_\beta)$.

Fix an $\alpha \in C$. Let $a' > a_\alpha$ and a' be a point of continuity of a function F_{x_α,x_0}. Then, $F_{x_\alpha,x_\beta}(a') \geq F_{x_\alpha,x_\beta}(a_\alpha) \geq h(a_\alpha)$. Hence and by Lemma 3, $F_{x_\alpha,x_0}(a') = \lim_{\beta\in C} F_{x_\alpha,x_\beta}(a') \geq h(a_\alpha)$. By Lemma 4, we get that $F_{x_\alpha,x_0}(a) \geq h(a_\alpha)$ for all $a > a_\alpha$. Since F_{x_α,x_0} is right-continuous, we may infer that $F_{x_\alpha,x_0}(a_\alpha) \geq h(a_\alpha)$. That means $x_0\, (= \lim C)$ is an upper bound of C.

□

Th. 11 is an immediate consequence of Th. 9 and Prop. 3.

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References

[1] Amann, H., *Order Structures and Fixed Points*, Ruhr-Universität, Bochum, 1977 (mimeographed lecture notes).

[2] Angelov, V. G., *Fixed point theorems in uniform spaces and applications*, Czechoslovak Math. J. **37** (1987), 19-33.

[3] ———, *On the Synge equations in a three-dimensional two-body problem of classical electrodynamics*, J. Math. Anal. Appl. **151** (1990), 488-511..

[4] Baranga, A., *The contraction principle as a particular case of Kleene's fixed point theorem*, Discrete Math. **98** (1991), 75-79.

[5] Boyd, D. W. and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.

[6] Browder, F. E., *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27-35.

[7] Br≤ndsted, A., *On a lemma of Bishop and Phelps*, Pacific J. Math. **55** (1974), 335-341.

[8] Büber, T. and W. A. Kirk, *A constructive proof of a fixed point theorem of Soardi*, Math. Japon. **41** (1995), 233-237.

[9] Caristi, J., *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241-251.

[10] Downing, D and W. A. Kirk, *A generalization of Caristi's theorem with applications to nonlinear mapping theory*, Pacific J. Math. **69** (1977), 339-345.

[11] Dugundji, J. and A. Granas, *Fixed Point Theory I*, PAN, Warszawa 1982.

[12] Ekeland, I. On the variational principle, J. Math. Anal. Appl. **47** (1974), 324-353.

[13] Fuchssteiner, B., *Iterations and fixpoints*, Pacific J. Math. **68** (1977), 73-80.

[14] Jachymski, J. R., *An iff fixed point criterion for continuous self-mappings of a complete metric space*, Aequationes Math. **48** (1994), 163-170.

[15] ———, *An extension of A. Ostrovski's theorem on the round-off stability of iterations*, Aequationes Math. **53** (1997), 242-253.

[16] ———, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. (1997 (to appear)).

[17] ———, *Fixed point theorems in metric and uniform spaces via the Knaster-Tarski Principle*, Nonlinear Anal. T. M. A. (1997 (to appear)).

[18] ———, *Some consequences of the Tarski-Kantorovitch ordering theorem in metric fixed point theory*, Quaestiones Math. to appear.

[19] ———, *Caristi's fixed point theorem and selections of set-valued contractions (submitted)*.

[20] Khamsi, M. A. and V. Y. Kreinovich, *Fixed point theorems for dissipative mappings in complete probabilistic metric spaces*, Math. Japon. **44** (1996), 513-520.

[21] Kirk, W. A., *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly **72** (1965), 1004-1006.

[22] Mańka, R., *Some forms of the axiom of choice*, Jahrb. Kurt Gödel Ges. **1** (1988), 24-34, Wien..

[23] Matkowski, J., *Integrable solutions of functional equations*, Dissertationes Math. (Rozprawy Mat.) **127**, 1975.

[24] Nadler,S. B. Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-488.

[25] Rakotch, E., *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459-465.

[26] Schweizer, B and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics **5**, 1983.

[27] Smithson, R. E., *Fixed points of order preserving multifunctions*, Proc. Amer. Math. Soc. **28** (1971), 304-310.

[28] Soardi, P., *Existence of fixed points of nonexpansive mappings in certain Banach lattices*, Proc. Amer. Math. Soc. **73** (1979), 25-29.

[29] Tarafdar, E., *An approach to fixed-point theorems on uniform spaces*, Trans. Amer. Math. Soc. **191** (1974), 209-225.

[30] Tsatchev, Ts. and V. G. Angelov, *Fixed points of nonself-mappings and applications*, Nonlinear Anal. T. M. A. **21** (1993), 9-16.

[31] Zeidler, E., *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, Berlin, 1986.

[32] Zermelo, E., *Beweis, das jede Menge wohlgeordnet werden kann*, Math. Ann. **59** (1904), 514-516.

[33] ———, *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Math. Ann. **65** (1908), 107-128.

[34] Zitarosa, A., *Una generalizzazione del teorema di Banach sulle contrazioni*, Matematiche **23** (1968), 417-424.

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